Strong theories, weight, and the independence property

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Strong theories, weight, and the IP

**Theorem S**
Let $T$ be simple, $\bar{a}$ finite, and $(\bar{b}_i)_{i<\kappa}$ an independent sequence of cofinality $\text{cf} \kappa > |T|$. Then for some $\alpha < \kappa$:
\[ \bar{a} \downarrow (\bar{b}_i)_{\alpha < i < \kappa}. \]

**Theorem D**
Let $T$ be dependent, $\bar{a}$ finite, and $(\bar{b}_i)_{i<\kappa}$ an indiscernible sequence of cofinality $\text{cf} \kappa > |T|$. Then for some $\alpha < \kappa$:
\[ (\bar{b}_i)_{\alpha < i < \kappa} \text{ is indiscernible over } \bar{a}. \]

If $T$ is supersimple, then Theorem S even holds for $\kappa = \omega$.

If $T$ is superdependent, then Theorem D even holds for $\kappa = \omega$.

What is wrong on this slide?
Strong theories, weight, and the IP

Contents:

1. Strong theories
2. Strongly simple theories
3. Strongly dependent theories
4. Strongly stable theories
5. Shelah’s conjecture on NIP fields
6. inp-minimality
7. Tree property of the second kind
**Strong theories (1/7)**

**Definition**

Inp-pattern (‘independent partitions’):

\[ \varphi^\alpha(\bar{x}; \bar{y}^\alpha) \text{ and } k^\alpha, \] where \( \alpha < \kappa \),

for which there is an array \( \bar{b}_i^\alpha \) s.t.:

- Rows \( \{ \varphi^\alpha(\bar{x}; \bar{b}_i^\alpha) \mid i < \omega \} \) are \( k^\alpha \)-inconsistent.
- Paths \( \{ \varphi^\alpha(\bar{x}; \bar{b}_i^\alpha) \mid \alpha < \kappa \} \), are consistent.

(A row is given by \( \alpha < \kappa \), a path by \( \eta \in \omega^\kappa \).)

**Definition** (Shelah 1978)

\( \kappa_{\text{inp}} = \text{smallest } \kappa \)

s.t. no inp-pattern of depth \( \kappa \) exists.

**Definition** (Shelah 1980)

Tree property of the second kind: \( \kappa_{\text{inp}} = \infty \).

**Definition**

Strong: \( \kappa_{\text{inp}} = \omega \).
Strongly simple theories

Theorem SS
Equivalent for simple $T$:
1. $T$ is strong.
2. $\forall \bar{a}$ finite $\forall (\vec{b}_i)_{i<\omega}$ independent$/C$
   $\exists n < \omega$:
   $$\bar{a} \downarrow_C (\vec{b}_i)_{n<i<\omega}.$$
3. $\forall \bar{a}$ finite $\forall (\vec{b}_i)_{i<\omega}$ independent$/C$
   $\exists n < \omega$:
   $$\bar{a} \downarrow_C \vec{b}_i \text{ for } i > n.$$

Definition
Strongly simple: strong + simple.
I.e. simple and finite weight.

Examples
• Supersimple theories.
• Simple theories with no dense forking chains.
Strongly dependent theories

**Theorem SD** (Shelah)
Equivalent for dependent $T$:
1. $T$ is strong.
2. $\forall \vec{a}$ finite $\forall (\vec{b}_i)_{i<\omega}$ indiscernible$/C$
   $\exists n < \omega$:
   $$(\vec{b}_i)_{n<i<\omega} \text{ indiscernible}/C\vec{a}.$$
3. $\forall \vec{a}$ finite $\forall (\vec{b}_i)_{i<\omega}$ indiscernible$/C$
   $\exists n < \omega$:
   $$(\vec{b}_i)_{n<i<\omega} \text{ has constant type}/C\vec{a}.$$

**Definition** (Shelah)
Strongly dependent: strong $+$ dependent.

**Examples**
- Superstable theories.
- $O$-minimal theories.
Strongly stable theories

Corollary
For stable theories, all the conditions of Theorems SS and SD are equivalent.

Definition (Shelah)
Strongly stable: strong + stable.

Remark
Strongly stable = strong + simple + NIP
= strongly simple + strongly dependent.

Examples
• Superstable theories.
• Stable theories with no dense forking chains.
**Shelah’s conjecture on NIP fields (5/7)**

**Theorem** (Shelah, Sh783)
Every superstable or o-minimal theory is strongly\(^+\) dependent.

**Theorem** (Shelah/Hrushovski, Sh783+Sh863)
The theory of a \(p\)-adic field is strongly dependent but not strongly\(^+\) dependent.

**Conjecture** (Shelah, Sh863)
Every strongly\(^+\) dependent field is
- algebraically closed or
- real closed.

**Conjecture** (Shelah, Sh863)
Every strongly dependent field is
- algebraically closed or
- real closed or
- a valuation field
  (similar to the \(p\)-adic fields).
inp-minimality

**Definition**

**inp-minimal:**
no inp-pattern of depth 2 for a single variable.

I.e. no $k$-inconsistent formulas
\[ \varphi(x, \bar{b}_0), \varphi(x, \bar{b}_1), \varphi(x, \bar{b}_2), \ldots \]
and $k'$-inconsistent formulas
\[ \psi(x, \bar{c}_0), \psi(x, \bar{c}_1), \psi(x, \bar{c}_2), \ldots \]
such that each $\varphi(x, \bar{b}_i) \land \psi(x, \bar{c}_j)$ is consistent.

**Definition** (Shelah, Onshuus-Usvyatsov)

**dp-minimal:** inp-minimal and dependent.

**Examples**

- strongly minimal theories.
- o-minimal theories.
- Simple theories s.t. every nonalgebraic 1-type has weight 1.
**Tree property of the second kind (7/7)**

**Definition** (Shelah 1980)
\[ \text{TP}_2: \kappa_{\text{inp}} = \infty. \]

**Theorem** (Shelah 1978)
The tree property = SOP$_2$ or TP$_2$.

**Remark**
Simple or dependent \( \Rightarrow \) NTP$_2$.

**Definition** (Casanovas 1999)
\[ \text{NT}(\kappa, \lambda) = \sup \text{cardinalities of antichains of partial types with } \leq \kappa \text{ formulas over a set of cardinality } \leq \lambda. \]

**Remark**
\[ \text{TP}_2 \Rightarrow \text{NT}(\kappa, \lambda) = \lambda^\kappa. \]