The role of matroids in model theory

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$\frac{1}{12}$ Categoricity

Definition

A theory T is categorical if any two models of T are isomorphic.

Theorem (Löwenheim-Skolem)

Let T be a countable complete first-order theory.
If T has models of size κ for some infinite cardinal κ,
then T has models of size κ for all infinite cardinals κ.

Corollary

Let T be a countable complete first-order theory.
T is categorical if and only if T has a finite model.

κ-categoricity

Definition

T is <u>κ-categorical</u> if any two models of size κ are isomorphic.

Theorem (Morley)

Let T be a countable complete first-order theory.
If T is κ-categorical for some uncountable cardinal κ,
then T is κ-categorical for all uncountable cardinals κ.

Examples of uncountably categorical theories:

- Infinite set with no additional structure.
- Vector space over a given finite (or countable) field.
- Projective space over a given finite (or countable) field.
- Algebraically closed field of a given characteristic.

$\frac{3}{12}$ Zilber's Conjecture

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Conjecture (Zilber)

Every uncountably categorical countable first-order theory is essentially of one of the above types.

Hrushovski found uncountably many counterexamples.

$\frac{4}{12}$ Matroids

Definition

A matroid is a set M equipped with a finitary closure operator $\mathrm{cl}\colon \mathcal{P}(M) \to \mathcal{P}(M)$ that satisfies the exchange property.

$$A \subseteq \mathrm{cl}\, A$$
$$A \subseteq B \implies \mathrm{cl}\, A \subseteq \mathrm{cl}\, B$$
$$\mathrm{cl}(\mathrm{cl}\, A) = \mathrm{cl}\, A$$
$$\mathrm{cl}\, A = \bigcup\{\mathrm{cl}\, F \mid F \subseteq_{\text{finite}} A\}$$

$$a \in \mathrm{cl}(C \cup \{b\}) \setminus \mathrm{cl}\, C \implies b \in \mathrm{cl}(C \cup \{a\}).$$

A basis of a matroid is a minimal generating set / maximal independent set. Its dimension is the cardinality of any basis.

Examples of matroids:

- Trivial closure (i.e. $\mathrm{cl}\, A = A$) on any set.
- Linear hull on a vector space. A matrix.
- Algebraic closure on an algebraically closed field.
- The edges of a graph.

Algebraic closure

Definition

Let M be a model of a first-order theory T, $\bar{b} \in M$.
A first-order formula $\varphi(\bar{x}, \bar{b})$ is called algebraic if it has only finitely many realisations.
An element $a \in M$ is called algebraic over a set $B \subseteq M$ if there is a tuple $\bar{b} \in B$ and an algebraic formula $\varphi(x, \bar{b})$ such that $M \models \varphi(a, \bar{b})$.
The algebraic closure of a set $B \subseteq M$ is the set

$$\operatorname{acl} B = \{a \in M \mid a \text{ algebraic over } B\}.$$

Examples of algebraic closure:

- In the theory of an infinite set (empty signature): $\operatorname{acl} B = B$.
- In any theory of a vector space: linear hull.
- In any algebr. closed field: algebr. closure in the sense of algebra.

Strong minimality

Definition

A complete first-order theory is called strongly minimal if for all models

- algebraic closure is a matroid, and
- for all n, all independent n-tuples have the same type.

Theorem

Let T be a countable complete first-order theory.

If T is strongly minimal, then T is uncountably categorical and the cardinality of an uncountable model equals its dimension.

More generally, T is uncountably categorical if and only if the restriction of T to a certain formula is strongly minimal and every model of T is prime over this strongly minimal part.

$\frac{7}{12}$ Independence for more general theories

To extend this dimension theory beyond uncountably categorical theories, we can:

1. drop or weaken the condition on independent *n*-types
2. generalise the notion of matroid, or
3. allow other closure operators instead of acl.

Real closed fields are an example of a pregeometric theory: Algebraic closure is a matroid. But the theory is not uncountably categorical, and this is an example of 1.

We will interpret a part of the machinery of stability theory as doing 2 in order to get 3.

Semimodular lattices

Definition (Wilcox)

(A, B) is a modular pair if for all $C \in [A \wedge B, B]$ we have $(A \vee C) \wedge B = C$.
A lattice is semimodular (M-symmetric) if being a modular pair is a symmetric relation.

• The lattice of closed sets of a matroid is semimodular; the closures of elements are its atoms.
• A semimodular lattice that is generated by its atoms can be interpreted as a matroid.

$\frac{9}{12}$ Independence in a semimodular lattice

$$A \perp_C B \iff (A \vee C) \wedge (C \vee B) = C, \text{ and } (A, B) \text{ is a modular pair}$$
$$\iff \text{for all } D \in [C, B \vee C]\colon (A \vee D) \wedge (B \vee C) = D.$$

Fact

In a matroid, $A \perp_C B$ is equivalent to the condition that every subset $A_0 \subset A$ which is independent over C is also independent over $B \cup C$.
In an arbitrary semimodular lattice, $\perp$ still deserves the name 'independence'.

For example in a vector space (or just a module), $A \perp_C B$ means: A and B are linearly independent over C. In this case the lattice is modular, so all pairs are modular pairs.

Weight

Definition

In a semimodular lattice, the weight of a lattice element A is the maximal n such that there is an independent sequence $B_0, B_1, \ldots, B_{n-1}$ with $A \not\perp B_0, \ldots, A \not\perp B_{n-1}$.

Fact

The weight 1 elements of a semimodular lattice form a matroid under the following closure operator:

$$\operatorname{cl} A = \{b \mid \exists A_0 \subseteq A \colon A_0 \text{ independent and } A_0 \not\perp b\}.$$

If there are ‘enough’ weight 1 elements, and if there is some control over the independent sets of weight 1 elements, then the matroid of weight 1 elements helps to understand the models of a theory. This is the case for superstable theories.

Thorn-forking

- A recent notion due to Scanlon, Onshuus and Ealy.
- Equivalent to Shelah's notion of forking, whenever that is well-behaved.
- More generally well-behaved than forking.

Definition

$A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_C B$ iff, up to isomorphism over $B \cup C$, $\operatorname{acl} A \perp_{\operatorname{acl} C} \operatorname{acl} B'$ for all $B' \supseteq B$.

A complete first-order theory is called rosy if $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ is symmetric.

If a theory has a semimodular lattice of algebraically closed sets, then the theory is rosy.

(The definition must be read in the monster model of T^{eq}.)

Outlook: Dependent theories

Definition

A complete first-order theory is said to be dependent or NIP if all formulas have bounded Vapnik-Chervonenkis dimension.

- This class includes all stable theories.
- Current research in model theory concentrates on unstable case.
- Forking seems to be important here, although it is not symmetric.
- If there is a suitable generalisation of matroids for this situation, it will probably not be symmetric, either.
- Greedoids seem to be a natural candidate.