Let $T \subseteq \omega \times \kappa$ be a homogeneous tree with measures $\{\mu_s\}_{s \in \omega^{<\omega}}$. So, $A = p[T]$ is κ-Suslin and $B = \omega^\omega - A$ is co-κ-Suslin. Given sets $\vec{A} = \{A_s\}_{s \in \omega^{<\omega}}$ where $A_s \subseteq T_s$, let $T^{\vec{A}} = \{(s, \vec{\alpha}) \in T \colon \forall i \leq |s|\ \vec{\alpha} \restriction i \in A_{s \restriction i}\}$. For $x \in B$, let $f_x \colon T_x \to \mathrm{On}$ be the canonical ranking function, and let $f_{x,n} \colon T_{x \restriction n} \to \mathrm{On}$ be the subfunction of f_x (i.e., the restriction of f_x to $T_{x \restriction n}$. Similarly, given $\{A_{x \restriction n}\}$ (where $A_{x \restriction n} \subseteq T_{x \restriction n}$) we let $f_x^{\vec{A}} \colon T_x^{\vec{A}} \to \mathrm{On}$ be the canonical ranking function of $T_x^{\vec{A}}$ and let $f_{x,n}^{\vec{A}}$ be the corresponding subfunctions.

Definition. The homogeneous tree $(T, \{\mu_s\})$ is *stable* if there measure one sets A_s with respect to the μ_s such that for any $x \in B$ and any measure one sets $B_{x \restriction n}$ (with respect to $\mu_{x \restriction n}$) we have that

$$[f_{x,n}^{\vec{A}}]_{\mu_{x \restriction n}} \leq [f_{x,n}^{\vec{B}}]_{\mu_{x \restriction n}}$$

for all n.

Thus, the homogeneous tree is stable if there is a fixed countable sequence $\{A_s\}$ of measure one sets which suffices to minimize the ranking functions for all x with T_x wellfounded.

The significance of this definition lies in the following easy fact.

Fact. Let $(T, \{\mu_s\})$ be a stable homogeneous tree as witnessed by the measure one sets $\{A_s\}$. Let T' be the Martin-Solovay tree with $B = p[T']$ constructed from $(T^{\vec{A}}, \{\mu_s\})$. Let $\{\varphi_n\}$ be the corresponding semi-scale, that is, for $x \in B$ we have $\varphi_n(x) = [f_{x,n}^{\vec{A}}]_{\mu_{x \restriction n}}$. Then $\{\varphi_n\}$ is a scale on B.

We next state our main result. The proof is similar to the Martin-Woodin proof that all trees are weakly homogeneous.

Theorem (AD). *Every homogeneous tree $(T, \{\mu_s\})$ on $\kappa < \Theta$ is stable.*

Proof. Let ν be a fine measure on $\mathcal{P}_{\omega_1}(\cup_n \mathcal{P}(\kappa^n))$. We show that for ν almost all σ that if A^σ denotes the measure one sets defined by σ (that is, $A^\sigma = \{A_s^\sigma\}_{s \in \omega^{<\omega}}$ where $A_s^\sigma = \cap\{A \in \sigma \colon \mu_s(A) = 1\}$), then the sets A^σ witness the stability of $(T, \{\mu_s\})$. Suppose this fails. For almost all σ we define, uniformly in σ, a tree U_σ on $\omega \times \lambda$ for some λ ($\lambda = \sup j_{\mu_s}(\kappa)$). Actually, the tree will be on $\omega \times \lambda^{<\omega}$, but we can identify $\lambda^{<\omega}$ with λ if we like.

We set $((s(0), \dots, s(n-1)), (\beta_0, \dots, \beta_n)) \in U_\sigma$ iff the following hold:

1.) $\beta_0 \in \omega$.

2.) Each β_i for $i > 0$ codes a finite sequence of integers t_i extending s (roughly, a commitment that later extensions of s must follow t_i) along with a finite sequence of ordinals $(\beta_0^i, \dots, \beta_{|t_i|-1}^i)$.

3.) s must be compatible with all t_i for $i < |s|$.

4.) Let f_j^i represent β_j^i with respect to $\mu_{s \restriction i}$. Then for almost all $\eta_0, \dots \eta_{i-1}$ with respect to $\mu_{s \restriction i}$ we have that $(f_0^i(\vec{\eta}), \dots, f_{|t_i|-1}^i) \in T^{A^\sigma}$.

5.) For almost all $\vec{\eta} = (\eta_0, \dots, \eta_i)$ with respect to $\mu_{s \restriction (i+1)}$ we have that $(f_0^{i+1}(\vec{\eta}), \dots, f_{|t_{i+1}|-1}^{i+1}(\vec{\eta}))$ extends in $T_{t_{i+1}}$ the sequence $(f_0^i(\vec{\eta} \restriction i), \dots, f_{|t_i|-1}^i(\vec{\eta} \restriction i))$.

6.) For $i < \beta_0$, each f_j^i is almost everywhere the identity function, and for $i = \beta_0$ we have that almost everywhere that $(f_0^i(\vec{\eta}), \dots, f_{|t_i|-1}^i(\vec{\eta}))$ is a proper extension in $T_{|t_i|}$ of $\vec{\eta}$ (i.e, for $i = \beta_0$ the f_j^i give a proper extension of the identity function).

7.) We weave in the Martin-Solovay tree into U_σ as well. Say at every even level of U_σ we put in ordinals from the Martin-Solovay tree (so a branch through $(U_\sigma)_x$ also gives a branch through $(T')_x$ and so proves $x \in B$).

For $x \in B$ the tree $(U_\sigma)_x$ is attempting to produce ordinals $[f^i_j]_{\mu_{x\restriction i}}$ which witness that the sets A^σ have not yet attained the minimal ranking functions. They do this, roughly speaking, by describing embeddings of T_x (on measure one sets) into proper initial segments of $T^{A^\sigma}_x$.

Claim. For any x, $(U_\sigma)_x$ is illfounded iff $x \in B$ and there are measure one sets $B_{x\restriction n} \subseteq A^\sigma_{x\restriction n}$ (with respect to $\mu_{x\restriction n}$) such that for some m, $[f^{\vec{B}}_x]_{\mu_{x\restriction m}} < [f^{A^\sigma}_x]_{\mu_{x\restriction m}}$.

Proof. If $(U_\sigma)_x$ is illfounded, we have $x \in B$ and get ordinals β^i_j for all i and $j < |t_i|$. Let f^i_j be functions representing these ordinals with respect to $\mu_{x\restriction i}$. Let $B_{x\restriction n}$ be measure one sets with respect to $\mu_{x\restriction n}$ on which (4)-(6) above hold. Then the f^i_j give an order-preserving map from $T^{\vec{B}}_x$ into $T^{A^\sigma}_x$ and from (6) we have that $[f^{\vec{B}}_x]_{\mu_{x\restriction\beta_0}} < [f^{A^\sigma}_x]_{\mu_{x\restriction\beta_0}}$. Conversely, suppose $x \in B$ and we can find measure one sets $B_{x\restriction n} \subseteq A^\sigma_{x\restriction n}$ where for some m we have $[f^{\vec{B}}_x]_{\mu_{x\restriction m}} < [f^{A^\sigma}_x]_{\mu_{x\restriction m}}$. Note that we always have $[f^{\vec{B}}_x]_{\mu_{x\restriction k}} \leq [f^{A^\sigma}_x]_{\mu_{x\restriction k}}$ for all k, Let $\beta_0 = m$, and for $i < \beta_0$ let $\beta^i_j = [\mathrm{id}]_{\mu_{x\restriction i}}$. Since $[f^{\vec{B}}_x]_{\mu_{x\restriction m}} < [f^{A^\sigma}_x]_{\mu_{x\restriction m}}$, we can find a function g^m_x with domain $\mathrm{dom}(\mu_{x\restriction m})$ such that almost everywhere $g^m_x(\vec{\eta})$ is a proper extension of $\vec{\eta}$ in $T^{A^\sigma}_x$ and $|T^{A^\sigma}_x(g^m_x(\vec{\eta}))| \geq |T^{\vec{B}}_x(\vec{\eta})|$. By countable additivity g^m_x has length ℓ almost everywhere, for some $\ell > m$. Let $t_m = x \restriction \ell$, and let $f^m_j(\vec{\eta}) = g^m_x(\vec{\eta}) \restriction j$. This defines the ordinal β_m. Continuing in this manner produces a branch $\vec{\beta}$ through U_x. □

Returning to the proof of the theorem, we are assuming that for ν almost all σ that there is an x such that $(U_\sigma)_x$ is illfounded. That is, $\forall^*_\nu \sigma(U_\sigma \text{ is illfounded })$. For almost all σ, let x^σ, $\vec{\beta}^\sigma$ be the leftmost branch through U_σ. By countable additivity we may fix $x \in B$ so that $x^\sigma = x$ almost everywhere. Now fix any σ in the measure one set where $x^\sigma = x$ and also σ contains measure one sets $C_{x\restriction n}$ (w.r.t. the $\mu_{x\restriction n}$) which attain the minimal values for $[f^{\vec{C}}_x]_{\mu_{x\restriction n}}$ (which we can do by countable choice). But then the branch $\vec{\beta}^\sigma$ gives a proper embedding from $T^{\vec{D}}_x$ into $T^{A^\sigma}_x \subseteq T^{\vec{C}}_x$ for some measure one sets $D_{x\restriction n}$. This contradicts the choice of the $C_{x\restriction n}$. [More precisely we have an m such that $[f^{\vec{D}}_{x\restriction m}]_{\mu_{x\restriction m}} < [f^{\vec{C}}_{x\restriction m}]_{\mu_{x\restriction m}}$.] This completes the proof of the theorem. □

The above theorem allows us to fill the gap in the scale analysis from AD, and get the complete analysis of scales and Suslin cardinals from just AD (we need AD^+ to get a largest Suslin cardinal as usual). In particular, we have the following.

Corollary (AD)**.** Let κ be a Suslin cardinal with $\boldsymbol{\Gamma} = S(\kappa)$ (the κ-Suslin sets) closed under quantifiers. Assume κ is not the largest Suslin cardinal. Then $\mathrm{Scale}(S(\kappa))$ (known before) and every $\check{\boldsymbol{\Gamma}}$ set admits a scale with all norms in $\mathrm{Env}(\boldsymbol{\Gamma}, \kappa)$ (the envelope). Also, every set in $\mathrm{Env}(\boldsymbol{\Gamma}, \kappa)$ admits a scale with all norms in $\mathrm{Env}(\boldsymbol{\Gamma}, \kappa)$. Finally, $\mathrm{Scale}(\boldsymbol{\Sigma}^\lambda_0)$, where $\boldsymbol{\Sigma}^\lambda_0 = \bigcup_\omega \mathrm{Env}(\boldsymbol{\Gamma}, \kappa)$.