Strong theories and weight

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Strong theories (1/6)

$((\varphi^\alpha(x,y^\alpha),k^\alpha)\mid\alpha<\kappa)$ is an inp-pattern (inp = ‘independent partitions’) for $p(x)$, if an array $(b^\alpha_i \mid i<\omega,\ \alpha<\kappa)$ witnesses it in the following sense:

- Every row $\{\varphi^\alpha(x,b^\alpha_i)\mid i<\alpha\}$ is k^α-inconsistent.
- Every path $\{\varphi^\alpha(x,b^\alpha_{\eta(\alpha)})\mid\alpha<\kappa\}$ is consistent with $p(x)$.

$\kappa_{\mathsf{inp}}(p)$ is the minimal κ such that no inp-pattern of length κ exists for p.

$\kappa_{\mathsf{inp}}(T)=\sup_p\kappa_{\mathsf{inp}}(p)$ is the minimal κ such that no inp-pattern of length κ exists.

T is strong if $\kappa_{\mathsf{inp}}(T)=\aleph_0$, i.e. no infinite inp-patterns exist.

(T does not have the tree property of the second kind if $\kappa_{\mathsf{inp}}(T)<\infty$.)

Proposition 1. *Given an inp-pattern for* p *and a set* B*, there is always a witnessing array such that each sequence* $(b_i^\alpha \mid i < \omega)$ *is indiscernible over* $B \cup \{b_i^\beta \mid i < \omega,\, \beta \neq \alpha\}$*.*

So if p is a partial type over B, then
$\kappa_{\mathsf{inp}}(p(x)) = \mathsf{sup}\{\kappa_{\mathsf{inp}}(q(x)) \mid p \subseteq q \in \mathsf{S}(B)\}$.

The simple case (3/6)

Recall the definition of preweight (T simple):

$$\text{pwt}(a/C) = \sup\{\kappa \mid \exists (b_i \mid i < \kappa) \text{ independent over } C, \text{ s.t. } \forall i: a \not\downarrow_C b_i.$$

Proposition 2. *(T simple)* *$p(x)$ partial type over C. Then*

$$\kappa_{\text{inp}}(p) = \Big(\sup\{\text{pwt}(q) \mid p \subseteq q \in \mathsf{S}(B),\, B \supseteq C\}\Big)^{(+)}.$$

Corollary 3. *A simple theory is strong if and only if every type has finite weight. Thus supersimple theories (and even theories with no dense forking chains) are strong.*

The NIP case (4/6)

$(\varphi^\alpha(x, y^\alpha) \mid \alpha < \kappa)$ is an ict-pattern (ict = 'independent contradictory types') for $p(x)$, if an array $(b_i^\alpha \mid i < \omega, \alpha < \kappa)$ witnesses it in the following sense:

For every path $\eta \in \omega^\kappa$ the set

$$\{\varphi^\alpha(x, b_i^\alpha) \mid \alpha < \kappa, i < \omega; \eta(\alpha) = i\} \cup$$
$$\{\neg\varphi^\alpha(x, b_i^\alpha) \mid \alpha < \kappa, i < \omega; \eta(\alpha) \neq i\}$$

is consistent.

$\kappa_{\text{ict}}(p)$ is the minimal κ such that no ict-pattern of length κ exists for p.

$\kappa_{\text{ict}}(T) = \sup_p \kappa_{\text{ict}}(p)$ is the minimal κ such that no ict-pattern of length κ exists.

T is strongly dependent if $\kappa_{\text{ict}}(T) = \aleph_0$, i.e. no infinite ict-patterns exist.

$\kappa_{\text{ict}}(T) = \infty$ if and only if T has the independence property. (Easy.)

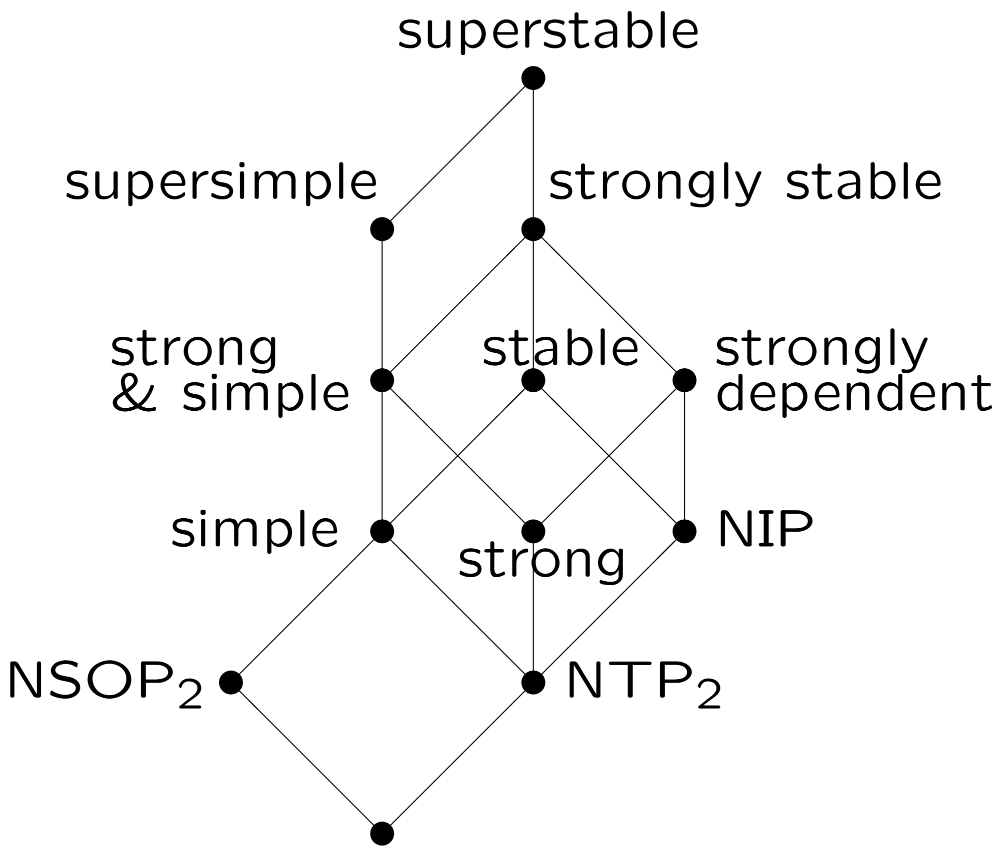
Proposition 4. *If T does not have the independence property, then $\kappa_{\text{inp}} = \kappa_{\text{ict}} < \infty$. Otherwise $\kappa_{\text{inp}} \leq \kappa_{\text{ict}} = \infty$.*

Corollary 5. *A theory is strongly dependent if and only if it is strong and does not have the independence property.*

Corollary 6. *A stable theory is strongly dependent if and only if every type has finite weight.*

Fact 7. *O-minimal theories and p-minimal theories are strongly dependent.*

A picture (6/6)


superstable
supersimple
strongly stable
strong
& simple
stable
strongly
dependent
simple
strong
NIP
NSOP$_2$
NTP$_2$

References

Shelah.
Classification Theory and the Number of Non-Isomorphic Models (1978).

Shelah (Sh93).
Simple unstable theories. Ann. Math. Logic 19 (1980) 177–203.

Shelah (Sh783).
Dependent first order theories, continued. Israel J. Math., accepted.

Shelah (Sh863).
Strongly dependent theories. Israel J. Math., submitted.

Onshuus and Usvyatsov.
On dp-minimal stable theories. Early draft.

Adler.
Strong theories and weight. Early draft.