Model theory tutorial
Part 1 – Morley's Theorem

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Overview

1. **Morley's Theorem** (Wednesday)
2. Forking and thorn-forking (Thursday)
3. Dependent theories I (Friday)
4. Dependent theories II (Saturday)

Goals of this tutorial: Motivation, general idea of how things work.

Not in scope at all: Complete overview of the field, details of proofs.

Recapitulation of basic first-order model theory

- Given a signature (a.k.a. language) we consider structures (models) and first-order formulas.
- Three major styles of model theory can be distinguished by the morphisms they use:
    - weak homomorphisms (finite model theory, Gaifman's Theorem, universal algebra)
    - substructure embeddings (decidability and quantifier elimination questions)
    - **elementary embeddings** (classification theory / stability theory).
- Compactness Theorem: A set of formulas whose finite subsets are consistent is consistent.
- Löwenheim-Skolem Theorem: An infinite model has elementary sub-/superstructures of (essentially) arbitrary infinite cardinality.
- Omitting Types Theorem: Any countable set of non-isolated types is omitted in some model.
- With these and a few other theorems, some people once thought that first-order model theory had been essentially completed.

Categorical theories

From now on, in this tutorial we will only consider countable theories, although in constructions we may occasionally add uncountably many constants to the signature.

T is categorical if all models $M \models T$ are isomorphic.
By Löwenheim-Skolem, it follows that the unique model of T is finite.

Originally, logicians found this shocking.
Can the concept of categoricity be rescued somehow for infinite models?

Let κ be infinite. T is κ-categorical if all models $M \models T$ of size $|M| = \kappa$ are isomorphic. By Löwenheim-Skolem, it follows that T is complete.

ω-categoricity

- ω-categoricity is relatively well understood, but has not been as fruitful so far as κ-categoricity for $\kappa > \omega$.
  (Is this because Shelah is less interested in the concept?)
- Theorem of Ryll-Nardzewski, Engeler and Svenonius: Characterisation of ω-categorical theories.
- Fraïssé limit: Every class of finite structures in a fixed finite relational signature, closed under substructures and satisfying JEP and AP, gives rise to an ω-categorical theory with QE.

Examples of κ-categorical theories

- Infinite set with no structure: κ-categorical for all infinite κ.
- Dense linear order without endpoints: (only) ω-categorical.
- Random graph: (only) ω-categorical.
- (More generally, any Fraïssé limit is ω-categorical, though not necessarily κ-categorical for any uncountable κ.)
- Infinite-dimensional vector space over a finite field:
  κ-categorical for all infinite κ.
  (Similarly for affine and projective spaces.)
- Vector space over an infinite field:
  κ-categorical for all uncountable κ, but not ω-categorical.
  (Similarly for affine and projective spaces.)
- Any algebraically closed field:
  κ-categorical for all uncountable κ, but not ω-categorical.

Uncountable categoricity

- Examples show: ω-categoricity and κ-categoricity for an uncountable κ have very different flavours.
- Uncountable categoricity appears to be much rarer than ω-categoricity, and every known uncountably categorical theory has a nice notion of dimension.
- Łoś's Conjecture:
  If T is κ-categorical for one uncountable κ, then T is κ-categorical for all uncountable κ.
- Zilber's Conjecture:
  Every uncountably categorical T is essentially the theory of an infinite set, a vector space / projective space, or of an algebraically closed field.
  (Boris Zilber denies responsibility for this version of the conjecture, which was refuted by Ehud Hrushovski.)

Morley's Theorem

Michael Morley proved Łoś's Conjecture in 1965. His proof techniques opened up a completely new, technically difficult but rewarding field of mathematics. Saharon Shelah soon became, and still is, the major driving force.

Let $\kappa, \lambda > \omega$.

If T is κ-categorical, then T is λ-categorical.

Lemma 1
If T is κ-categorical,
then T is totally transcendental and admits no Vaught pairs.

Lemma 2
If T is totally transcendental and admits no Vaught pairs,
then T is λ-categorical.

Lemma 1 – total transcendentality

T is totally transcendental if there is no tree of formulas with parameters that has inconsistent branching but consistent branches.

Fact: For any T there are arbitrarily big models $M \models T$ such that for every countable $B \subseteq M$, M realises only countably many types over B. (Tricky; involves Skolem hull of an indiscernible sequence.)

If T is κ-categorical, the κ-sized model must have this property.
With Löwenheim-Skolem it follows that T is ω-stable:
Over any countable set, T has only countably many types.
Total transcendentality now follows easily.

Fact: Totally transcendental theories have prime models, and these are even very well behaved.

Lemma 1 – no Vaught pairs

A Vaught pair is $M \prec N$ such that $M \neq N$ but for some formula $\varphi(x)$ (with parameters in M) we have $\varphi^M = \varphi^N$.

Vaught's Two-Cardinal Theorem: If there is a Vaught pair, then there is a Vaught pair $M \prec N$ such that M is countable and N is uncountable.

If T is totally transcendental, then we can even make N as big as we want. (A bit tricky; use prime models to build a chain of Vaught extensions.)

There is always a κ-sized model N in which every formula with parameters has finitely many or κ solutions. (Simple union of chains argument.)
It follows that κ-categorical T cannot have a Vaught pair $M \prec N$ with $|M| = \omega$, $|N| = \kappa$, so it cannot have any Vaught pair.

Lemma 2 – strongly minimal formula

A formula $\mu(x)$ is minimal in M if it has infinitely many solutions, but for every $\varphi(x)$ with parameters in M, either $\mu(x) \wedge \varphi(x)$ or $\mu(x) \wedge \neg\varphi(x)$ has only finitely many solutions. ($\rightarrow$ generic type)

If T is totally transcendental and $M \models T$, there is a minimal formula in M.
(The formula may have parameters in M, but this is only a superficial problem because we can take M to be a prime model of T. Let's assume it has no parameters.)

If T has no Vaught pairs, every minimal formula is strongly minimal.
(Proof uses elimination of $\exists^{\infty}$.)

Lemma 2 – matroid

A matroid $(\mathcal{M}, \mathrm{cl})$ (a.k.a. pregeometry) is a set $\mathcal{M}$ together with a finitary closure operator cl on $\mathcal{M}$ which satisfies the exchange law: $a \in \mathrm{cl}(C \cup \{b\}) \setminus \mathrm{cl}\, C \implies b \in \mathrm{cl}(C \cup \{a\})$.

Examples: Linear hull in a vector space, algebraic closure in a field, graphical matroid.

Generating set: $\mathrm{cl}\, A = \mathcal{M}$. Independent set: $a \notin \mathrm{cl}(A \setminus \{a\})$ for all $a \in A$.
Basis = minimal generating set = maximal independent set.
All bases have the same cardinality: rank/dimension of the matroid.

If $\mu(x)$ is a strongly minimal formula, then (μ^M, acl) is a matroid.
Moreover, any two independent sequences have the same type.
(The converse is also true.)

Lemma 2 – proving λ-categoricity

Let $N_1, N_2 \models T$ such that $|N_1| = |N_2| = \lambda$.

By Löwenheim-Skolem and non-existence of Vaught pairs, $\left|\mu^{N_i}\right| = |N_i| = \lambda$. Let $B_i \subseteq \mu^{N_i}$ be a basis of μ^{N_i}. Then $|B_i| = \left|\mu^{N_i}\right| = \lambda$.

So B_1 and B_2 have the same type, i.e. there is a partial isomorphism between them. It extends to a partial isomorphism between μ^{N_1} and μ^{N_2}.

By existence of prime models and non-existence of Vaught pairs, N_1 is prime over μ^{N_1}. The partial isomorphism therefore extends to an embedding of N_1 into N_2. By non-existence of Vaught pairs, this must be an isomorphism $N_1 \cong N_2$.

Corollary

Lemma 2 can be extended to the case $\lambda = \omega$. In this case we only know that $|B_i| \leq \omega$. Therefore only countably many dimensions of μ^{N_i} are possible, and so there are only countably many isomorphism types of countable models.

Further analysis shows: If μ^{N_i} can have dimension n, then it can also have dimension $n+1$. It follows that the number of countable models up to isomorphism is either 1 or ω.

Observations

κ-categoricity could a priori depend on properties of the universe of set theory. (Two non-isomorphic models might have the same cardinality in one universe but not in another.) But total transcendence and existence of Vaught pairs are *robust notions*. Mainstream model theory so far has been most interested in such robust notions and has been avoiding advanced set theory.

Counting models of a given size, or counting types over a set of a given size are examples of a general technique that often gives rise to *dividing lines* between order/control and chaos/complexity. A good dividing line is robust and allows us to say a lot both on the order side and on the complexity side.

In a sense, classification theory is about *classifying fields of mathematics*. It shows deep similarities or differences between mathematical fields which can be represented by first-order theories.

A large part of classical classification theory is motivated by algebraic geometry, with first-order formula playing the role of polynomials/varieties.

References

- Michael Morley: Categoricity in power.
  Original paper proving Morley's Theorem. Still very readable, as terminology and notation have not changed much.
- Wilfrid Hodges: Model theory.
  Section 12.2 has a proof of Morley's Theorem in a series of exercises.
- Steven Buechler: Essential stability theory.
  Proves Morley's Theorem very early, using few prerequisites.
- Anand Pillay: Introduction to stability theory.
  Somewhat dated but quite readable. Cheap new Dover edition.
- Saharon Shelah: Classification theory.
  Standard work. A small number of researchers are able to read it. A few more are able to use it as a reference with some effort. Contains a proof of Morley's Theorem in the uncountable case.

Tomorrow

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We will see how to generalise the idea of a matroid so it applies to much more general theories. This is due to Shelah (forking, 1969) and became a bit clearer with the discovery of thorn-forking (Scanlon, Onshuus 2003). But very similar things were done much earlier in lattice theory after John von Neumann proved a sensational theorem in the 1930s, similar to Hrushovski's group construction in model theory.