Two aspects of stability theory

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January 2009

Prologue: Morley's Theorem

$\frac{1}{2}$ Categoricity in first-order model theory

Definition
A theory T is categorical if any two models of T are isomorphic.

Theorem (Löwenheim-Skolem)
Let T be a countable complete first-order theory.
If T has models of size κ for some infinite cardinal κ,
then T has models of size κ for all infinite cardinals κ.

Corollary
Let T be a countable complete first-order theory.
T is categorical if and only if T has a finite model.

$\frac{2}{2}$ κ-categoricity and Morley's Theorem

Definition

T is κ-categorical if any two models of size κ are isomorphic.

Theorem (Morley)

Let T be a countable complete first-order theory.
If T is κ-categorical for some uncountable cardinal κ,
then T is κ-categorical for all uncountable cardinals κ.

Examples of uncountably categorical theories:

- Infinite set with no additional structure.
- Vector space over a given finite (or countable) field.
- Projective space over a given finite (or countable) field.
- Algebraically closed field of a given characteristic.

Part I: Geometry

$\frac{1}{9}$ Zilber's Conjecture

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Conjecture (Zilber)

Every uncountably categorical countable first-order theory is essentially of one of the above types.

Hrushovski found uncountably many counterexamples.

The definition of "essentially" involves matroids. These occur in the proof of Morley's theorem.

Matroids

Definition

A matroid is a set M equipped with a finitary closure operator $\mathrm{cl}\colon \mathcal{P}(M) \to \mathcal{P}(M)$ that satisfies the exchange property.

$$A \subseteq \mathrm{cl}\, A$$
$$A \subseteq B \implies \mathrm{cl}\, A \subseteq \mathrm{cl}\, B$$
$$\mathrm{cl}(\mathrm{cl}\, A) = \mathrm{cl}\, A$$
$$\mathrm{cl}\, A = \bigcup\{\mathrm{cl}\, F \mid F \subseteq_{\text{finite}} A\}$$

$$a \in \mathrm{cl}(C \cup \{b\}) \setminus \mathrm{cl}\, C \implies b \in \mathrm{cl}(C \cup \{a\}).$$

A basis of a matroid is a minimal generating set / maximal independent set. Its dimension is the cardinality of any basis.

Examples of matroids:

- Trivial closure (i.e. $\mathrm{cl}\, A = A$) on any set.
- Linear hull on a vector space. A matrix.
- Algebraic closure on an algebraically closed field.
- The edges of a graph.

Algebraic closure

Definition

Let M be a model of a first-order theory T, $\bar{b} \in M$.
A first-order formula $\varphi(\bar{x}, \bar{b})$ is called algebraic if it has only finitely many realisations.
An element $a \in M$ is called algebraic over a set $B \subseteq M$ if there is a tuple $\bar{b} \in B$ and an algebraic formula $\varphi(x, \bar{b})$ such that $M \models \varphi(a, \bar{b})$.
The algebraic closure of a set $B \subseteq M$ is the set

$$\operatorname{acl} B = \{a \in M \mid a \text{ algebraic over } B\}.$$

Examples of algebraic closure:

- In the theory of an infinite set (empty signature): $\operatorname{acl} B = B$.
- In any theory of a vector space: linear hull.
- In any algebr. closed field: algebr. closure in the sense of algebra.

Strong minimality

Definition

A complete first-order theory is called strongly minimal if for all models

- algebraic closure is a matroid, and
- for all n, all independent n-tuples have the same type.

Theorem

Let T be a countable complete first-order theory.

If T is strongly minimal, then T is uncountably categorical and the cardinality of an uncountable model equals its dimension.

More generally, T is uncountably categorical if and only if the restriction of T to a certain formula is strongly minimal and every model of T is prime over this strongly minimal part.

$\frac{5}{9}$ Independence for more general theories

To extend this dimension theory beyond uncountably categorical theories, we can:

1. drop or weaken the condition on independent n-types
2. generalise the notion of matroid, or
3. allow other closure operators instead of acl.

Real closed fields are an example of a pregeometric theory: Algebraic closure is a matroid. But the theory is not uncountably categorical, and this is an example of 1.

We will interpret a part of the machinery of stability theory as doing 2 in order to get 3.

Semimodular lattices

Definition (Wilcox)

(A, B) is a modular pair if for all $C \in [A \wedge B, B]$ we have $(A \vee C) \wedge B = C$.

A lattice is semimodular (M-symmetric) if being a modular pair is a symmetric relation.

- The lattice of closed sets of a matroid is semimodular; the closures of elements are its atoms.
- A semimodular lattice that is generated by its atoms can be interpreted as a matroid.

Independence in a semimodular lattice

$$A \perp_C B \iff (A \vee C) \wedge (C \vee B) = C, \text{ and } (A, B) \text{ is a modular pair}$$
$$\iff \text{for all } D \in [C, B \vee C]\colon (A \vee D) \wedge (B \vee C) = D.$$

Fact

In a matroid, $A \perp_C B$ is equivalent to the condition that every subset $A_0 \subset A$ which is independent over C is also independent over $B \cup C$.
In an arbitrary semimodular lattice, $\perp$ still deserves the name 'independence'.

For example in a vector space (or just a module), $A \perp_C B$ means: A and B are linearly independent over C. In this case the lattice is modular, so all pairs are modular pairs.

Weight

Definition

In a semimodular lattice, the weight of a lattice element A is the maximal n such that there is an independent sequence $B_0, B_1, \ldots, B_{n-1}$ with $A \not\perp B_0, \ldots, A \not\perp B_{n-1}$.

Fact

The weight 1 elements of a semimodular lattice form a matroid under the following closure operator:

$$\operatorname{cl} A = \{b \mid \exists A_0 \subseteq A \colon A_0 \text{ independent and } A_0 \not\perp b\}.$$

If there are 'enough' weight 1 elements, and if there is some control over the independent sets of weight 1 elements, then the matroid of weight 1 elements helps to understand the models of a theory. This is the case for superstable theories.

Thorn-forking

- A recent notion due to Scanlon, Onshuus and Ealy.
- Equivalent to Shelah's notion of forking, whenever that is well-behaved.
- More generally well-behaved than forking.

Definition

$A \mathop{\smile\!\!\!\!\!\!|}^{þ}_C B$ iff, up to isomorphism over $B \cup C$, $\operatorname{acl} A \perp_{\operatorname{acl} C} \operatorname{acl} B'$ for all $B' \supseteq B$.

A complete first-order theory is called rosy if $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ is symmetric.

If a theory has a semimodular lattice of algebraically closed sets, then the theory is rosy.

(The definition must be read in the monster model of T^{eq}.)

Part II: Combinatorics

$\frac{1}{9}$ Back to Morley's Theorem

Theorem (Morley)

Let T be a countable complete first-order theory.
If T is κ-categorical for some uncountable cardinal κ,
then T is κ-categorical for all uncountable cardinals κ.

Counting models

Classify complete first-order theories T according to their model spectrum:

$$I(\kappa, T) = \text{number of models of } T \text{ of size } \kappa \text{ up to isomorphism.}$$

Characterise the possible functions by structural, model theoretic properties of T.
($\kappa > |T|$ avoids Vaught's Conjecture and less essential variations.)

The problem was solved by Shelah. Some of the structural properties are very intricate.

$\frac{3}{9}$ Counting types

Classification according to the stability function is much easier:

$$g_T(\kappa) = \sup_{M \models T,\, |M| = \kappa} |S(M)| .$$

The six possible stability functions are (Keisler 1974):

$$\kappa, \quad \kappa + 2^{\aleph_0}, \quad \kappa^{\aleph_0},$$
$$\operatorname{ded} \kappa, \quad (\operatorname{ded} \kappa)^{\aleph_0}, \quad 2^{\kappa}.$$

• Under GCH, the last three are actually the same.
• It is consistent with ZFC that $(\operatorname{ded} \kappa)^{\aleph_0} < 2^{\kappa}$ for some κ.
(Baumgartner 1976)
• Is $\operatorname{ded} \kappa < (\operatorname{ded} \kappa)^{\aleph_0}$ possible?

$\frac{4}{9}$ Dividing lines

Shelah's project: Classify (complete, first-order) theories by identifying fundamental combinatorial 'dividing lines' between 'order' and 'chaos'.

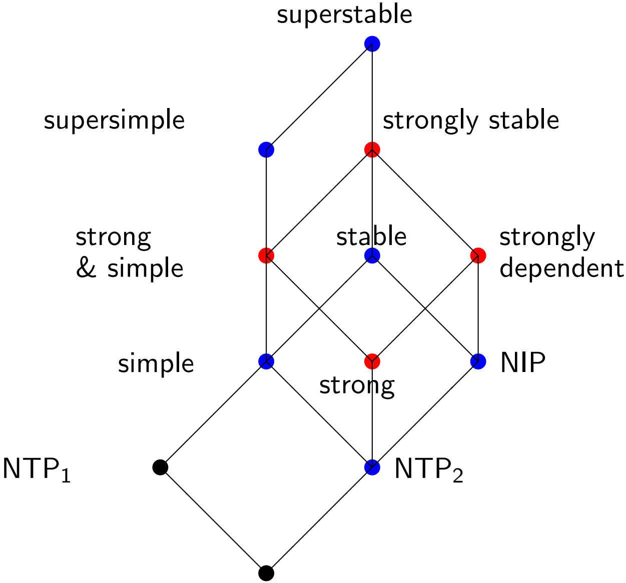
Some of the most fruitful dividing lines can be expressed in terms of the stability function:

- Totally transcendental theories: $g_T(\kappa) = \kappa$
- Superstable theories: $g_T(\kappa) \leq \kappa + 2^{\aleph_0}$
- Stable theories: $g_T(\kappa) \leq \kappa^{\aleph_0}$
- Dependent (NIP) theories: $g_T(\kappa) < 2^\kappa$.

Some others (simplicity, supersimplicity) can be expressed by a variant:

$$\begin{aligned}\mathrm{NT}_T(\kappa, \lambda) = \sup |\{A \mid A \text{ antichain of partial types} \\ \text{with } \leq \kappa \text{ formulas over a set of size } \leq \lambda\}|.\end{aligned}$$

$\frac{5}{9}$ A lattice of dividing lines



$\frac{6}{9}$ Geometric consequences of combinatorial properties

- Superstability, stability, supersimplicity, simplicity can be expressed geometrically.
- For strong, strongly dependent, NIP, NTP_2 this is likely.
- For NTP_1 there is probably no geometric characterisation.
- There is an entire hierarchy of properties between simplicity and NTP_1. Of these, the 'partial order property' seems to be the only candidate for a good geometric definition.

NIP/dependent theories

Definition

A complete first-order theory is said to be dependent or NIP if all formulas have bounded Vapnik-Chervonenkis dimension.

- This class includes all stable theories.
- It also includes many unstable theories that are important in algebra.
- Current research in model theory concentrates on the unstable case.
- Forking seems to be important here, although it is not symmetric.
- If there is a suitable generalisation of matroids for this situation, it will probably not be symmetric, either.
- Greedoids seem to be a natural candidate.

$\frac{8}{9}$ Hierarchy of order properties

$$
\begin{array}{ccccccccc}
\text{strict OP} & \Rightarrow & \text{FPOP} & \Rightarrow & \text{POP} & & & & \\
\Downarrow & & \Downarrow & & \Downarrow & & & & \\
\text{FFSOP} & \Rightarrow & \text{FSOP} & \Rightarrow & \text{SOP} & & & & \\
 & & & & \Downarrow & & & & \\
 & & & & \vdots & & & & \\
 & & & & \Downarrow & & & & \\
 & & & & \text{SOP}_4 & & & & \\
 & & & & \Downarrow & & & & \\
 & & & & \text{SOP}_3 & \Rightarrow & \text{TP}_1 & \Rightarrow & \text{TP} \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \text{OP} \\
 & & & & & & & & \Downarrow \\
 & & & & & & & & \text{inf. model}
\end{array}
$$

A geometrically simple theory with the strong OP

Signature: Binary relations $R^1, R^2, R^3, \ldots$.

Axioms of T_0:

$$\forall xy(R^n xy \to R^{n+1} xy),$$
$$\forall xyz(R^m xy \wedge R^n yz \to R^{m+n} xz),$$
$$\forall x(\neg R^n xx).$$

T: theory of the Fraïssé limit of the finite models of T_0.
In T, $R^n = (R^1)^n$.

T is 'mock stable':
$A \mathop{\smile\!\!\!\!\!\!|}_C B$ if $A \cap B \subseteq C$ and for $a \in A \setminus C$, $b \in B \setminus C$ only unavoidable relations $R^n ab$ or $R^n ba$ hold.