

The role of matroids in model theory

Hans Adler

Leeds University

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$\frac{1}{12}$ Categoricity

Definition

A theory T is categorical if any two models of T are isomorphic.

Theorem (Löwenheim-Skolem)

*Let T be a countable complete first-order theory.
If T has models of size κ for some infinite cardinal κ ,
then T has models of size κ for all infinite cardinals κ .*

Corollary

*Let T be a countable complete first-order theory.
 T is categorical if and only if T has a finite model.*

$\frac{2}{12}$ κ -categoricity

Definition

T is κ -categorical if any two models of size κ are isomorphic.

Theorem (Morley)

*Let T be a countable complete first-order theory.
If T is κ -categorical for some uncountable cardinal κ ,
then T is κ -categorical for all uncountable cardinals κ .*

Examples of uncountably categorical theories:

- Infinite set with no additional structure.
- Vector space over a given finite (or countable) field.
- Projective space over a given finite (or countable) field.
- Algebraically closed field of a given characteristic.

$\frac{3}{12}$ Zilber's Conjecture

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Conjecture (Zilber)

Every uncountably categorical countable first-order theory is essentially of one of the above types.

Hrushovski found uncountably many counterexamples.

$\frac{4}{12}$ Matroids

Definition

A matroid is a set M equipped with a finitary closure operator $\text{cl}: \mathcal{P}(M) \rightarrow \mathcal{P}(M)$ that satisfies the exchange property.

$$A \subseteq \text{cl } A$$

$$A \subseteq B \implies \text{cl } A \subseteq \text{cl } B$$

$$\text{cl}(\text{cl } A) = \text{cl } A$$

$$\text{cl } A = \bigcup \{ \text{cl } F \mid F \subseteq_{\text{finite}} A \}$$

$$a \in \text{cl}(C \cup \{b\}) \setminus \text{cl } C \implies b \in \text{cl}(C \cup \{a\}).$$

A basis of a matroid is a minimal generating set / maximal independent set. Its dimension is the cardinality of any basis.

Examples of matroids:

- Trivial closure (i.e. $\text{cl } A = A$) on any set.
- Linear hull on a vector space. A matrix.
- Algebraic closure on an algebraically closed field.
- The edges of a graph.

$\frac{5}{12}$ Algebraic closure

Definition

Let M be a model of a first-order theory T , $\bar{b} \in M$.

A first-order formula $\varphi(\bar{x}, \bar{b})$ is called algebraic if it has only finitely many realisations.

An element $a \in M$ is called algebraic over a set $B \subseteq M$ if there is a tuple $\bar{b} \in B$ and an algebraic formula $\varphi(x, \bar{b})$ such that $M \models \varphi(a, \bar{b})$.

The algebraic closure of a set $B \subseteq M$ is the set

$$\text{acl } B = \{a \in M \mid a \text{ algebraic over } B\}.$$

Examples of algebraic closure:

- In the theory of an infinite set (empty signature): $\text{acl } B = B$.
- In any theory of a vector space: linear hull.
- In any algebraically closed field: algebraic closure in the sense of algebra.

$\frac{6}{12}$ Strong minimality

Definition

A complete first-order theory is called strongly minimal if for all models

- algebraic closure is a matroid, and
- for all n , all independent n -tuples have the same type.

Theorem

Let T be a countable complete first-order theory.

If T is strongly minimal, then T is uncountably categorical and the cardinality of an uncountable model equals its dimension.

More generally, T is uncountably categorical if and only if the restriction of T to a certain formula is strongly minimal and every model of T is prime over this strongly minimal part.

$\frac{7}{12}$ Independence for more general theories

To extend this dimension theory beyond uncountably categorical theories, we can:

1. drop or weaken the condition on independent n -types
2. generalise the notion of matroid, or
3. allow other closure operators instead of acl.

Real closed fields are an example of a pregeometric theory:

Algebraic closure is a matroid. But the theory is not uncountably categorical, and this is an example of 1.

We will interpret a part of the machinery of stability theory as doing 2 in order to get 3.

$\frac{8}{12}$ Semimodular lattices

Definition (Wilcox)

(A, B) is a modular pair if for all $C \in [A \wedge B, B]$ we have $(A \vee C) \wedge B = C$.

A lattice is semimodular (M-symmetric) if being a modular pair is a symmetric relation.

- The lattice of closed sets of a matroid is semimodular; the closures of elements are its atoms.
- A semimodular lattice that is generated by its atoms can be interpreted as a matroid.

$\frac{9}{12}$ Independence in a semimodular lattice

$$\begin{aligned} A \perp_C B &\iff (A \vee C) \wedge (C \vee B) = C, \text{ and } (A, B) \text{ is a modular pair} \\ &\iff \text{for all } D \in [C, B \vee C]: (A \vee D) \wedge (B \vee C) = D. \end{aligned}$$

Fact

In a matroid, $A \perp_C B$ is equivalent to the condition that every subset $A_0 \subset A$ which is independent over C is also independent over $B \cup C$.

In an arbitrary semimodular lattice, \perp still deserves the name 'independence'.

For example in a vector space (or just a module), $A \perp_C B$ means: A and B are linearly independent over C . In this case the lattice is modular, so all pairs are modular pairs.

Definition

In a semimodular lattice, the weight of a lattice element A is the maximal n such that there is an independent sequence B_0, B_1, \dots, B_{n-1} with $A \not\leq B_0, \dots, A \not\leq B_{n-1}$.

Fact

The weight 1 elements of a semimodular lattice form a matroid under the following closure operator:

$$\text{cl } A = \{b \mid \exists A_0 \subseteq A: A_0 \text{ independent and } A_0 \not\leq b\}.$$

If there are 'enough' weight 1 elements, and if there is some control over the independent sets of weight 1 elements, then the matroid of weight 1 elements helps to understand the models of a theory. This is the case for superstable theories.

$\frac{11}{12}$ Thorn-forking

- A recent notion due to Scanlon, Onshuus and Ealy.
- Equivalent to Shelah's notion of forking, whenever that is well-behaved.
- More generally well-behaved than forking.

Definition

$A \perp_C^b B$ iff, up to isomorphism over $B \cup C$, $\text{acl } A \perp_{\text{acl } C} \text{acl } B'$ for all $B' \supseteq B$.

A complete first-order theory is called rosy if \perp_C^b is symmetric.

If a theory has a semimodular lattice of algebraically closed sets, then the theory is rosy.

(The definition must be read in the monster model of T^{eq} .)

$\frac{12}{12}$ Outlook: Dependent theories

Definition

A complete first-order theory is said to be dependent or NIP if all formulas have bounded Vapnik-Chervonenkis dimension.

- This class includes all stable theories.
- Current research in model theory concentrates on unstable case.
- Forking seems to be important here, although it is not symmetric.
- If there is a suitable generalisation of matroids for this situation, it will probably not be symmetric, either.
- Greedoids seem to be a natural candidate.