Model theory tutorial
Part 2 – Forking and thorn-forking

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Overview

1. Morley's Theorem (Wednesday)
2. **Forking and thorn-forking** (Thursday)
3. Dependent theories I (Friday)
4. Dependent theories II (Saturday)

Recall the goals of this tutorial: Motivation, general idea of how things work. For filling in the proofs see literature recommendations at the end of the talk.

Morley sequences

Recall the key step in the proof of Morley's theorem: Every model is determined by a basis $(b_i)_{i<\lambda}$ of its strongly minimal set (a matroid). Every such basis has the following properties, which we will take as starting points today:

- $(b_i)_{i<\lambda}$ is an indiscernible sequence.
- $(b_i)_{i<\lambda}$ is an independent sequence.

Indiscernibility makes sense in any theory, but the notion of an independent sequence so far only makes sense when we have a matroid. Adapting it to more general contexts is hard.

Once we have done this, we will refer to a sequence that is simultaneously indiscernible and independent as a Morley sequence.

Independence relation of a matroid

Given any matroid $(\mathcal{M}, \mathrm{cl})$ and any set $C \subset \mathcal{M}$, a quotient $(\mathcal{M}, \mathrm{cl})/C$ can be defined as $(\mathcal{M}, \mathrm{cl}_C)$, where $\mathrm{cl}_C(X) = \mathrm{cl}(C \cup X)$.
We speak of independence / generating / basis *over* C.

Now we can define what it means for two arbitrary sets A and B to be independent *from each other* over a third set:
$A \mathop{\smile\!\!\!\!\!\!|}_C B$ if every subset $A' \subseteq A$ which is independent over C is also independent over BC. (BC is an abbreviation for $B \cup C$.)
On the next slide we will collect some properties of the relation $\mathop{\smile\!\!\!\!\!\!|}$.

From this we can recover the independence notion of the matroid:
A is independent $\iff a \mathop{\smile\!\!\!\!\!\!|}_\emptyset (A \setminus \{a\})$ for all $a \in A$.

Properties of ternary independence in a matroid

(symmetry)
$A \mathop{\smile\!\!\!\!\!\!\,|}_C B \iff B \mathop{\smile\!\!\!\!\!\!\,|}_C A$.

(monotonicity)
If $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!\,|}_C B'$.

(base monotonicity)
Suppose $D \subseteq C \subseteq B$. If $A \mathop{\smile\!\!\!\!\!\!\,|}_D B$, then $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$.

(transitivity)
Suppose $D \subseteq C \subseteq B$. If $B \mathop{\smile\!\!\!\!\!\!\,|}_C A$ and $C \mathop{\smile\!\!\!\!\!\!\,|}_D A$, then $B \mathop{\smile\!\!\!\!\!\!\,|}_D A$.

(normality)
$A \mathop{\smile\!\!\!\!\!\!\,|}_C B$ implies $AC \mathop{\smile\!\!\!\!\!\!\,|}_C B$.

(finite character)
If $A_0 \mathop{\smile\!\!\!\!\!\!\,|}_C B$ for all finite $A_0 \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$.

(local character)
For every A there is a cardinal $\kappa(A)$ such that for any set B there is a subset $C \subseteq B$ of cardinality $|C| < \kappa(A)$ such that $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$.

Free pseudoplane

The free pseudoplane is a forest consisting of trees that branch infinitely in every node. It has the following nice independence relation:

$A \mathop{\smile\!\!\!\!\!\!|}_C B \iff$ every path from A to B meets $\operatorname{acl} C$.

In the free pseudoplane, $\operatorname{acl} C$ is just the convex hull of C.

Independence relations in model theory

In the matroid of a strongly minimal set we also have the following two properties, which must be read in a monster model of the theory:

(extension)
If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $\hat{B} \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C \hat{B}$.

(full existence)
For any A, B and C there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$

Independence relation: Any relation with these properties.

The axioms are not independent. For example, full existence follows from local character and extension.
More surprisingly, symmetry can also be proved from the other axioms.
The proof illustrates the use of Morley sequences.

Stability

Stable theory: no infinite set of tuples can be linearly ordered by a formula.
For example, totally transcendental theories, all theories of modules, and the free pseudoplane are all stable.
The dense linear order, the random graph, the reals and Peano arithmetic are all examples of theories that are not stable.

Stable theories are precisely the theories admitting an independence relation with stationarity: Given a model M and two tuples $\bar{a}$ and $\bar{b}$, $\mathrm{tp}(\bar{a}\bar{b}/M)$ is determined by $\mathrm{tp}(\bar{a}/M)$ and $\mathrm{tp}(\bar{b}/M)$ if we also know that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}$.

Moreover, there can only be one independence relation with stationarity, and we can get it either as forking independence or as thorn-forking independence.

Although forking is the earlier notion and seems more useful in unstable theories, we will start with thorn-forking, which is easier to grasp.

Modular case

The simplest candidate for an independence relation is the following relation:

$$A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} B \iff \operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl} C.$$

This relation always satisfies all axioms except possibly base monotonicity. The only axiom that is not trivial to prove is full existence. (Extension then follows together with transitivity.)

$\mathop{\smile\!\!\!\!\!\!|}^{a}$ satisfies base monotonicity (and is an independence relation) if and only if the lattice of algebraically closed sets in the monster model is modular.
This is the case in vector spaces (even modules), but not in fields.

Digression – von Neumann's continuous lattices

$\mathop{\smile}\limits^{|}{}_{a}$ is a purely lattice theoretical notion. In a sense, it was used in what one would expect to be a very different context half a century before it arose in model theory.

In the 1930s, John von Neumann examined a certain type of Hilbert spaces because of their significance for quantum theory. He was particularly interested in the lattices of subspaces of these Hilbert spaces. He referred to lattices which share certain important properties with these lattices as continuous lattices. In particular, continuous lattices are modular and admit a relation (perspectivity) that is somewhat analogous to having the same type.

He proved a very deep theorem about continuous lattices: Every continous lattice which has sufficiently high 'dimension' to contain what model theorists call the group configuration, is isomorphic to the lattice of ideals of a ring with pseudoinverses (a von Neumann regular ring).

Ehud Hrushovski much later independently invented a very similar construction in model theory. The respective proofs are based on the same results of classical finite geometry, and are largely parallel.

Semimodularity

John von Neumann's theorem was totally unexpected. Lattice theory had been considered a framework for expressing trivial mathematical facts rather than a deep field of mathematics.

Now the field suddenly attracted excellent researchers such as Birkhoff, Mac Lane and Wilcox, who tried to extend von Neumann's results beyond the modular case. They came up with several versions of 'semimodularity', which are all equivalent for atomic lattices but different in general.

M-symmetry (due to Wilcox) is what we need here.

Independence in an M-symmetric lattice

If base monotonicity does not hold, let's enforce it:
$A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{M}}_{C} B \iff$ for all $B' \subseteq B$: $A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{a}}_{CB'} B$.
(If base monotonicity does hold, then $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{M}} = \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{a}}$.)

In lattice theoretical terms:
$A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{M}}_{C} B \iff A \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{a}}_{C} B$ and $(\mathrm{acl}(AC), \mathrm{acl}(BC))$ is a modular pair.

Surprisingly, the axioms for independence relations are so robust, that most of them do not break under this operation: They hold for $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{M}}$ because they hold for $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{a}}$.

But some axioms, such as symmetry or existence, can break in general. M-symmetry just says that $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{M}}$ satisfies symmetry. Although not trivial, it then follows that $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathsf{M}}$ also satisfies existence and is in fact an independence relation.

Examples

It turns out that in the case of a strongly minimal set, $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ is precisely the independence relation that we defined earlier based on the matroid. In particular, the lattice of algebraically closed sets for an algebraically closed field is M-symmetric.

But sometimes a meaningful independence relation exists even in the absence of M-symmetry.

Recall the free pseudoplane with its independence relation:

$A \mathop{\smile\!\!\!\!\!\mid}_C B \iff$ every path from A to B meets acl C.

$\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}}$ does not satisfy the extension axiom in this case:
If a and b are neighbours, then $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}} b$ but there is no neighbour c of b such that $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{M}} bc$.

Proving symmetry from the other axioms

Morley sequence in $\mathrm{tp}(a/BC)$ over C:
$(a_i)_{i<\omega}$ indiscernible over BC, all a_i realising $\mathrm{tp}(a/BC)$, and $a_{<n} \mathop{\smile\!\!\!\!\!\!\mid}_C a_n$ for all n.

Lemma 1
If $a \mathop{\smile\!\!\!\!\!\!\mid}_C B$,
then there is a Morley sequence in $\mathrm{tp}(a/BC)$ over C.

Lemma 2
If there is a Morley sequence in $\mathrm{tp}(a/BC)$ over C,
then $B \mathop{\smile\!\!\!\!\!\!\mid}_C a$.

Thorn-forking

Tom Scanlon and Alf Onshuus in 2002 defined the most general independence relation: thorn-forking.

Our approach is to simply repeat what has worked before. If the extension axiom does not hold for $\mathop{\smile\!\!\!\!\!\!\downarrow}^{M}$, let's enforce it.

$A \mathop{\smile\!\!\!\!\!\!\downarrow}^{þ}_{C} B \iff$ for all $\hat{B} \supseteq B$ there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\downarrow}^{M}_{C} \hat{B}$.

Again, surprisingly most axioms do *not* break under this operation. One can show that $\mathop{\smile\!\!\!\!\!\!\downarrow}^{þ}$ satisfies local character if and only if it satisfies symmetry if and only if it is an independence relation.

Rosy theory: $\mathop{\smile\!\!\!\!\!\!\downarrow}^{þ}$ is an independence relation.

(There is some complication here related to imaginaries, which we will ignore.)

Forking

$A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{d}}_C B \iff$ for all sequences $(\bar{b}_i)_{i<\omega}$ of C-indiscernibles with $\bar{b}_0 \in B$ there is $A' \equiv_{C\bar{b}_0} A$ such that the sequence is AC-indiscernible.
(Dividing independence.)

$A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{f}}_C B \iff$ for all $\hat{B} \supseteq B$ there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{d}}_C \hat{B}$.
(Forking independence.)

Like $\mathop{\smile\!\!\!\!\!\!\,|}^{\text{þ}}$, $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{f}}$ in an arbitrary theory satisfies most axioms.
It satisfies symmetry if and only if it satisfies local character, if and only if it is an independence relation.
Simple theory: $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{f}}$ is an independence relation.

Totally transcendental $\Rightarrow$ stable $\Rightarrow$ simple $\Rightarrow$ rosy.

Universal property of forking independence:
If $A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{f}}_C B$ and $\mathop{\smile\!\!\!\!\!\!\,|}$ is *any* independence relation, then $A \mathop{\smile\!\!\!\!\!\!\,|}_C B$.
(Thorn-forking satisfies a dual universal property.)

References

- Hans Adler: A geometric introduction to forking and thorn-forking. Thoroughly explains forking and thorn-forking via their independence relations. See this paper and the sequel for pointers to the original literature.
- Enrique Casanovas: Simple theories.
  This recent book is a good introduction and gives a good first idea of how independence relations are used.

Tomorrow

1. Morley's Theorem (Wednesday)
2. Forking and thorn-forking (Thursday)
3. **Dependent theories** (Friday)
4. Forking in dependent theories (Saturday)

A theory is stable if non-forking is an independence relation and satisfies stationarity. It is simple if non-forking is just an independence relation. And it is dependent (or NIP) if non-forking just satisfies the stationarity condition. This is tricky, because in this context non-forking is not even symmetric.

Before we can see on Saturday how non-forking and Morley sequences seem to be meaningful even in the asymmetric case, tomorrow we will look at examples of dependent theories (dense linear order, the reals) and see how well-behaved indiscernible sequences are there.