

Thorn-forking and generalised semimodularity

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Morley's theorem

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Let T be a first-order theory in a countable language.

Theorem (Löwenheim-Skolem)

If T has an infinite model, then for every infinite cardinal κ , T has a model of size κ .

Theorem (Morley)

If there is an uncountable cardinal κ such that all models of T of cardinality κ are isomorphic, then the same is true for every uncountable cardinal.

Examples

- Algebraically closed fields of a fixed characteristic
- Vector spaces over a fixed field.

Algebraic closure

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The algebraic closure of a subset A of a model of a theory T consists of all elements which satisfy a formula with parameters in A that has only finitely many solutions. E.g. algebraic closure in algebraically closed fields, linear hull in vector spaces.

For every model of a theory T , algebraic closure is a finitary closure operator:

- $A \subseteq \text{acl } A$.
- $A \subseteq B \Rightarrow \text{acl } A \subseteq \text{acl } B$.
- $\text{acl}(\text{acl } A) = \text{acl } A$.
- $\text{acl } A = \bigcup_{\text{finite } A_0 \subseteq A} \text{acl } A_0$.

Strongly minimal theories

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Sometimes algebraic closure also satisfies the exchange law:

- $a \in \text{acl}(Cb) \setminus \text{acl } C \quad \Rightarrow \quad b \in \text{acl}(Ca).$

Finitary closure operator + exchange law
= matroid.

In a matroid, maximal independent set = minimal generating set. We have a good notion of basis and dimension.

A theory is strongly minimal if algebraic closure satisfies the exchange law and the type of n independent elements depends only on n .

Every strongly minimal theory is an example for Morley's theorem. The converse is morally true.

Projective geometry

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A projective space consists of a set of points P and a set of subsets of P , called lines, subject to the following axioms:

- Any two distinct points a, b are in exactly one line $\text{cl}(a, b)$.
- If a, b, c, d are distinct points and no 3 of them are collinear, then $\text{cl}(a, b) \cap \text{cl}(c, d) \neq \emptyset$ implies $\text{cl}(a, c) \cap \text{cl}(b, d) \neq \emptyset$.

Every projective space is a matroid (but not conversely).

Fundamental theorem of projective geometry

Given a projective space of vectors over a field, the field can be reconstructed from the matroid, provided the dimension is ≥ 3 .

Matroids can be seen as semimodular atomistic lattices.

In the 1930s, John von Neumann's work on quantum theory made him interested in certain lattices of subspaces of a Hilbert space.

A continuous lattice is a complete lattice in which for every element a and linearly ordered set B the following hold:

- $a \wedge \bigvee B = \bigvee_{b \in B} a \wedge b.$
- $a \vee \bigwedge B = \bigwedge_{b \in B} a \vee b.$

(This can be seen as an infinite modularity condition.)

Theorem (von Neumann) Every irreducible continuous lattice with a homogeneous basis of order ≥ 5 is isomorphic to the lattice of ideals of a von Neumann regular ring.

Generalising modularity

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An active field in the 1930s.

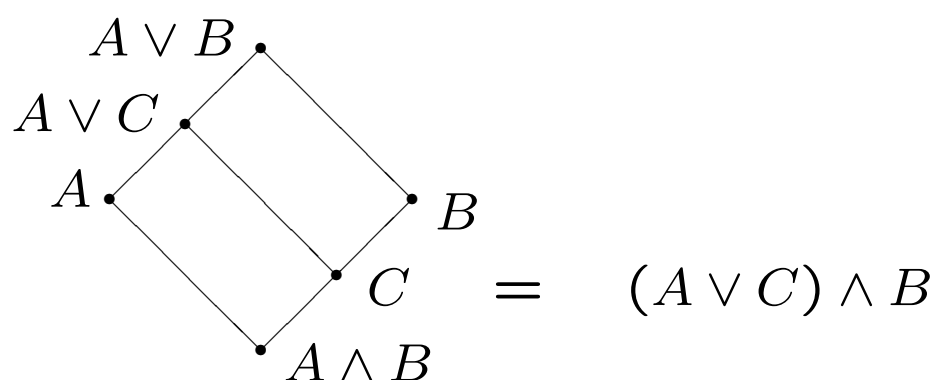
Several attempts:

- Birkhoff's condition (Birkhoff 1935)
- semimodularity (Birkhoff 1935)
- Mac Lane's condition (Mac Lane 1938)
- M-symmetry (Wilcox 1938)

All conditions agree for finite lattices.

All conditions differ in general.

$M(A, B)$ if A and B form a modular pair:



M-symmetry: $M(A, B) \iff M(B, A)$.

Independence

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In an M-symmetric lattice, define $A \perp_C^M B$ by $(A \vee C) \wedge (B \vee C) = C$ and $M(A \vee C, B \vee C)$.

This relation was used by John von Neumann, and also by others to prove generalisations of his theorem to certain M-symmetric lattices.

In a model of T s.t. the lattice of algebraically closed sets is M-symmetric, we can define $A \perp_C^M B$ for general sets A, B, C by $\text{acl}(AC) \perp_{\text{acl } C}^M \text{acl}(BC)$.

The relation \perp^M then satisfies the axioms of a strict independence relation.

(A theory which admits a strict independence relation is called rosy.)

Strict independence relations

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$$a \perp_B a \Rightarrow a \in \text{acl } B. \quad [\text{anti-reflexivity}]$$

$$A \perp_C B \iff \begin{array}{l} \text{[finite character]} \\ A_0 \perp_C B_0 \text{ for all finite } A_0 \subseteq A, B_0 \subseteq B. \end{array}$$

$$\begin{array}{l} \text{For } D \subseteq C \subseteq B: \quad \text{[full transitivity]} \\ A \perp_D B \iff A \perp_C B \text{ and } A \perp_D C. \end{array}$$

$$A \perp_C B \iff B \perp_C A \quad [\text{symmetry}]$$

$$\forall A, B, C \exists B' \equiv_C B: A \perp_C B'. \quad [\text{full existence}]$$

$$\begin{array}{l} A \perp_C B \subseteq \hat{B} \Rightarrow \text{[extension]} \\ \exists \hat{B}' \equiv_{ABC} \hat{B} : A \perp_C \hat{B}'. \end{array}$$

$$\begin{array}{l} \forall A, B \exists C \subseteq B: \quad \text{[local character]} \\ A \perp_C B \text{ and } |C| \leq \kappa(|A|). \end{array}$$

(We can remove extension and local character, as they follow from the other axioms.)

Free pseudoplane

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A tree or forest in which every vertex has infinite degree.

In this example the exchange law does not hold (a maximal independent set need not be generating). \perp^M does not satisfy symmetry and extension. However the following is a strict independence relation:

$A \perp_C B \iff$
every path connecting A and B
passes through $\text{acl } C$.

($\text{acl } C$ is the convex hull of C .)

For the free pseudoplane \perp^M is no good because symmetry and extension are broken. Let's fix extension!

$$A \perp_C^b B \iff \forall \hat{B} \supseteq B \exists \hat{B}' \equiv_{ABC} \hat{B} : A \perp_C^M \hat{B}'.$$

This does it for the free pseudoplane. How about the general case?

Theorem

1. \perp^b always satisfies all axioms except perhaps symmetry and local character.
2. For \perp^b , symmetry and local character are equivalent.
3. If \perp^b is not a strict independence relation, then there is none. If it is one, it is the weakest.

Theorem

The following are equivalent for all T :

- There are arbitrarily big cardinals κ such that over every model of size κ there are only κ complete types.
- No model of T contains an infinite set of tuples which are linearly ordered by a formula.
- Thorn-forking is an independence relation, and if $\bar{a} \perp_C^b \bar{b}$, the type $\text{tp}(\bar{a}\bar{b}/C)$ is determined by $\text{tp}(\bar{a}/C)$ and $\text{tp}(\bar{b}/C)$.

For stable theories, Ehud Hrushovski has proved results similar to the fundamental theorem of projective geometry and to von Neumann's theorem.