Shrinking indiscernibles

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Contents:

The independence property (1/7)

$\varphi(x;y)$ has the independence property
if φ can code arbitrarily big finite sets:

$$\forall n \quad \exists a_1 \ldots a_n \quad \forall S \subseteq \{1, \ldots, n\}:$$
$$\{\varphi(a_i; y) \mid i \in S\} \cup \{\neg\varphi(a_i; y) \mid i \notin S\}$$
is consistent.

If φ does not have the independence property,
we say 'φ is NIP'.

T is NIP if all formulas $\varphi(\bar{x}; \bar{y})$ are NIP.

(By a classical result, a theory is stable iff it is NIP and does not have the strict order property. Since every stable theory is even simple, it follows that a theory is stable iff it is NIP and simple.)

Indiscernibles (2/7)

I a linear order, $J \subseteq I$.

$(a_i)_{i \in I}$ is uniform modulo J for $\varphi(x_1 \ldots x_m)$:

$$\forall u, v \in I^m \text{ s.t. } u \equiv_J^{\mathrm{qf}} v \text{ in } (I, <): \quad \varphi(a_{u_1} \ldots a_{u_m}) \iff \varphi(a_{v_1} \ldots a_{v_m}).$$

$(a_i)_{i \in I}$ is indiscernible modulo J over B:

> $(a_i)_{i \in I}$ is uniform modulo J
> for all formulas over B.

We drop J if $J = \emptyset$.
We drop B if $B = \emptyset$.

(Note: The tuples in this definition are not required to be strictly descending. In order to simplify the rest of the talk we will assume that they are.)

Poizat's classical result

« T a la propriété d'indépendance si et seulement si il existe une suite indicernable dans l'ordre (sur $\emptyset$) sécable. »

Here is a free translation (of one direction).

Theorem (Poizat 1981)
Suppose T is NIP. Given:

- I, a complete linear order;
- $(a_i)_{i \in I}$ indiscernible;
- $\varphi(x; b)$.

Then $\exists$ finite $J \subseteq I$ s.t.
$(a_i)_{i \in I}$ is uniform modulo J for $\varphi(x; b)$.

Proof
Suppose not.
Then $\forall n$ $\exists$ an indiscernible sequence $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, b)$ iff i is even.
For all $S \subseteq \{1, \ldots, n\}$,
$\{\varphi(a_i; y) \mid i \in S\} \cup \{\neg\varphi(a_i; y) \mid i \notin S\}$
is consistent, so φ is not NIP.

The Baldwin-Benedikt result (4/7)

“If M lacks IP and I is order-indiscernible with order type a complete dense linear order, then for every L-formula $\varphi(\bar{x}, \bar{y})$ there is a quantifier-free $<$-formula $\psi(\bar{w}, \bar{y})$ such that for every $\bar{m}$ there is a $\bar{c}_{\bar{m}} \in I$ such that $\forall \bar{y} \in P[\psi(\bar{c}_{\bar{m}}, \bar{y}) \equiv \varphi(\bar{m}, \bar{y})]$.”

The following is a free translation.

Theorem (Baldwin-Benedikt 2000)
Suppose T is NIP. Given:

- I, a complete linear order;
- $(a_i)_{i \in I}$ indiscernible;
- $\varphi(x_1 \ldots x_m; b)$.

Then $\exists$ finite $J \subseteq I$ s.t.
$(a_i)_{i \in I}$ is uniform modulo J for $\varphi(x_1 \ldots x_m; b)$.

Note: The case $m = 1$ is precisely Poizat’s classical result.

Proof (5/7)

We may assume that I is also dense.
$J = \{j \in I \mid j \text{ critical}\}$.
j is critical if there is $\bar{u} = u_1 \dots u_m$, $j = u_k$, s.t. *(U denotes an open interval in I)*
$\forall U \ni j \ \exists j' \in U$: $\models \varphi(a_{\bar{u}}; b) \leftrightarrow \neg\varphi(a_{\bar{u}[j'/j]}; b)$.

- $(a_i)_{i \in I}$ is uniform modulo J for $\varphi(x_1 \dots x_m; b)$.

Show $\bar{u} \equiv_J^{\text{qf}} \bar{v}$ implies $\models \varphi(a_{\bar{u}}; b) \leftrightarrow \varphi(a_{\bar{v}}; b)$ by induction on the last k s.t. $u_k \neq v_k$.

- J is finite.

Otherwise there are infinitely many critical points with the same k.
Consider $(M, I, <, f)$, where $M \models T$ contains $(a_i)_{i \in I}$ and $f(i) = a_i$. Work in a monster model extending this.

There is an indiscernible sequence of $\bar{u}^n$ witnessing that u_k^n is critical; with distinct u_k^n. Using density and criticality, we can shift every second element a bit, preserving indiscernibility, but changing the truth value of $\varphi(f(\bar{u}^n); b)$. Contradiction to case $m = 1$.

Corollary

Suppose T is NIP.

Given:

- B, a finite set of parameters;
- I, a complete linear order;
- $(a_i)_{i\in I}$ indiscernible.

Then $\exists\ J \subseteq I$, of size $|J| < |T|^+$, s.t.
$(a_i)_{i\in I}$ is indiscernible modulo J over B.

Corollary

Suppose T is NIP.

Given a finite set B,

every indiscernible sequence $(a_i)_{i<|T|^+}$

has an end piece indiscernible over B.

Definition

If in the second corollary we can replace $|T|^+$ by $\aleph_0$, then T is called strongly NIP.

If the same is true even for the first corollary, then T is called strongly$^+$ NIP.

Shelah's conjecture on NIP fields (7/7)

Theorem (Shelah, Sh783)
Every superstable or o-minimal theory is strongly^{+} NIP.

Theorem (Shelah/Hrushovski, Sh783+Sh863)
The theory of a p-adic field is strongly NIP but not strongly^{+} NIP.

Conjecture (Shelah, Sh863)
Every strongly^{+} NIP field is

- algebraically closed or
- real closed.

Conjecture (Shelah, Sh863)
Every strongly NIP field is

- algebraically closed or
- real closed or
- a valuation field
  (similar to the p-adic fields).

References

Poizat.
Théories instables. JSL 46(3):513–522, 1981.

Baldwin and Benedikt.
Stability theory, permutations of indiscernibles, and embedded finite models. Trans. AMS 352(11):4937–4969, 2000.

Shelah (Sh715).
Classification theory for elementary classes with the dependence property—a modest beginning. Sci. Math. Japon. 59(2):265–316, 2004.

Shelah (Sh783).
Dependent first order theories, continued. Israel J. Math., accepted.

Shelah (Sh863).
Strongly dependent theories. Israel J. Math., submitted.