Strong theories, weight, and the independence property

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Strong theories, weight, and the IP

Theorem S
Let T be simple, $\bar{a}$ finite, and $(\bar{b}_i)_{i<\kappa}$ an independent sequence of cofinality $\operatorname{cf}\kappa > |T|$.
Then for some $\alpha < \kappa$:

$$\bar{a} \perp (\bar{b}_i)_{\alpha<i<\kappa}.$$

Theorem D
Let T be dependent, $\bar{a}$ finite, and $(\bar{b}_i)_{i<\kappa}$ an indiscernible sequence of cofinality $\operatorname{cf}\kappa > |T|$.
Then for some $\alpha < \kappa$:

$$(\bar{b}_i)_{\alpha<i<\kappa} \text{ is indiscernible over } \bar{a}.$$

If T is supersimple,
then Theorem S even holds for $\kappa = \omega$.

If T is superdependent,
then Theorem D even holds for $\kappa = \omega$.

What is wrong on this slide?

Strong theories, weight, and the IP

Contents:

Definition

Inp-pattern ('independent partitions'):

$\varphi^\alpha(\bar{x};\bar{y}^\alpha)$ and k^α, where $\alpha < \kappa$,

for which there is an array $\bar{b}^\alpha_i$ s.t.:

- Rows $\{\varphi^\alpha(\bar{x};\bar{b}^\alpha_i) \mid i < \omega\}$ are k^α-inconsistent.
- Paths $\{\varphi^\alpha(\bar{x};\bar{b}^\alpha_{\eta(\alpha)}) \mid \alpha < \kappa\}$, are consistent.

(A row is given by $\alpha < \kappa$, a path by $\eta \in \omega^\kappa$.)

Definition (Shelah 1978)

κ_{inp} = smallest κ

s.t. no inp-pattern of depth κ exists.

Definition (Shelah 1980)

Tree property of the second kind: $\kappa_{\mathrm{inp}} = \infty$.

Definition

Strong: $\kappa_{\mathrm{inp}} = \omega$.

Theorem SS

Equivalent for simple T:

1. T is strong.
2. $\forall\, \bar{a}$ finite $\forall\, (\bar{b}_i)_{i<\omega}$ independent/C $\exists\, n < \omega$:
$$\bar{a} \perp_C (\bar{b}_i)_{n<i<\omega}.$$
3. $\forall\, \bar{a}$ finite $\forall\, (\bar{b}_i)_{i<\omega}$ independent/C $\exists\, n < \omega$:
$$\bar{a} \perp_C \bar{b}_i \text{ for } i > n.$$

Definition

<u>Strongly simple:</u> strong + simple.

I.e. simple and finite weight.

Examples

- Supersimple theories.
- Simple theories with no dense forking chains.

Theorem SD (Shelah)

Equivalent for dependent T:

1. T is strong.
2. $\forall\, \bar{a}$ finite $\forall\, (\bar{b}_i)_{i<\omega}$ indiscernible/C $\exists\, n < \omega$:

$$(\bar{b}_i)_{n<i<\omega} \text{ indiscernible}/C\bar{a}.$$

3. $\forall\, \bar{a}$ finite $\forall\, (\bar{b}_i)_{i<\omega}$ indiscernible/C $\exists\, n < \omega$:

$$(\bar{b}_i)_{n<i<\omega} \text{ has constant type}/C\bar{a}.$$

Definition (Shelah)

Strongly dependent: strong + dependent.

Examples

- Superstable theories.
- O-minimal theories.

Strongly stable theories (4/7)

Corollary
For stable theories, all the conditions of Theorems SS and SD are equivalent.

Definition (Shelah)
Strongly stable: strong + stable.

Remark
Strongly stable = strong + simple + NIP
= strongly simple + strongly dependent.

Examples
- Superstable theories.
- Stable theories with no dense forking chains.

Shelah's conjecture on NIP fields (5/7)

Theorem (Shelah, Sh783)
Every superstable or o-minimal theory is strongly$^+$ dependent.

Theorem (Shelah/Hrushovski, Sh783+Sh863)
The theory of a p-adic field is strongly dependent but not strongly$^+$ dependent.

Conjecture (Shelah, Sh863)
Every strongly$^+$ dependent field is

- algebraically closed or
- real closed.

Conjecture (Shelah, Sh863)
Every strongly dependent field is

- algebraically closed or
- real closed or
- a valuation field
  (similar to the p-adic fields).

Definition

inp-minimal:

no inp-pattern of depth 2 for a single variable.

I.e. no k-inconsistent formulas

$\varphi(x, \bar{b}_0)$, $\varphi(x, \bar{b}_1)$, $\varphi(x, \bar{b}_2)$, . . .

and k'-inconsistent formulas

$\psi(x, \bar{c}_0)$, $\psi(x, \bar{c}_1)$, $\psi(x, \bar{c}_2)$, . . .

such that each $\varphi(x, \bar{b}_i) \wedge \psi(x, \bar{c}_j)$ is consistent.

Definition (Shelah, Onshuus-Usvyatsov)

dp-minimal: inp-minimal and dependent.

Examples

- Strongly minimal theories.
- o-minimal theories.
- Simple theories s.t. every nonalgebraic 1-type has weight 1.

Tree property of the second kind (7/7)

Definition (Shelah 1980)
TP$_2$: $\kappa_{\text{inp}} = \infty$.

Theorem (Shelah 1978)
Tree property = SOP$_2$ **or** TP$_2$.

Remark
Simple or dependent $\Rightarrow$ NTP$_2$.

Definition (Casanovas 1999)
NT(κ, λ) = supremum of cardinalities of antichains of partial types with $\leq \kappa$ formulas over a set of cardinality $\leq \lambda$.

Remark
TP$_2$ $\Rightarrow$ NT$(\kappa, \lambda) = \lambda^\kappa$.