Kim's Lemma for NTP_2 Theories

A simpler proof of a result by Chernikov and Kaplan ("Forking and dividing in NTP_2 theories").

Hans Adler, February 2011

<u>Definition</u> $\varphi(x,y)$ has TP_2 if the following exists:

$$
\begin{array}{cccc}
\varphi(x,b_{00}) & \varphi(x,b_{01}) & \varphi(x,b_{02}) & \dots \\
\varphi(x,b_{10}) & \varphi(x,b_{11}) & \varphi(x,b_{12}) & \dots \\
\varphi(x,b_{20}) & \varphi(x,b_{21}) & \varphi(x,b_{22}) & \dots \\
\vdots & \vdots & \vdots &
\end{array}
$$

- Each row k-inconsistent for some k (always the same k).
- For every function $f:\omega\to\omega$, $\{\varphi(x,b_{if(i)}) \mid i<\omega\}$ consistent.

<u>Remark</u> $TP_2 \Rightarrow$ tree property (just use the same row repeatedly in every tree level)

$TP_2 \Rightarrow$ independence property (observe that for every subset of formulas in the first column there is a tuple a making precisely these true)

<u>Definition</u> $a \mathop{\downarrow}^{f}_{C} B$ $:\Leftrightarrow$ $tp(a/BC)$ does not fork over C.

$a \mathop{\downarrow}^{i}_{C} B$ $:\Leftrightarrow$ $tp(a/BC)$ has a global extension that is invariant over C (or equivalently: that does not split over C).

<u>Definition</u> C is an <u>invariance base</u> if for all A, B there is $A' \equiv_C A$ s.t. $A' \mathop{\downarrow}^{i}_{C} B$.

All models are invariance bases.
In dependent theories, $\downarrow^i = \downarrow^f$.

<u>Definition</u> A global type $p(x)$ is <u>strictly invariant</u> over C if it is invariant over C and for all $B \supseteq C$, all $a \models p\upharpoonright B$: $B \mathop{\downarrow}^{f}_{C} a$.
(The first condition says $a \mathop{\downarrow}^{i}_{C} B$.)

A <u>strict Morley sequence</u> over C is a sequence that is generated by a global type $p(x)$ strictly invariant over C.
Generated means: $a_0 \models p\upharpoonright C$, $a_1 \models p\upharpoonright Ca_0$, $a_2 \models p\upharpoonright Ca_0a_1$, ...

Kim's Lemma for NTP$_2$ theories (Chernikov, Kaplan)

In an NTP$_2$ theory, for any formula $\varphi(x,b)$ and any invariance base M the following are equivalent:

1. Every strict Morley sequence in $tp(b/M)$ witnesses that $\varphi(x,b)$ divides over M.
2. Some strict Morley sequence in $tp(b/M)$ witnesses that $\varphi(x,b)$ divides over M.
3. $\varphi(x,b)$ divides over M.
4. $\varphi(x,b)$ forks over M.

We will prove this in a series of lemmas. ~~Note that this lemma gives new information even for a simple theory.~~

Resilience Lemma I (NTP$_2$)

If $\varphi(x,b)$ divides over C and $q(y) \supset tp(b/C)$ is a strictly invariant global extension, then every sequence generated by q over C (is a strict Morley sequence over C and) witnesses that $\varphi(x,b)$ divides over C.

Proof Choose $N \supseteq C$ $(|T|+|C|)^+$-saturated and such that $b \models q \restriction N$.

Choose any sequence $\bar{b}_0 = (b_{0i})_{i\in\omega}$, indiscernible over C, which witnesses that $\varphi(x,b)$ divides over C.

Since $N \mathop{\downarrow}^f_C b$, we may assume that $\bar{b}_0$ is indiscernible over N. Moreover, we may choose $\bar{b}_0$ so that $b \models q \restriction N\bar{b}_0$.

Choose any sequence $\bar{b}_1 = (b_{1i})_{i\in\omega}$, indiscernible over C, such that $\bar{b}_0 \equiv \bar{b}_1$.

Since $N\bar{b}_0 \mathop{\downarrow}^f_C b$, we may assume that $\bar{b}_1$ is indiscernible over N. Moreover, we may choose $\bar{b}_1$ so that $b \models q \restriction N\bar{b}_0\bar{b}_1$.

Continuing in this way, we get a sequence $\bar{b}_0, \bar{b}_1, \bar{b}_2, \ldots$, giving rise to a matrix

$$\begin{matrix} \varphi(x,b_{00}) & \varphi(x,b_{01}) & \varphi(x,b_{02}) & \cdots \\ \varphi(x,b_{10}) & \varphi(x,b_{11}) & \varphi(x,b_{12}) & \cdots \\ \varphi(x,b_{20}) & \varphi(x,b_{21}) & \varphi(x,b_{22}) & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

with k-inconsistent rows (for some k).

Since $(b_{if(i)})_{i\in\omega}$ is generated over C by q, $\{\varphi(x,b_{if(i)}) \mid i\in\omega\}$ is consistent either for all $f:\omega\to\omega$ or for none. In the former case the matrix witnesses TP$_2$, a contradiction. In the latter case we are finished. $\square$

Resilience Lemma II (NTP_2).

If $\varphi(x,b)$ divides over an invariance base M, then there is an $\downarrow^i$-Morley sequence over M which witnesses this.

Proof For big enough κ, let $\bar{b} = (b_i)_{i<\kappa}$ witness that $\varphi(x,b)$ divides over M.

Choose $N \supset M$ $(|T|+|M|)^+$-saturated set.

$b \downarrow^i_M N$.

Extract from $(b_i)_{i<\kappa}$ a sequence of indiscernibles over N and replace $\bar{b}$ by this new sequence, $\bar{b} = (b_i)_{i<\omega}$. Note $\bar{b} \downarrow^i_M N$ still holds, by finite character of $\downarrow^i$. $tp(\bar{b}/N)$ generates over M: $\bar{b}_0, \bar{b}_1, \bar{b}_2, \ldots$

For all n, $\bar{b}_n$ is indiscernible over $M\bar{b}_{<n} \subset N$.

because this is true for $\bar{b}$.

$$\left.\begin{array}{l} \text{because this is true for } \bar{b}. \\ \bar{b}_{\geq n} \downarrow^i_M \bar{b}_{<n} \underset{\text{monotonicity}}{\overset{\text{base}}{\Longrightarrow}} \bar{b}_{>n} \downarrow^i_{M\bar{b}_{<n}} \bar{b}_n \end{array}\right\} \Rightarrow \bar{b}_n \text{ indiscernible over } M\bar{b}_{\neq n}.$$

We get a matrix as in the previous proof, with k-inconsistent rows for some k. Since the rows are mutually indiscernible over M, again $(b_{i f(i)})_{i<\omega}$ has the same type over M for all $f: \omega \to \omega$. In the same way as before we see that $\{\varphi(x, b_{i0}) \mid i<\omega\}$ must be k'-inconsistent for some k', so the $\downarrow^i$-Morley sequence$/_M$ $(b_{i0})_{i<\omega}$ witnesses that $\varphi(x,b)$ divides over M. □

Vacuum Cleaner Lemma (NTP_2)

Let $p(x)$ be a partial global type that is invariant over an invariance base M.

Suppose $p(x) \vdash \psi(x,b) \vee \bigvee_{i<n} \varphi^i(x,c)$, where $b \downarrow^i_M c$ and each $\varphi^i(x,c)$ divides over M.

Then $p(x) \vdash \psi(x,b)$.

Corollary A consistent partial global type that is invariant over an invariance base M does not fork over M. (set $\psi = \perp$)

Proof Trivial for $n=0$. Suppose the lemma holds for n, and $p(x) \vdash \psi(x,b) \vee \bigvee_{i\leq n} \varphi^i(x,c)$, where $b \downarrow^i_M c$ and each $\varphi^i(x,c)$ divides over M.

Let $(c_i)_{i<\omega}$ be an $\downarrow^i$-Morley sequence over M $\rightsquigarrow k$ which witnesses that $\varphi^n(x,c)$ divides over M.

Since $b \downarrow^i_M c = c_0$, we may assume $b \downarrow^i_M (c_i)_{i<\omega}$, and in particular $(c_j)_{j<\omega}$ is indiscernible over Mb.

By invariance of p,

$$p(x) \vdash \psi(x,b) \vee \bigwedge_{\substack{j<k' \\ \uparrow \\ \text{appropriate } k^*}} \bigvee_{i\leq n} \varphi^i(x,c_j).$$

Here k is chosen so that $\bigwedge_{j<k} \varphi^n(x, b_j)$ is inconsistent. It follows that

$$p(x) \vdash \psi(x,b) \vee \bigvee_{\substack{i<n \\ j<k}} \varphi^i(x, c_j).$$

For each j,

$$\left. \begin{array}{l} b \downarrow^i_M c_{\geq j} \Rightarrow b \downarrow^i_{Mc_{>j}} c_j \\ c_{>j} \downarrow^i_M c_j \end{array} \right\} \Rightarrow bc_{>j} \downarrow^i_M c_j$$

Applying the induction hypothesis k times, we get

$$p(x) \vdash \psi(x,b) \vee \bigvee_{1\leq j<k} \bigvee_{i<n} \varphi^i(x,c_j)$$

$$p(x) \vdash \psi(x,b) \vee \bigvee_{2\leq j<k} \bigvee_{i<n} \varphi^i(x,c_j)$$

$$\vdots$$

$$p(x) \vdash \psi(x,b) \vee \bigvee_{k-1\leq j<k} \bigvee_{i<n} \varphi^i(x,c_j)$$

$$p(x) \vdash \psi(x,b).$$

□

Existence Lemma (NTP_2)

Every type over an invariance base M has a strictly invariant global extension.

Proof Given a complete type $p(x) = tp(a/M)$, consider the partial global type

$$p(x) \cup \{\neg\varphi(x,b) \mid \varphi(a,y) \text{ forks over } M\}$$
$$\cup \{\psi(x,c) \leftrightarrow \psi(x,c') \mid c \equiv_M c'\}.$$

We need to show that this partial type is consistent.

If not, then $p(x) \vdash \varphi(x,b) \vee \bigvee_{i<n} (\psi_i(x,c_i) \not\leftrightarrow \psi_i(x,c_i'))$ where $\varphi(a,y)$ forks over M and $c_i \equiv_M c_i'$.

Since $\varphi(a,y)$ forks over M, the partial global type $q(y) = \{\varphi(a',y) \mid a' \equiv_M a\}$ also forks over M.

As it is invariant over M, by the (corollary to) the Vacuum Cleaner Lemma, $q(y)$ is inconsistent.

Let $a_0, a_1, \dots, a_{m-1} \models tp(a/M)$ be s.t. $\{\varphi(a_i,y) \mid i<m\}$ is inconsistent. Since M is an invariance base, $tp(a_0,\dots,a_{m-1}/M)$ has a global extension $p^*(x_0,\dots,x_{m-1})$ that is invariant over M. Each $p^*\restriction x_j$ is invariant over M, and $p^*\restriction x_j \supset \cancel{p(x_j) \vdash \varphi(x_j,b) \vee \bigvee_{i<n}(\psi_i(x,c_i) \not\leftrightarrow \psi_i(x,c_i'))}$.

It follows that

$$p^*(x_0,\dots,x_{m-1}) \vdash \varphi(x_0,b) \wedge \dots \wedge \varphi(x_{m-1},b),$$

a contradiction. □

Proof of Kim's Lemma $1 \xrightarrow[\text{Lemma}]{\text{Existence}} 2 \Rightarrow 3 \Rightarrow 4$

$\Downarrow$ Resilience Lemma I.

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