

Strong theories and weight

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Strong theories

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$((\varphi^\alpha(x, y^\alpha), k^\alpha) \mid \alpha < \kappa)$ is an inp-pattern
(inp = 'independent partitions')
for $p(x)$, if an array $(b_i^\alpha \mid i < \omega, \alpha < \kappa)$ witnesses it in the following sense:

- Every row $\{\varphi^\alpha(x, b_i^\alpha) \mid i < \alpha\}$ is k^α -inconsistent.
- Every path $\{\varphi^\alpha(x, b_{\eta(\alpha)}^\alpha) \mid \alpha < \kappa\}$ is consistent with $p(x)$.

$\kappa_{\text{inp}}(p)$ is the minimal κ such that no inp-pattern of length κ exists for p .

$\kappa_{\text{inp}}(T) = \sup_p \kappa_{\text{inp}}(p)$ is the minimal κ such that no inp-pattern of length κ exists.

T is strong if $\kappa_{\text{inp}}(T) = \aleph_0$,
i.e. no infinite inp-patterns exist.

(T does not have the tree property of the second kind if $\kappa_{\text{inp}}(T) < \infty$.)

Strong theories, continued

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Proposition 1. *Given an inp-pattern for p and a set B , there is always a witnessing array such that each sequence $(b_i^\alpha \mid i < \omega)$ is indiscernible over $B \cup \{b_i^\beta \mid i < \omega, \beta \neq \alpha\}$.*

So if p is a partial type over B , then

$$\kappa_{\text{inp}}(p(x)) = \sup\{\kappa_{\text{inp}}(q(x)) \mid p \subseteq q \in \mathcal{S}(B)\}.$$

The simple case

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Recall the definition of preweight (T simple):

$$\begin{aligned} \text{pwt}(a/C) = \\ \sup\{\kappa \mid \exists(b_i \mid i < \kappa) \text{ independent over } C, \\ \text{s.t. } \forall i : a \not\perp_C b_i.\} \end{aligned}$$

Proposition 2. (T simple)

$p(x)$ partial type over C . Then

$$\kappa_{\text{inp}}(p) = \left(\sup\{\text{pwt}(q) \mid p \subseteq q \in \mathcal{S}(B), B \supseteq C\} \right)^{(+)}.$$

Corollary 3. A simple theory is strong if and only if every type has finite weight. Thus supersimple theories (and even theories with no dense forking chains) are strong.

The NIP case

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$(\varphi^\alpha(x, y^\alpha) \mid \alpha < \kappa)$ is an ict-pattern
(ict = 'independent contradictory types')
for $p(x)$, if an array $(b_i^\alpha \mid i < \omega, \alpha < \kappa)$ witnesses it in the following sense:

For every path $\eta \in \omega^\kappa$ the set

$$\{\varphi^\alpha(x, b_i^\alpha) \mid \alpha < \kappa, i < \omega; \eta(\alpha) = i\} \cup \\ \{\neg\varphi^\alpha(x, b_i^\alpha) \mid \alpha < \kappa, i < \omega; \eta(\alpha) \neq i\}$$

is consistent.

$\kappa_{\text{ict}}(p)$ is the minimal κ such that no ict-pattern of length κ exists for p .

$\kappa_{\text{ict}}(T) = \sup_p \kappa_{\text{ict}}(p)$ is the minimal κ such that no ict-pattern of length κ exists.

T is strongly dependent if $\kappa_{\text{ict}}(T) = \aleph_0$,
i.e. no infinite ict-patterns exist.

$\kappa_{\text{ict}}(T) = \infty$ if and only if T has the independence property. (Easy.)

The NIP case, continued

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Proposition 4. *If T does not have the independence property, then $\kappa_{\text{inp}} = \kappa_{\text{ict}} < \infty$. Otherwise $\kappa_{\text{inp}} \leq \kappa_{\text{ict}} = \infty$.*

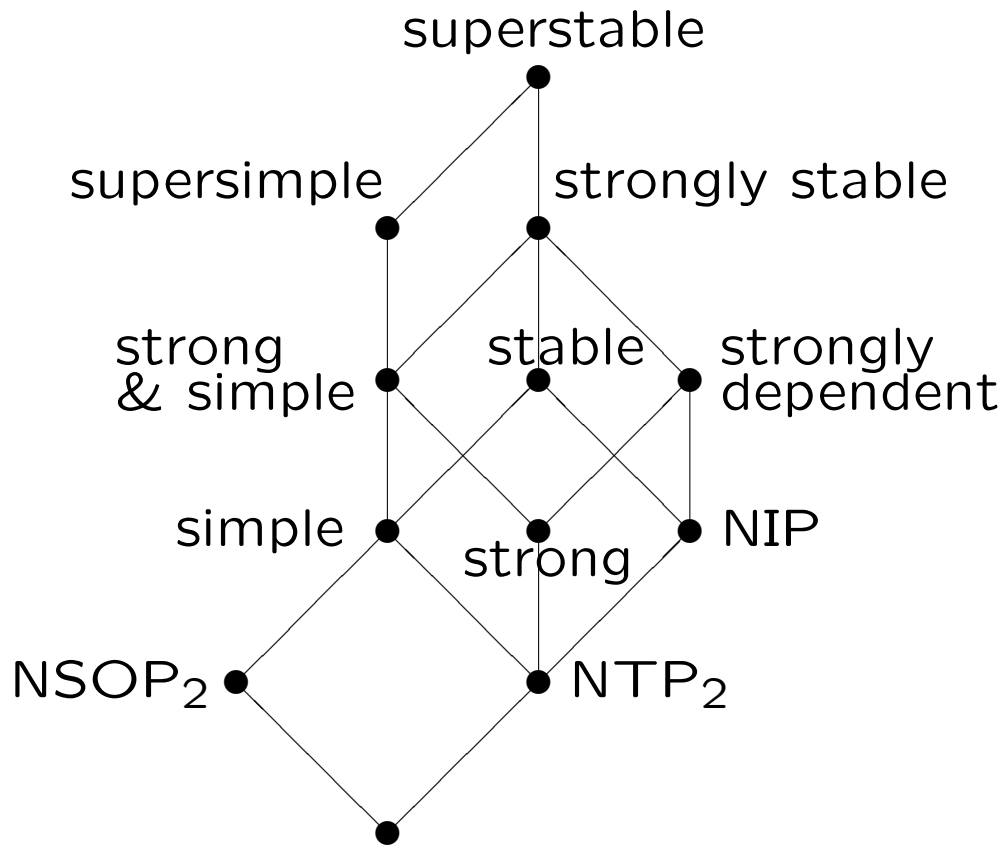
Corollary 5. *A theory is strongly dependent if and only if it is strong and does not have the independence property.*

Corollary 6. *A stable theory is strongly dependent if and only if every type has finite weight.*

Fact 7. *O-minimal theories and p-minimal theories are strongly dependent.*

A picture

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