Strong theories, burden and weight

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Outline

An *independent forking system* for a over C is a set $\mathcal{B}$ of tuples such that

- $\mathcal{B}$ is independent over C, and
- $a \not\downarrow_C b$ for all $b \in \mathcal{B}$.

$$\mathrm{pwt}(a/C) = \sup\{|\mathcal{B}| : \mathcal{B} \text{ indep. fork. syst. for } a \text{ over } C\}$$
$$\mathrm{wt}(a/C) = \sup\{\mathrm{pwt}(a/C') \mid C' \supseteq C,\ a \underset{C}{\downarrow} C'\}$$

Fact

If $\mathrm{wt}(a/C)$ *is a successor cardinal or* $\aleph_0$, *the supremum is attained.*

But how about $\mathrm{wt}(a/C) = \aleph_\omega$?

Theorem

If $\mathrm{wt}(a/C)$ *is finite, then there are* $C' \supseteq C$ *s.t.* $a \mathop{\smile\!\!\!\!\!\!|}_C C'$ *and* $b_0, \ldots, b_{n-1}$*, independent over* C'*, such that* $\mathrm{wt}(b_i/C') = 1$ *and* a *is domination equivalent to* $b_0 \ldots b_{n-1}$ *over* C*.*

In supersimple theories we can choose $\mathrm{U}(b_i/C') = \omega^\alpha$. In simple theories without dense forking chains we can choose $\mathrm{tp}(b_i/C')$ regular.

Fact

If X *is a set of tuples of weight 1 over* C*, then the following operator defines a matroid ('pregeometry') on* X*:*

$$b \in \mathrm{cl}(A) \iff \exists A_0 \subseteq A,\ A_0 \text{ indep. over } C,\ A_0 \not\mathop{\smile\!\!\!\!\!\!|}_C b.$$

Outline

To speak of independence, one needs a ternary relation that has at least the following properties:

(invariance)
: If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $(A', B', C') \equiv (A, B, C)$, then $A' \mathop{\smile\!\!\!\!\!\!\mid}_{C'} B'$.

(monotonicity)
: If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!\mid}_C B'$.

(transitivity)
: Suppose $D \subseteq C \subseteq B$. If $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$ and $C \mathop{\smile\!\!\!\!\!\!\mid}_D A$, then $B \mathop{\smile\!\!\!\!\!\!\mid}_D A$.

(strong finite character)
: If $A \not\mathop{\smile\!\!\!\!\!\!\mid}_C B$, then there is a finite tuple $\bar{a} \in A$ and a formula $\varphi(\bar{x}, \bar{b}\bar{c})$ with parameters in BC such that
  - $\models \varphi(\bar{a}, \bar{b}\bar{c})$, and
  - $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!\mid}_C B$ for all $\bar{a}'$ satisfying $\models \varphi(\bar{a}', \bar{b}\bar{c})$.

We call it a *preindependence relation* if it also satisfies the following:

(base monotonicity)
: Suppose $D \subseteq C \subseteq B$. If $A \mathop{\smile\!\!\!\!\!\!\mid}_D B$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

(normality)
: $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ implies $AC \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

It is *extensible* if it also satisfies:

(extension)
: If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $\hat{B} \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\mid}_C \hat{B}$.

Fact

In any complete first-order theory, dividing, thorn-dividing and M-dividing are preindependence relations, and therefore(!) forking and thorn-forking are extensible preindependence relations.

An extensible preindependence relation is an *independence relation* if it also satisfies the following axioms:

(local character)

For every A there is a cardinal $\kappa(A)$ with the following property:
For any set B there is a subset $C \subseteq B$ of cardinality $|C| < \kappa(A)$ such that $A \downarrow_C B$.

(symmetry)

$$A \downarrow_C B \iff B \downarrow_C A.$$

(existence)

For all A, B, C there is $B' \equiv_C B$ such that $A \downarrow_C B'$.

Theorem

It's enough to check extension and local character, or to check symmetry and existence.

Note: The easy direction is not true without *strong* finite character.

A theory is simple iff forking independence is an independence relation. Forking independence in a simple theory is the only independence relation that satisfies the independence theorem over models, or equivalently the *chain condition*:

(chain condition) If $(\bar{b}_i)_{i<\kappa}$ sequence of C-indiscernibles, $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_C \bar{b}_0$ ($\bar{a}$ and $\bar{b}_i$ may be infinite), then there is $\bar{a}' \equiv_{\bar{b}_0 C} \bar{a}$ such that $(\bar{b}_i)_{i<\kappa}$ is $\bar{a}'C$-indiscernible and $\bar{a}' \mathop{\smile\!\!\!\!\!\!\mid}_C (\bar{b}_i)_{i<\kappa}$.

(The official definition of the chain condition is different, but equivalent.)

A theory is rosy iff thorn-forking independence is an independence relation, iff there is an independence relation on $\mathbb{M}^{\mathrm{eq}}$ that satisfies $a \mathop{\smile\!\!\!\!\!\!|}_B a \implies a \in \mathrm{acl}^{\mathrm{eq}} B$. Thorn-forking in a rosy theory is the weakest such independence relation.

Theorem

Forking and thorn-forking agree in every simple theory with elimination of hyperimaginaries.

Is there any complete first-order theory with two such independence relations?

Fact

Simple and o-minimal theories are rosy.
C-minimal theories are not rosy.

Outline

An $\mathfrak{p}$-*independent* $\mathfrak{p}$-*forking system* for a over C is a set $\mathcal{B}$ of tuples such that

- $\mathcal{B}$ is $\mathfrak{p}$-independent over C, and
- $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{p}}_C b$ for all $b \in \mathcal{B}$.

$$\mathrm{pwt}^{\mathfrak{p}}(a/C) = \sup\{|\mathcal{B}| : \mathcal{B} \text{ indep. fork. syst. for } a \text{ over } C\}$$
$$\mathrm{wt}^{\mathfrak{p}}(a/C) = \sup\{\mathrm{pwt}^{\mathfrak{p}}(a/C') \mid C' \supseteq C,\ a \mathop{\smile\!\!\!\!\!\!|}_C C'\}$$

Fact

If $\mathrm{wt}^{\mathfrak{p}}(a/C)$ *is a successor cardinal or* $\aleph_0$*, the supremum is attained.*

How about $\mathrm{wt}^{\mathfrak{p}}(a/C) = \aleph_\omega$?

Theorem

If $\mathrm{wt}^{\text{þ}}(a/C)$ *is finite, then there are* $C' \supseteq C$ *s.t.* $a \mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}_{C} C'$ *and* $b_0, \ldots, b_{n-1}$, *þ-independent over* C', *such that* $\mathrm{wt}^{\text{þ}}(b_i/C') = 1$ *and* a *is domination equivalent to* $b_0 \ldots b_{n-1}$ *over* C.

In superrosy theories we can choose $\mathrm{U}(b_i/C') = \omega^{\alpha}$. In rosy theories without dense forking chains we can choose $\mathrm{tp}(b_i/C')$ regular.

Fact

If X *is a set of tuples of þ-weight 1 over* C, *then the following operator defines a matroid ('pregeometry') on* X:

$$b \in \mathrm{cl}(A) \iff \exists A_0 \subseteq A,\ A_0 \text{ þ-indep. over } C,\ A_0 \not\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}_{C} b.$$

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- Rosiness and thorn-forking are **not** good notions because they were not first found by Shelah. In other words, they are (apparently) not directly related to 'dividing lines'.

Outline

An *inp-pattern* for a partial type $p(x)$ is a sequence

$$(\varphi^0(x,y^0),k^0),(\varphi^1(x,y^1),k^1),(\varphi^2(x,y^2),k^2),\dots$$

for which there are mutually (over $\operatorname{dom} p$) indiscernible sequences

$$\begin{array}{llll} b^0 = b^0_0, & b^0_1, & b^0_2, & \dots \\ b^0 = b^1_0, & b^1_1, & b^1_2, & \dots \\ b^0 = b^2_0, & b^2_1, & b^2_2, & \dots \end{array}$$

such that for the associated array of formulas

$$\begin{array}{llll} \varphi^0(x,b^0_0), & \varphi^0(x,b^0_1), & \varphi^0(x,b^0_2), & \dots \\ \varphi^1(x,b^1_0), & \varphi^1(x,b^1_1), & \varphi^1(x,b^1_2), & \dots \\ \varphi^2(x,b^2_0), & \varphi^2(x,b^2_1), & \varphi^2(x,b^2_2), & \dots \end{array}$$

the following holds:

- each row i is k^i-inconsistent, and
- each 'path' is consistent with $p(x)$.

$\kappa_{inp}(T)$ is the smallest cardinal κ such that there is no inp-pattern of length κ for any $\bar{x} = \bar{x}$ ($\bar{x}$ finite).
T has TP_2, the *tree property of the second kind*, if $\kappa_{inp}(T) = \infty$.

Fact

T has the tree property (i.e. is not simple) if and only if

- *T has the TP_2, or*
- *T has the SOP_2 (formerly known as TP_1).*

Burden of a type: $\mathrm{bdn}(p)$ is the supremum of the lengths of inp-patterns of p. (May or may not be attained!)

Theorem

If T is simple, then

$$\mathrm{bdn}(p) = \sup_{p' \supseteq p} \mathrm{pwt}(p) = \sup_{p' \supseteq p} \mathrm{wt}(p).$$

If T is rosy, then

$$\mathrm{bdn}(p) \geq \sup_{p' \supseteq p} \mathrm{pwt}^{\text{þ}}(p) = \sup_{p' \supseteq p} \mathrm{wt}^{\text{þ}}(p).$$

T is *strong* if $\kappa_{inp}(T) = \aleph_0$.
T is strong iff for every (finitary) type $p(x)$, either $\text{bdn}(p)$ is finite, or $\text{bdn}(p) = \aleph_0$ and the supremum is not attained.

Theorem

A simple theory is strong iff every type has finite weight.
A dependent (i.e. NIP) theory is strong iff it is strongly dependent.

Corollary

A stable theory is strongly dependent if and only if every type has finite weight.

T is *inp-minimal* if every 1-type has burden ≤ 1.
T is *dp-minimal* if T is NIP and inp-minimal.

Examples of dp-minimal theories:

- stable theories in which every 1-type has weight ≤ 1; e.g. strongly minimal theories
- (weakly) o-minimal theories
- C-minimal theories
- p-minimal theories
- (weakly) D-minimal theories, i.e. theories controlled by a family of Vapnik-Chervonenkis codimension 1.