Thorn-forking and generalised semimodularity

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Morley's theorem

Let T be a first-order theory in a countable language.

Theorem (Löwenheim-Skolem)
If T has an infinite model, then for every infinite cardinal κ, T has a model of size κ.

Theorem (Morley
If there is an uncountable cardinal κ such that all models of T of cardinality κ are isomorphic, then the same is true for every uncountable cardinal.

Examples

- Algebraically closed fields of a fixed characteristic
- Vector spaces over a fixed field.

The algebraic closure of a subset A of a model of a theory T consists of all elements which satisfy a formula with parameters in A that has only finitely many solutions. E.g. algebraic closure in algebraically closed fields, linear hull in vector spaces.

For every model of a theory T, algebraic closure is a finitary closure operator:

- $A \subseteq \text{acl}\, A$.
- $A \subseteq B \quad \Rightarrow \quad \text{acl}\, A \subseteq \text{acl}\, B$.
- $\text{acl}(\text{acl}\, A) = \text{acl}\, A$.
- $\text{acl}\, A = \bigcup_{\text{finite } A_0 \subseteq A} \text{acl}\, A_0$.

Strongly minimal theories

Sometimes algebraic closure also satisfies the exchange law:

- $a \in \mathrm{acl}(Cb) \setminus \mathrm{acl}\, C \quad \Rightarrow \quad b \in \mathrm{acl}(Ca)$.

Finitary closure operator + exchange law
= matroid.

In a matroid, maximal independent set = minimal generating set. We have a good notion of basis and dimension.

A theory is strongly minimal if algebraic closure satisfies the exchange law and the type of n independent elements depends only on n.

Every strongly minimal theory is an example for Morley's theorem. The converse is morally true.

A projective space consists of a set of points P and a set of subsets of P, called lines, subject to the following axioms:

- Any two distinct points a, b are in exactly one line $\mathrm{cl}(a, b)$.

- If a, b, c, d are distinct points and no 3 of them are collinear, then $\mathrm{cl}(a, b) \cap \mathrm{cl}(c, d) \neq \emptyset$ implies $\mathrm{cl}(a, c) \cap \mathrm{cl}(b, d) \neq \emptyset$.

Every projective space is a matroid (but not conversely).

Fundamental theorem of projective geometry

Given a projective space of vectors over a field, the field can be reconstructed from the matroid, provided the dimension is ≥ 3.

Continuous lattices

Matroids can be seen as semimodular atomistic lattices.

In the 1930s, John von Neumann's work on quantum theory made him interested in certain lattices of subspaces of a Hilbert space.

A continuous lattice is a complete lattice in which for every element a and linearly ordered set B the following hold:

- $a \wedge \bigvee B = \bigvee_{b \in B} a \wedge b$.

- $a \vee \bigwedge B = \bigwedge_{b \in B} a \vee b$.

(This can be seen as an infinite modularity condition.)

Theorem (von Neumann) Every irreducible continuous lattice with a homogeneous basis of order ≥ 5 is isomorphic to the lattice of ideals of a von Neumann regular ring.

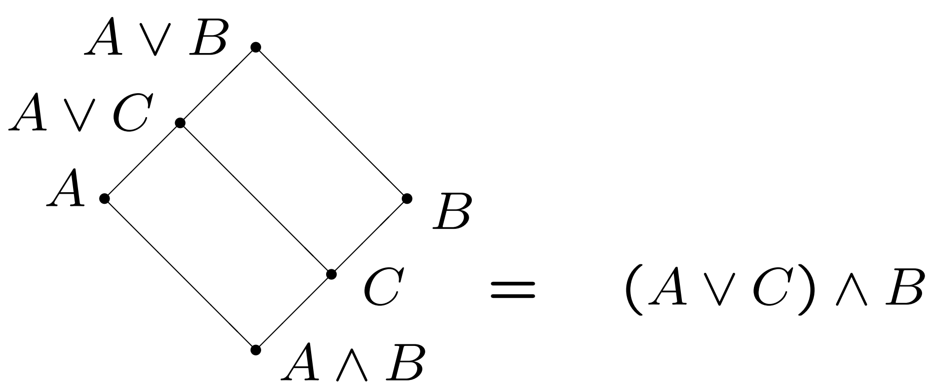
Generalising modularity

An active field in the 1930s.
Several attempts:

- Birkhoff's condition (Birkhoff 1935)
- semimodularity (Birkhoff 1935)
- Mac Lane's condition (Mac Lane 1938)
- M-symmetry (Wilcox 1938)

All conditions agree for finite lattices.
All conditions differ in general.

$M(A, B)$ if A and B form a modular pair:



M-symmetry: $M(A, B) \iff M(B, A)$.

In an M-symmetric lattice, define $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathsf{M}}_{C} B$ by $(A \vee C) \wedge (B \vee C) = C$ and $M(A \vee C, B \vee C)$.

This relation was used by John von Neumann, and also by others to prove generalisations of his theorem to certain M-symmetric lattices.

In a model of T s.t. the lattice of algebraically closed sets is M-symmetric, we can define $A \mathop{\smile\!\!\!\!\!\!\;|}^{\mathsf{M}}_{C} B$ for general sets A, B, C by $\mathrm{acl}(AC) \mathop{\smile\!\!\!\!\!\!\;|}^{\mathsf{M}}_{\mathrm{acl}\, C} \mathrm{acl}(BC)$.

The relation $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathsf{M}}$ then satisfies the axioms of a <u>strict independence relation</u>.

(A theory which admits a strict independence relation is called <u>rosy</u>.)

Strict independence relations

$a \mathop{\smile\!\!\!\!\!\!|}_B a \quad \Rightarrow \quad a \in \mathrm{acl}\, B.$ [anti-reflexivity]

$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff$ [finite character]
$A_0 \mathop{\smile\!\!\!\!\!\!|}_C B_0$ for all finite $A_0 \subseteq A$, $B_0 \subseteq B$.

For $D \subseteq C \subseteq B$: [full transitivity]
$A \mathop{\smile\!\!\!\!\!\!|}_D B \iff A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_D C$.

$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff B \mathop{\smile\!\!\!\!\!\!|}_C A$ [symmetry]

$\forall A, B, C \;\; \exists B' \equiv_C B$: $A \mathop{\smile\!\!\!\!\!\!|}_C B'$. [full existence]

$A \mathop{\smile\!\!\!\!\!\!|}_C B \subseteq \hat{B} \quad \Rightarrow$ [extension]
$\exists \hat{B}' \equiv_{ABC} \hat{B} : A \mathop{\smile\!\!\!\!\!\!|}_C \hat{B}'$.

$\forall A, B \;\; \exists C \subseteq B :$ [local character]
$A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $|C| \leq \kappa(|A|)$.

(We can remove extension and local character, as they follow from the other axioms.)

Free pseudoplane

A tree or forest in which every vertex has infinite degree.

In this example the exchange law does not hold (a maximal independent set need not be generating). $\mathop{\smile\!\!\!\!\!\!|}^{\mathsf{M}}$ does not satisfy symmetry and extension. However the following is a strict independence relation:

$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff$
every path connecting A and B
passes through $\operatorname{acl} C$.

($\operatorname{acl} C$ is the convex hull of C.)

Thorn-forking

For the free pseudoplane $\mathop{\smile\!\!\!\!\!\!|}^{\mathsf{M}}$ is no good because symmetry and extension are broken. Let's fix extension!

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathsf{þ}}_{C} B \iff \forall \hat{B} \supseteq B \;\; \exists \hat{B}' \equiv_{ABC} \hat{B} : A \mathop{\smile\!\!\!\!\!\!|}^{\mathsf{M}}_{C} \hat{B}.$$

This does it for the free pseudoplane. How about the general case?

Theorem

1. $\mathop{\smile\!\!\!\!\!\!|}^{\mathsf{þ}}$ always satisfies all axioms except perhaps symmetry and local character.
2. For $\mathop{\smile\!\!\!\!\!\!|}^{\mathsf{þ}}$, symmetry and local character are equivalent.
3. If $\mathop{\smile\!\!\!\!\!\!|}^{\mathsf{þ}}$ is not a strict independence relation, then there is none. If it is one, it is the weakest.

Theorem
The following are equivalent for all T:

• There are arbitrarily big cardinals κ such that over every model of size κ there are only κ complete types.

• No model of T contains an infinite set of tuples which are linearly ordered by a formula.

• Thorn-forking is an independence relation, and if $\bar{a} \mathop{\smile\!\!\!\!\!|}^{\text{þ}}_{C} \bar{b}$, the type tp($\bar{a}\bar{b}/C$) is determined by tp($\bar{a}/C$) and tp($\bar{b}/C$).

For stable theories, Ehud Hrushovski has proved results similar to the fundamental theorem of projective geometry and to von Neumann's theorem.