Model theory tutorial
Part 4 – Forking in dependent theories

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Overview

1. Morley's Theorem (Wednesday)
2. Forking and thorn-forking (Thursday)
3. Dependent theories (Friday)
4. **Forking in dependent theories** (Saturday)

Forking formulas

$\varphi(\bar{x}, \bar{b})$ *divides over* C:
There is a sequence $(\bar{b}_i)_{i<\omega}$ indiscernible over C such that $\bar{b}_0 = \bar{b}$ and $\{\varphi(\bar{x}, \bar{b}_i) \mid i < \omega\}$ is inconsistent.

$\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B \iff \mathrm{tp}(\bar{a}/BC)$ contains a formula that divides over C.

$\varphi(\bar{x}, \bar{b})$ *forks over* C:
There are finitely many formulas $\psi_j(\bar{x}, \bar{c}_j)$ that divide over C and such that $\varphi(\bar{x}, \bar{b}) \vdash \bigvee_j \psi_j(\bar{x}, \bar{c}_j)$.

$\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}^{f}_{C} B \iff \mathrm{tp}(\bar{a}/BC)$ contains a formula that forks over C.

Non-forking in a dependent theory

Proposition

In a dependent theory,

if $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^{f}_{C} B$ and $(\bar{b}_i)_{i<\omega}$ is indiscernible over C, with $\bar{b}_0, \bar{b}_1 \in B$,

then $\bar{b}_0 \equiv_{\bar{a}C} \bar{b}_1$.

Corollary

In a dependent theory,

a type that does not fork over a model is invariant over the model:

If $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}^{f}_{M} \bar{b}_0\bar{b}_1$ and $\bar{b}_0 \equiv_M \bar{b}_1$, then $\bar{b}_0 \equiv_{\bar{a}M} \bar{b}_1$.

Corollary

In a dependent theory,

every type has only a bounded number of non-forking global extensions.

Kim's Lemma

A dependent theory may also be simple. (Simple + dependent = stable.)

Kim's Lemma
Let T be simple.
For any $\varphi(x, b)$ and any C the following are equivalent:

1. $\varphi(x, b)$ divides over C.
2. $\varphi(x, b)$ forks over C.
3. Every Morley sequence in $\mathrm{tp}(b/C)$ witnesses that $\varphi(x, b)$ divides over C.
4. Some Morley sequence in $\mathrm{tp}(b/C)$ witnesses that $\varphi(x, b)$ divides over C.

Surprisingly, a version of this lemma also holds in all dependent theories. In fact, it holds in all NTP_2 theories. NTP_2 is a natural common generalisation of dependent and simple.

Studying NTP_2 theories is a strategy for studying dependent theories. (Basic stability theory got easier through the generalisation to simple theories.)

NTP_2

$\varphi(x, y)$ has TP_2 if a matrix of instances exists as follows:

$$\begin{array}{cccc} \varphi(x, b_{00}) & \varphi(x, b_{01}) & \varphi(x, b_{02}) & \dots \\ \varphi(x, b_{10}) & \varphi(x, b_{11}) & \varphi(x, b_{12}) & \dots \\ \varphi(x, b_{20}) & \varphi(x, b_{21}) & \varphi(x, b_{22}) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

- For some k, each row is k-inconsistent.
- For every $f : \omega \to \omega$, $\{\varphi(x, b_{i,f(i)}) \mid i < \omega\}$ is consistent.

Fact: If such an array exists, then we can make the rows mutually indiscernible and the sequence of rows indiscernible.

Strict invariance

Strictly invariant over C *global type* $p(x)$:
Invariant over C and
$\forall B \supseteq C \forall a \models p{\restriction}B: \quad B \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{C} a$

Strict Morley sequence over C:
A sequence that is generated by a global type that is strictly invariant over C.

Kim-Chernikov-Kaplan Lemma

Theorem

Let T be NTP_2.

For any $\varphi(x, b)$ and any model M the following are equivalent:

1. $\varphi(x, b)$ divides over M.
2. $\varphi(x, b)$ forks over M.
3. Every strict Morley sequence in $tp(b/M)$ witnesses that $\varphi(x, b)$ divides over M.
4. Some strict Morley sequence in $tp(b/M)$ witnesses that $\varphi(x, b)$ divides over M.

The theorem also holds over more general sets than models (invariance bases), though we will not consider this case.

Proof sketch

- $4 \Rightarrow 1 \Rightarrow 2$ is obvious.
- Lemmas 1 and 2
  - Proof of Lemma 2
  - Lemma 2 says that $1 \Rightarrow 3$;
    $2 \Rightarrow 3$ is a simple corollary
  - Skipped proof of Lemma 1 is similar
- Existence Lemma
  - Skipped proof uses Vacuum Cleaner Lemma
  - Implies $3 \Rightarrow 4$
- Vacuum Cleaner Lemma
  - Skipped proof of Vacuum Cleaner Lemma uses Lemma 1

Lemmas 1 and 2

Suppose $\varphi(x, b)$ divides over M.

Lemma 1
There is a Morley sequence over M
which witnesses that $\varphi(x, b)$ divides over M.

Lemma 2
Let $q(y) \supset \mathrm{tp}(b/M)$ be a strictly invariant global extension.
Then every strict Morley sequence generated by q over M
witnesses that $\varphi(x, b)$ divides over M.

Proof of Lemma 2

Choose any M-indiscernible sequence $\bar{b}_0 = (b_{0i})_{i<\omega}$
witnessing that $\varphi(x, b)$ divides over M.
We may choose $\bar{b}_0$ so that $b \models q \upharpoonright M\bar{b}_0$.
Using $\bar{b}_0 \mathop{\smile\!\!\!\!\!\!|}^{f}_{M} b$, we can find an $M\bar{b}_0$-indiscernible sequence
$\bar{b}_1 \equiv_M \bar{b}_0$ in $\mathrm{tp}(b/M\bar{b}_0) = q \upharpoonright M\bar{b}_0$.
We may also assume $b \models q \upharpoonright M\bar{b}_0\bar{b}_1$.
Continuing in this way, we get a matrix

$$\begin{array}{cccc} b_{00} & b_{01} & b_{02} & \dots \\ b_{10} & b_{11} & b_{12} & \dots \\ b_{20} & b_{21} & b_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

such that for each row the φ-instances are k-inconsistent.
All vertical paths are generated by q and so have the same type.
By NTP_2, the φ-instances on vertical paths cannot all be consistent, so they are inconsistent.

Existence Lemma

Lemma
Every type over M has a strictly invariant global extension.
In other words:
In every type over M there is a strict Morley sequence.

The proof is straightforward once you know that a global type invariant over M does not fork over M.

. . . which is obvious.

Except that we need it for partial global types, in which case it is surprisingly hard to prove.

Vacuum Cleaner Lemma

Lemma

Let $p(x)$ be a partial global type, invariant over M.
Suppose $p(x) \vdash \psi(x,b) \vee \bigvee_{i<n} \varphi^i(x,c)$,
where $\mathrm{tp}(b/Mc)$ has a global extension invariant over M
and each $\varphi^i(x,c)$ divides over M.
Then $p(x) \vdash \psi(x,b)$.

Corollary

A consistent partial global type that is invariant over M
does not fork over M.

Proof of corollary

Let $p(x)$ be a partial global type invariant over M.
If p forks over M, then $p(x) \vdash \bot \vee \bigvee_{i<n} \varphi^i(x,c)$,
where each $\varphi^i(x,c)$ divides over M.
Note that $\mathrm{tp}(\emptyset/Mc)$ has a global extension invariant over M.
By the lemma, $p(x) \vdash \bot$.

References

- Hans Adler: Introduction to theories without the independence property.
- Artem Chernikov, Itay Kaplan: Forking and dividing in NTP_2 theories.