

Thorn-forking

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Shrinking indiscernibles

Contents:

1. **Pregeometries**
2. Independence by pregeometry
3. **Strict independence relations**
4. Infinite forest
5. **First candidate**
6. Full transitivity
7. **Second candidate**
8. **Third candidate**
9. **Stable theories**
10. The lattice connection

Pregeometries

1/X

(Model theoretic) algebraic closure always satisfies the axioms of an algebraic closure operator:

- $A \subseteq \text{acl } A$.
- $A \subseteq B \Rightarrow \text{acl } A \subseteq \text{acl } B$.
- $\text{acl}(\text{acl } A) = \text{acl } A$.
- $\text{acl } A = \bigcup_{\text{finite } A_0 \subseteq A} \text{acl } A_0$.

Sometimes it also satisfies the exchange law:

- $a \in \text{acl}(Cb) \setminus \text{acl } C \Rightarrow b \in \text{acl}(Ca)$.

Algebraic closure operator + exchange law
= pregeometry.

Examples: vector spaces, alg. closed fields.
In a pregeometry, maximal independent set
= minimal generating set. We have a good
notion of basis and dimension.

Independence by pregeometry

2/X

Exchange law:

- $a \in \text{acl}(Cb) \setminus \text{acl } C \quad \Rightarrow \quad b \in \text{acl}(Ca).$

This implies atomic points!

Without atomic points we can still have a notion of independence like linear or algebraic independence.

How can we find it?

In a pregeometry as follows:

$$A \downarrow_C B \iff$$

for all $B_0 \subseteq B$: $\dim(B_0/C) = \dim(B_0/AC).$

Strict independence relations

3/X

$$a \perp_B a \Rightarrow a \in \text{acl } B. \quad [\text{anti-reflexivity}]$$

$$A \perp_C B \iff \begin{array}{l} \text{[finite character]} \\ A_0 \perp_C B_0 \text{ for all finite } A_0 \subseteq A, B_0 \subseteq B. \end{array}$$

$$\text{For } D \subseteq C \subseteq B: \quad \text{[full transitivity]} \\ A \perp_D B \iff A \perp_C B \text{ and } A \perp_D C.$$

$$A \perp_C B \iff B \perp_C A \quad \text{[symmetry]}$$

$$\forall A, B, C \exists B' \equiv_C B: A \perp_C B'. \quad \text{[full existence]}$$

$$A \perp_C B \subseteq \hat{B} \Rightarrow \begin{array}{l} \text{[extension]} \\ \exists \hat{B}' \equiv_{ABC} \hat{B} : A \perp_C \hat{B}'. \end{array}$$

$$\forall A, B \exists C \subseteq B: \quad \text{[local character]} \\ A \perp_C B \text{ and } |C| \leq \kappa(|A|).$$

(We can remove full existence, as it follows from the other axioms.)

Infinite forest

4/X

A tree or forest in which every vertex has infinite degree.

$A \perp_C B \iff$
every path connecting A and B
passes through $\text{acl } C$.

($\text{acl } C$ is the convex hull of C .)

- This is a strict independence relation.
- The exchange law does not hold.
- A maximal independent set need not be generating (no good notion of basis).

Can we define good candidates for strict independence relations in a general way, when there is no pregeometry?

First candidate

5/X

$$A \downarrow_C^a B \iff \text{acl}(AC) \cap \text{acl}(BC) = \text{acl } C.$$

This is a partial success. We win all modules but lose the fields.

Modules:

This is a strict independence relation. (In case of vector spaces it's the one we had before.)

Algebraically closed fields:

Full transitivity never holds, other axioms do.

Let's have a closer look at full transitivity.

Full transitivity

6/X

For $D \subseteq C \subseteq B$: [full transitivity]

$$A \perp_D B \iff A \perp_C B \text{ and } A \perp_D C.$$

This is really three axioms combined into one.

$$A \perp_D B \Rightarrow A \perp_C B. \quad [\text{base monoton.}]$$

$$A \perp_D B \Rightarrow A \perp_D C. \quad [\text{monotonicity}]$$

$$A \perp_C B \text{ and } A \perp_D C \Rightarrow A \perp_D B. \quad [\text{partial trans.}]$$

Problematic for \perp^a : only base monotonicity.
It holds iff the lattice of algebraically closed sets is modular.

What can we do?

Second candidate

7/X

If base monotonicity is broken, let's fix it!

$$A \perp_D^M B \iff A \perp_C^a B \text{ for all } C \text{ s.t. } D \subseteq C \subseteq \text{acl}(BD).$$

- Now we can only prove partial transitivity on the other side.
- Really problematic: symmetry, extension, local character.

Theorem If \perp^M is symmetric, then it's a strict independence relation.

Proof Straightforward but long. Describe \perp^M in terms of local forking and redo large parts of Byunghan Kim's thesis.

\perp^M is good whenever \perp^a is good (e.g. modules). \perp^M is also good whenever we have a pregeometry.

But not for the infinite forest.

Third candidate

8/X

For the infinite forest \perp^M is no good because symmetry and extension are broken. Let's fix extension!

$$A \perp_C^b B \iff \forall \hat{B} \supseteq B \exists \hat{B}' \equiv_{ABC} \hat{B} : A \perp_C^M \hat{B}'.$$

This does it for the infinite forest. How about the general case?

Theorem

1. \perp^b always satisfies all axioms except perhaps symmetry and local character.
2. For \perp^b , symmetry and local character are equivalent.
3. If \perp^b is not a strict independence relation, then there is none. If it is one, it is the weakest.

Proof

1. Check that most axioms are preserved when passing from \perp^M to \perp^b .
2. As for forking in Byunghan Kim's thesis.
3. Easy.

Stable theories

9/X

Theorem (Harnik-Harrington)

T is stable iff there is a (necessarily unique) strict independence relation s.t.

$$A \equiv_M A', A \perp_M B, A' \perp_M B \Rightarrow A \equiv_{MB} A'.$$

How can we show it must be \perp^b ?

(I.e. forking = thorn-forking in stable T ?)

Show that forking and thorn-forking have the same Morley sequences. A Morley sequence $(a_i)_{i \in \omega + \omega^*}$ over B is B -indiscernible and satisfies $\text{acl}(B(a_i)_{i \in \omega}) \cap \text{acl}(B(a_i)_{i \in \omega^*}) = \text{acl } B$.

For forking in stable T this condition is sufficient because of canonical bases. So forking has at least as many Morley sequences as thorn-forking. Therefore forking is weaker than thorn-forking, hence equal.

(Proof also works for simple T with elimination of hyperimaginaries.)

The lattice connection

10/X

A certain part of stability theory can be described nicely in terms of the lattice of algebraically closed sets and its (elementary) automorphisms.

T trivial 1-based rosy \iff lattice is distributive.

T 1-based rosy \iff lattice is modular.

\perp^M is symmetric \iff lattice is M-symmetric.

T superrosy \iff lattice is arithmetic and T rosy with finite coding.