Strict orders prohibit elimination of hyperimaginaries

Hans Adler

Leeds University

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Part I

A non-eliminable infinitary hyperimaginary

Imaginaries

Imaginaries: $\bar{a}/E$ where $\bar{a}$ is a (finite) tuple and $E(\bar{x}, \bar{y})$ is a formula (over $\emptyset$) that defines an equivalence relation.

In stable theories, canonical bases of types are closed sets of imaginaries.

Elimination of imaginaries: For all $\bar{a}/E$ there is a set B of *real* elements such that $\bar{a}/E$ and B (pointwise) are fixed under the same automorphisms of $\mathbb{M}$.

Adding imaginaries to T yields T^{eq}, a theory that eliminates imaginaries.

Hyperimaginaries

Hyperimaginaries: $\bar{a}/E$ where $E(\bar{x}, \bar{y})$ is a partial type (over $\emptyset$) that defines an equivalence relation.

In simple theories, canonical bases of types are closed sets of imaginaries.

Elimination of hyperimaginaries: For all $\bar{a}/E$ there is a set B of *imaginary* elements such that $\bar{a}/E$ and B (pointwise) are fixed under the same automorphisms of $\mathbb{M}$.

Adding hyperimaginaries to T yields T^{heq}, a *cat* that eliminates hyperimaginaries.

Elimination of hyperimaginaries

Remark

$\aleph_0$-*categorical theories eliminate finitary hyperimaginaries.*

Theorem (Pillay, Poizat)

Stable theories eliminate hyperimaginaries.

Theorem (Buechler, Pillay, Wagner)

Supersimple theories eliminate hyperimaginaries.

The case of general simple theories is still open.

Two questions

Problem

Do $\aleph_0$-categorical theories eliminate all hyperimaginaries?

Problem

Characterise those dependent (i.e. NIP) theories which eliminate hyperimaginaries.

Two easy answers

Example

The dense linear order without endpoints does not eliminate hyperimaginaries.

Theorem

For a dependent (= NIP) theory T the following are equivalent:

- *T is stable.*
- *T is simple.*
- *T does not have the strict order property*
- *T eliminates hyperimaginaries.*

A non-eliminable hyperimaginary

In the theory of a dense linear order, let $(a_i)_{i\in\mathbb{Q}}$ be an ascending chain, i.e. $a_i < a_j$ for $i < j$. Let $p(\bar{x}) = \mathrm{tp}(\bar{a}/\emptyset)$.

A type-definable equivalence relation on the realisations of p:

$$E(\bar{x};\bar{y}) \quad = \quad \{(x_i < y_j) \wedge (y_i < x_j) \mid i,j \in \mathbb{Q},\ i < j\}$$

To see that $\bar{a}/E$ is not eliminable, assume E implies a definable equivalence relation ϵ. Show that ϵ is trivial.

Clearly E is not equivalent to the set of all such ϵ.

Another question

Problem

Find the weakest generalisation of the strict order property that is still inconsistent with elimination of hyperimaginaries.

Part II

Strong order properties, mock stability, and elimination of hyperimaginaries

Digraphs type-definable in the monster model

Any partial type $R(\bar{x}, \bar{y})$ with $|\bar{x}| = |\bar{y}|$ defines a directed graph.

- R has the *partial order property* (POP) if R defines a strict partial order with infinite chains.
- R has the *strong order property* (SOP) if R has infinite chains and no cycles.
- R has the *n-strong order property* (SOP_n) if R has infinite chains and no cycles of length $\leq n$.

Note on ambiguous notation:

- The abbreviation 'SOP' was formerly used for the *strict* order property.
- Shelah and Džamonja treat 'SOP_1' and 'SOP_2' as special cases. Their definitions are not at all equivalent to the above definitions.

Definitions of order properties

- T has POP, SOP, SOP_n if a partial type has it.
- T has FPOP, FSOP, FSOP_n if a finitary partial type has POP, FSOP, FSOP_n.
- T has FFPOP, FFSOP, FFSOP_n if a formula has POP, FSOP, FSOP_n.

Special cases:

- FFPOP = strict order property.
- $\mathrm{SOP}_n = \mathrm{FSOP}_n = \mathrm{FFSOP}_n$.
- SOP_2 = order property (i.e. instability).
- SOP_1 = infinity of the model.

Related properties:

- TP_1, the tree property of the first kind, more recently known as 'SOP_2'.
- A weaker(?) variant of TP_1, known as 'SOP_1'.
- TP, the tree property.

Hierarchy of order properties

$$
\begin{array}{ccccccccc}
\begin{array}{c}\text{FFPOP}\\ \text{(strict OP)}\end{array} & \Rightarrow & \text{FPOP} & \Rightarrow & \text{POP} & & & & \\
\Downarrow & & \Downarrow & & \Downarrow & & & & \\
\text{FFSOP} & \Rightarrow & \text{FSOP} & \Rightarrow & \text{SOP} & & & & \\
 & & & & \Downarrow & & & & \\
 & & & & \vdots & & & & \\
 & & & & \Downarrow & & & & \\
 & & & & \text{SOP}_4 & & & & \\
 & & & & \Downarrow & & & & \\
 & & & & \text{SOP}_3 & \Rightarrow & \text{TP}_1 & \Rightarrow & \text{TP} \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \text{SOP}_2 \text{ (i.e. OP)} \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \begin{array}{c}\text{SOP}_1\\ \text{(i.e. inf. model)}\end{array}
\end{array}
$$

Optimality of the noneliminable hyperimaginary

Theorem

- *A theory with POP cannot eliminate hyperimaginaries.*
- *There is a theory with FFSOP which eliminates hyperimaginaries.*

A mock stable theory with FFSOP ...

Signature: Binary relations $R^1, R^2, R^3, \ldots$.

Axioms of T_0:

$$\forall xy(R^n xy \to R^{n+1} xy),$$
$$\forall xyz(R^m xy \wedge R^n yz \to R^{m+n} xz),$$
$$\forall x(\neg R^n xx).$$

T: theory of the Fraïssé limit of the finite models of T_0.
In T, $R^n = (R^1)^n$.

T is 'mock stable':
$A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ if $A \cap B \subseteq C$ and for $a \in A \setminus C$, $b \in B \setminus C$ only unavoidable relations $R^n ab$ or $R^n ba$ hold.

. . . and elimination of hyperimaginaries

Given a hyperimaginary $\bar{a}/E$, let A be the intersection of $\operatorname{acl} \bar{b} = \operatorname{dcl} \bar{b}$ for all $\bar{b}$ s.t. $\models E(\bar{a}, \bar{b})$.

By Neumann's Lemma (and compactness) there is $\bar{b}$ s.t. $\models E(\bar{a}, \bar{b})$ and $\operatorname{acl}(\bar{a}A) \cap \operatorname{acl}(\bar{b}A) = A$.

Find $\bar{c} \equiv_{\bar{a}} \bar{b}$ s.t. $\bar{b} \perp_{\bar{a}} \bar{c}$. (At most R^1-relations.)
Find $\bar{d} \equiv_{\bar{b}} \bar{c}$ s.t. $\bar{c} \perp_{\bar{b}} \bar{d}$. (At most R^2-relations.)
. . .

By compactness, there is $\bar{a}' \equiv_A \bar{a}$ s.t. $\bar{a} \perp_A \bar{a}'$ and $\models E(\bar{a}, \bar{a}')$.
It easily follows that any $\bar{a}' \equiv_A \bar{a}$ satisfies $\models E(\bar{a}, \bar{a}')$.

Postscript

After the talk, Dugald Macpherson suggested that it is unlikely that SOP will be accepted as the new standard abbreviation for the strong order property, as opposed to its well-established meaning, the strict order property. I agree. Moreover, I am not proposing to change the meaning of SOP_1 and SOP_2 'officially' as in this talk. I am still thinking about possible solutions.

One way out could be to refer to the strong order property as 'acyclic order property' AOP, define AOP_n uniformly as in this talk, and refer to SOP_2 and SOP_1 as TP_1 and weak TP_1, respectively.