Theories controlled by formulas of VC codimension 1

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Organisation

1. D-relations and D-minimal theories
2. VC-minimal / weakly D-minimal theories
3. Swiss cheese decomposition and dp-minimality of VC-minimal theories.

(1/8) The idea behind •-minimality

T is T_0-minimal if T extends T_0, and for T every unary definable set (with parameters) is definable (with parameters) by a quantifier-free formula in the signature of T_0.

In other words:
• T is a 1-minimal expansion of T_0, i.e. for T every unary definable set is definable in the signature of T_0.
• T_1, the completion of T_0 that is the restriction of T to the signature of T_0, has 1-QE, i.e. every unary definable set is definable by a quantifier-free formula.

By appropriate choice of T_0, we get strong minimality, o-minimality, C-minimality, and p-minimality.

Simultaneous generalisation of strong minimality and o-minimality: T_0 the theory of partial orders $\leq$ such that comparability is an equivalence relation.
Can we also cover C-minimality?

(2/8) D-relations

To cover C-minimality as well, we could just relax the axioms of C-relations:

(C1) $C(a;bc) \Leftrightarrow C(a;cb)$
(C2) $C(a;bc) \Rightarrow \neg C(b;ac)$
(C3) $C(a;bc) \;\Rightarrow\; C(d;bc) \vee C(a;dc)$
(C4) $a \neq b \;\Rightarrow\; \exists c \neq b \colon C(a;bc)$
(C5) $\exists ab \colon a \neq b$.

Instead of just dropping axioms (C4) and (C5), we use D-relations, which are slightly more general than (C1)–(C3), and easier to understand:

(D1) $D(ab;cd) \;\Rightarrow\; D(ba;cd) \wedge D(cd;ab)$
(D2) $D(ab;cd) \;\Rightarrow\; \neg D(ac;bd)$
(D3) $D(ab;cd) \;\Rightarrow\; D(ae;cd) \vee D(ab;ce)$.

D-minimality is T_0-minimality, for T_0 the theory of D-relations.

(3/8) D-minimal theories

The meaning of a D-relation: If a, b, c, d are nodes of a 'tree', $D(ab; cd)$ means that the 'path' from a to b and the 'path' from c to d are disjoint.

The cones: sets of the form $D(xb; cd)^M$.
Every definable unary set is a Boolean combination of cones.

The cones generate a topology (as a subbasis):
Open sets = unions of finite intersections of cones.

This topology is:
- often a definable topology;
- the correct topology in the o-minimal and C-minimal cases;
- the discrete topology in the strongly minimal case.

(4/8) VC codimension

$\Omega \subseteq \mathcal{P}(M)$ shatters $X \subseteq M$ if $\Omega \restriction X = \mathcal{P}(X)$.

$\dim_{VC} M = \max \{ |X| \mid \Omega \text{ shatters } X \}$.

$\dim_{VC} \varphi(\bar{x}; \bar{y}) = \dim_{VC} \{ \varphi(\bar{x}; \bar{b})^M \mid \bar{b} \in M \}$.
$\dim_{VC}^{op} \varphi(\bar{x}; \bar{y}) = \dim_{VC} \{ \varphi(\bar{a}; \bar{y})^M \mid \bar{a} \in M \}$.

φ is VC-minimal if $\dim_{VC}^{op} \varphi = 1$.

$\dim_{VC}^{op} \varphi(x; \bar{y}) = 0 \iff$
every instance $\varphi(x; \bar{b})$ is trivial.

$\dim_{VC}^{op} \varphi(x; \bar{y}) \leq 1 \iff$
for all $A = \varphi(x; \bar{b}_1)^M$ and $B = \varphi(x; \bar{b}_2)^M$:
$A \subseteq B$, $A \supseteq B$, $A \cap B = \emptyset$ or $A \cup B = M$.

(5/8) VC-minimal theories

VC codimension (and VC dimension) can also be defined for families $\{\varphi_i(x; \bar{y}_i) \mid i \in I\}$ of formulas.

VC-minimal theory:

• Has a distinguished VC-minimal family.

• Every unary definable (with parameters) set is a Boolean combination of instances of the distinguished family.

We get a D-relation:

$D(ab; cd) \iff$

an instance $A = \varphi(x; \bar{e})^M$ separates a and b from c and d.

Moreover, $\neg D$ is type-definable.

(6/8) Weakly D-minimal theories

Weakly D-minimal theory:

• Has a distinguished incomplete type $p(xy; zw)$ whose complement is a D-relation.

• The definable (with parameters) unary sets are Boolean combinations of definable generalised cones.

Generalised cone:

A set $A \subseteq M^1$ such that $a, b \in A,\ c, d \notin A \ \Rightarrow\ D(ab; cd)$.

VC-minimal = weakly D-minimal:

In a VC-minimal theory:

Instances of the VC-minimal family are generalised cones.

In a weakly D-minimal theory:

Definable generalised cones form a VC-minimal family.

Its D-relation is the distinguished one.

(7/8) Swiss cheese decomposition

In a (weakly) D-minimal theory we can distinguish a node of the 'tree' described by the D-relation, by fixing a parameter. For every generalised cone and its complement we distinguish the one not containing the root as basic. Any two basic generalised cones are comparable or disjoint.

Swiss cheese:
$A \setminus (B_1 \cup \cdots \cup B_n)$, where A, $B_1, \ldots B_n$ are definable basic generalised cones, $B_1, \ldots B_n \subseteq A$, and $B_1, \ldots B_n$ are pairwise disjoint.

Every unary definable set is a disjoint union of Swiss cheeses.

(8/8) Dp-minimal theories

A theory is dp-minimal if the following does not exist:

- Formulas $\varphi(x;\bar{y})$ and $\psi(x;\bar{z})$.
- Parameters $\bar{b}_i$, $\bar{c}_j$ (where $i, j < \omega$).
- Elements a_{ij} such that
  $$\models \varphi(a_{ij};\bar{b}_n) \iff i = n \text{ and } \models \varphi(a_{ij};\bar{c}_n) \iff j = n.$$

Equivalently, T is dependent and the following does not exist:

- Formulas $\varphi(x;\bar{y})$ and $\psi(x;\bar{z})$.
- Parameters $\bar{b}_i$ ($i < \omega$) such that
  $\langle \varphi(x;\bar{b}_i) \mid i < \omega \rangle$ is k-inconsistent for some k.
- Parameters $\bar{c}_j$ ($j < \omega$) such that
  $\langle \psi(x;\bar{c}_j) \mid j < \omega \rangle$ is k-inconsistent for some k.
- Elements a_{ij} such that $\models \varphi(a_{ij};\bar{b}_i) \wedge \psi(a_{ij};\bar{c}_j)$.

A stable theory is dp-minimal iff every 1-type has weight ≤ 1.
O-minimal, C-minimal and p-minimal theories are also known to be dp-minimal.