Model theory tutorial
Part 3 – Dependent theories

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1 June 2012

Overview

1. Morley's Theorem (Wednesday)
2. Forking and thorn-forking (Thursday)
3. **Dependent theories** (Friday)
4. Forking in dependent theories (Saturday)

Theories minimal over another theory

Many definitions in classification theory can be put in the following form: A theory is X if there are no formulas $\varphi_i(\bar{x})$ with parameters in some model, such that a certain configuration involving the formulas does not exist.

Moreover, it is usually the case that to check that a theory is X, we only need to check formulas with a single free variable.
This explains why definitions of the following type turn out to be rather useful:

Given a fixed and well understood base theory T_0, a theory $T \supseteq T_0$ in a signature extending that of T_0 is minimal over T_0 if T has 1-QE down to the quantifier-free formulas of T_0: Modulo T, every formula $\varphi(x)$ with parameters in a model of T is equivalent to a quantifier-free formula $\varphi(x')$ in the signature of T_0, with parameters in the same model.

The theories minimal over the theory of an infinite set are precisely the strongly minimal theories.

O-minimal theories

O-minimal theory: minimal over the theory of linear orders.

O-minimal theories are not stable, in fact not even simple, but they have a number of other remarkable properties.

By definition, every unary definable set is a finite union of intervals. More generally, every definable set is a finite boolean combination of cells of a certain simple form.

O-minimal theories are rosy, and in fact for any o-minimal structure M, (M, acl) is a matroid.

Dependent theories

VC dimension:
$\mathrm{vc}(\varphi(\bar{x};\bar{y})) = \max\{n < \omega \mid \exists(\bar{a}_I)_{I\subseteq n}\exists(\bar{b}_j)_{j<n}(\models \varphi(\bar{a}_I,\bar{b}_j) \Leftrightarrow j \in I)\}$.

$\mathrm{vc}^*(\varphi(\bar{x};\bar{y})) = \mathrm{vc}(\varphi(\bar{y};\bar{x}))$.

Fact: $\mathrm{vc}^*(\varphi) < 2^{\mathrm{vc}(\varphi)+1}$.

A formula with $\mathrm{vc}^*(\varphi) = \infty$ is said to have the independence property.

Dependent or NIP theory:
No formula $\varphi(\bar{x},\bar{y})$ has the independence property.

VC dimension and the independence property were defined independently in 1971 by Vapnik and Chervonenkis and by Shelah. The connection was observed 20 years later by Laskowski.

O-minimal theories are dependent

Theorem

If no formula of the form $\varphi(x;\bar{y})$ has the independence property, then T is dependent.

By compactness in an o-minimal theory:

For every formula $\varphi(x;\bar{y})$ there is an upper bound k for the number of intervals required to describe an instance $\varphi(x;\bar{b})$. It follows that $\mathrm{vc}(\varphi(x;\bar{y})) \leq 2k+1$.

But how to prove the theorem? Shelah's original proof counted types, but required that two cardinals are distinct which may, in fact, be equal. So he also needed the fact that it is consistent for these cardinals to be different, as well as set theoretical absoluteness of independence.

Today, there is a much simpler proof.

Alternation number

Alternation number:
$\mathrm{alt}(\varphi(\bar{x}, \bar{y}))$ is the maximal number of times that $\varphi(\bar{a}_i, \bar{b})$ may change its truth value for a tuple $\bar{b}$ and an indiscernible sequence $(\bar{a}_i)_{i<\omega}$.

$\mathrm{alt}(\varphi) \leq 2\,\mathrm{vc}(\varphi) + 1$.

$\mathrm{vc}(\varphi)$ cannot be bounded in terms of $\mathrm{alt}(\varphi)$,
but $\mathrm{alt}(\varphi)$ is finite $\iff$ $\mathrm{vc}(\varphi)$ is finite.

Making indiscernibles more indiscernible

Lemma

If $\mathrm{alt}(\varphi(\bar{x}_1 \dots \bar{x}_k; \bar{y})) < \infty$, then for every $\bar{b}$, every indiscernible sequence $(\bar{a}_i)_{i<\omega}$ has an end piece $(\bar{a}_i)_{N \le i < \omega}$ such that the truth value of $\varphi(\bar{a}_{i_1} \dots \bar{a}_{i_k}; \bar{b})$ is the same whenever $N \le i_1 < \dots < i_k$.

Theorem

In a dependent theory, for any set B,
every indiscernible sequence $(\bar{a}_i)_{i<\kappa}$ with cf $\kappa > |T| + |B|$ has a non-trivial end piece that is indiscernible over B.
(Note: If κ is regular, then the end piece has the same length as the original sequence.)

Lemma

The following are equivalent:

1. No formula $\varphi(\bar{x}; \bar{y})$ with $|\bar{y}| \le n$ has the independence property.
2. If $(\bar{b}_i)_{i<|T|^+}$ is indiscernible and $|B| \le n$,
   then a non-trivial end piece is indiscernible over B.

The stability function

$g_T(\kappa) = \sup_{M \models T, |M| = \kappa} |S(M)|.$

For countable T, only the following six stability functions are possible:

- κ (totally transcendent)
- $\kappa + 2^{\aleph_0}$ (superstable, not totally transcendent)
- $\kappa^{\aleph_0}$ (stable, not superstable)
- $\operatorname{ded} \kappa$ (unstable, not multi-order)
- $(\operatorname{ded} \kappa)^{\aleph_0}$ (dependent, multi-order)
- 2^κ (not dependent).

Under the generalised continuum hypothesis, $\operatorname{ded} \kappa = 2^\kappa$. But it is consistent that this is not the case.

References

- Hans Adler: Introduction to theories without the independence property.

Tomorrow

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4. **Forking in dependent theories** (Saturday)

Yesterday we have seen how Morley sequences can be used to check whether two sets are independent with respect to some independence relation.
In dependent unstable theories, this does not work in the same way, because forking is not symmetric. But surprisingly, something like this still holds.