Hod up to $AD_\mathbb{R} + \Theta$ is measurable[*][†]

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Abstract

Suppose M is a transitive class size model of $AD_\mathbb{R} +$ "Θ is regular". M is a minimal model of $AD_\mathbb{R} +$ "Θ is measurable" if (i) $\mathbb{R}, Ord \subseteq M$ (ii) there is $\mu \in M$ such that $M \vDash$ "μ is a normal $\mathbb{R}$-complete measure on Θ" and (iii) for any transitive class size $N \subsetneq M$ such that $\mathbb{R} \subseteq N$, $N \vDash$ "there is no $\mathbb{R}$-complete measure on Θ". Continuing Trang's work in [8], we compute HOD of a minimal model of $AD_\mathbb{R} +$ "Θ is measurable".

The computation of HOD of models of determinacy has been one of the central themes in descriptive inner model theory. Steel's seminal [5] jumpstarted the project and a later work of Steel and Woodin (for instance see [3] or [4]) established connections with the Mouse Set Conjecture, core model induction and the inner model problem.

The analysis of HOD presented in the above papers, however, only computes V_Θ^{HOD}. Woodin computed the full HOD of $L(\mathbb{R})$ under $AD^{L(\mathbb{R})}$ (the proof can be found in [6]). Trang continued this work in [8]. He presented the exact inner model theoretic structure of HOD of models of determinacy that have the form $L(\wp(\mathbb{R}))$ and are contained inside the minimal model of $AD_\mathbb{R} + $ "Θ is regular". We extend Trang's work to minimal models of $AD_\mathbb{R} + $ "Θ is measurable". This notion, however, needs a definition.

Definition 0.1 *Assume AD^+. We say "Θ is measurable" if there is an $\mathbb{R}$-complete normal measure μ on Θ. We then let Θms be the theory $AD_\mathbb{R} + $ "Θ is measurable".*

Because we need to assume that the computation of $V_\Theta^{\text{HOD}^{L(\wp(\mathbb{R}))}}$ can be carried out, our current techniques can only work in some minimal setting, in a setting where the Mouse Set Conjecture (MSC, see Chapter 3.1 of [4]) and The Generation Of Pointclasses (see Conjecture 3.17 of [4]) are true. The first says that ordinal definable reals appear in mice and the second says that sufficiently closed pointclasses are generated by hod pairs. The models where our computation works are the minimal models of Θms.

Definition 0.2 *M is a minimal model of Θms if $M \vDash \Theta ms$, $\mathbb{R} \cup Ord \subseteq M$ and for any $N \subsetneq M$ such that $\mathbb{R} \cup Ord \subseteq N$, $N \vDash \neg\Theta ms$.*

We now have the following simple representation of minimal models of Θms.

Lemma 0.3 *Suppose that V is a minimal model of Θms. Let μ be a normal $\mathbb{R}$-complete measure on Θ. Then $V = L(\wp(\mathbb{R}))[\mu]$.*

Proof. We have that $L(\wp(\mathbb{R}))[\mu] \vDash \Theta ms$. Because $L(\wp(\mathbb{R}))[\mu] \subseteq V$, we have that $V = L(\wp(\mathbb{R}))[\mu]$. $\qquad\qquad\square$

It is not immediately clear that the existence of a minimal model of Θms follows from the existence of a model of Θms. This is because there could be $\subsetneq$-descending sequence of models $(M_i : i < \omega)$ such that for all $i < \omega$, $\Theta^{M_i} = \Theta^{M_{i+1}}$ and $M_i \vDash \Theta ms$. We will in fact show that this cannot happen (see Theorem 2.1). However, towards showing the aforementioned fact, we will need to analyze the *weakly minimal* models of Θms.

Definition 0.4 *We say M is a weakly minimal model of Θms if*

1. *$\mathbb{R} \cup Ord \subseteq M$ and $M \vDash \Theta ms$,*

2. *for some $\mu \in M$ such that $M \vDash$ "μ is a normal $\mathbb{R}$-complete measure on Θ", $M = L(\wp(\mathbb{R}))[\mu]$, and*

3. *for any transitive N such that $\wp(\mathbb{R})^N \subsetneq \wp(\mathbb{R})^M$ and $\mathbb{R} \cup Ord \subseteq N$, $N \vDash \neg \Theta ms$.*

It is important to note that it follows from the main results of [3] (see Section 6.4 and Section 6.1 of [3]) that if M is a weakly minimal model of Θms then both MSC and The Generation of Pointclasses hold in $L(\wp(\mathbb{R})^M)$. We will exploit this in the next section. We mention that while it is not clear that there are minimal models of Θms, it is not hard to show, modulo the existing literature, that weakly minimal models do exist.

Lemma 0.5 *Suppose M is a transitive model of Θms such that $\mathbb{R}, Ord \subseteq M$. Then there is a weakly minimal model of Θms.*

Proof. Let $\Gamma \subseteq \wp(\mathbb{R})^M$ be Wadge least initial segment of $\wp(\mathbb{R})^M$ such that for some filter μ on $w(\Gamma)$, $L(\Gamma)[\mu] \vDash \Theta ms$. It follows from the proof of Theorem 13.1 of [2] that divergent models of AD^+ contain Γ, implying that if N is any transitive model of Θms such that $\mathbb{R}, Ord \subseteq N$ then $\Gamma \subseteq N$. $\square$

1 The main theorem

Suppose V is a weakly minimal model of Θms. As was mentioned above, both MSC and The Generation of Pointclasses hold in V. We can then freely use the machinery of [3]. Let $M = L(\wp(\mathbb{R}))$ and working in M, let $\mathcal{F}$ be the set of hod pairs $(\mathcal{P}, \Sigma)$ such that Σ has branch condensation and is fullness preserving.

Following [3], for $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda) \in \mathcal{F}$, we let

$$(\mathcal{P}, \Sigma) \preceq (\mathcal{Q}, \Lambda)$$

if for some $\alpha \leq \lambda^{\mathcal{Q}}$, $(\mathcal{Q}(\alpha), \Lambda_{\mathcal{Q}(\alpha)})$ is a tail of $(\mathcal{P}, \Sigma)$, i.e., $\mathcal{Q}(\alpha) \in I(\mathcal{P}, \Sigma)$ and $\Lambda_{\mathcal{Q}(\alpha)} = \Sigma_{\mathcal{Q}(\alpha)}$. It follows from comparison theory of hod pairs (see Chapter 2 of [3])

that $\preceq$ is directed. We then let $\mathcal{M}_\infty$ be the direct limit of the system $\mathcal{F}$ under the maps $\pi_{\mathcal{P},\mathcal{Q}}^\Sigma : \mathcal{P} \to \mathcal{Q}(\alpha)$ where $\alpha \leq \lambda^\mathcal{Q}$ is such that $\mathcal{Q}(\alpha)$ is a Σ-iterate of $\mathcal{P}$.

It is shown in [3] (see Theorem 4.24) that in M, $\mathcal{M}_\infty = V_\Theta^{\mathrm{HOD}}$. The following is essentially the generic interpretability result of [3] (see Theorem 3.10 of [3]). Given a hod premouse $\mathcal{P}$ and X generic over $\mathcal{P}$, we let $\Sigma^{\mathcal{P}[X]}$ be the interpretation of $\Sigma^\mathcal{P}$ onto $\mathcal{P}[X]$ according to the procedure described in the proof of Theorem 3.10 of [3].

Lemma 1.1 *In M (and hence in V), $\mathcal{M}_\infty$ is (Θ, Θ)-iterable via a strategy Σ such that given any $\vec{\mathcal{T}}$ according to Σ,*

$$\Sigma(\vec{\mathcal{T}}) = b \leftrightarrow \mathcal{M}_\infty[\vec{\mathcal{T}}] \vDash \Sigma^{\mathcal{M}_\infty[\vec{\mathcal{T}}]}(\vec{\mathcal{T}}) = b.$$

We let Σ be the strategy of $\mathcal{M}_\infty$ described in Lemma 1.1. Next, we define a model extending $\mathcal{M}_\infty$. Given a Σ-premouse $\mathcal{N} \unrhd \mathcal{M}_\infty$ we say $\mathcal{N}$ is good if

1. $\mathcal{N}$ is sound, and

2. whenever

$$\pi : \bar{\mathcal{N}} \to \mathcal{N}$$

is elementary and $\bar{\mathcal{N}}$ is countable, then $\bar{\mathcal{N}}$ is ω_1-iterable above $\pi^{-1}(\Theta)$ as a Σ^π-premouse.

The next two lemmas are basic lemmas about good mice.

Lemma 1.2 *Suppose $\mathcal{N}_0, \mathcal{N}_1$ are good such that for some $\eta \in \mathcal{N}_0 \cap \mathcal{N}_1$, η is a cutpoint of both $\mathcal{N}_0$ and $\mathcal{N}_1$, $\mathcal{N}_0|\eta = \mathcal{N}_1|\eta$ and $\rho_\omega(\mathcal{N}_0), \rho_\omega(\mathcal{N}_1) \leq \eta$. Then either $\mathcal{N}_0 \unlhd \mathcal{N}_1$ or $\mathcal{N}_1 \unlhd \mathcal{N}_0$.*

Proof. Suppose that neither $\mathcal{N}_0 \unlhd \mathcal{N}_1$ nor $\mathcal{N}_1 \unlhd \mathcal{N}_0$ holds. Let then $\pi : H \to L_\xi(\mathcal{P}(\mathbb{R}))[\mu]$ be elementary such that $\xi\gamma\Theta$, $\mathcal{N}_i \in rng(\pi)$, $\Sigma \in rng(\pi)$, $\eta \in rng(\pi)$ and $|H| = \omega$.

We let $\bar{\mathcal{N}}_i = \pi^{-1}(\mathcal{N}_i)$ for $i = 0, 1$. Because Σ has hull condensation, it follows that $\Sigma \upharpoonright H = \Sigma^\pi$. It follows from elementarity that for $i = 0, 1$, $\rho_\omega(\bar{\mathcal{N}}_i) = \pi^{-1}(\eta)$. But now because $\bar{\mathcal{N}}_i$ are sound Σ^π-mice, we have that $\bar{\mathcal{N}}_0 \unlhd \bar{\mathcal{N}}_1$ or $\bar{\mathcal{N}}_1 \unlhd \bar{\mathcal{N}}_0$. $\square$

Lemma 1.3 *Suppose $\mathcal{N}$ is good such that $\rho_\omega(\mathcal{N}) \leq \Theta$. Then $\rho_\omega(\mathcal{N}) = \Theta$.*

4

Proof. Towards a contradiction assume $\rho(\mathcal{N}) < \Theta$. Let $\pi : H \rightarrow L_\xi(\mathcal{P}(\mathbb{R}))[\mu]$ be such that $H \cap \Theta \in \Theta$, $crit(\pi) > \rho(\mathcal{N})$ and $\mathcal{N} \in rng(\pi)$. Let $\mathcal{P} = \pi^{-1}(\mathcal{N})$. It follows from the proof of Lemma 1.2 that $M \vDash$ "$\mathcal{P}$ is OD". It then follows that $\mathcal{P} \trianglelefteq \mathcal{M}_\infty$. This is a contradiction. $\qquad\square$

We then let $\mathcal{M}^* = \cup\{\mathcal{N} : \mathcal{M}_\infty \trianglelefteq \mathcal{N}, \mathcal{N}$ is good and $\rho_\omega(\mathcal{N}) = \Theta\}$. Notice that we have that $\mathcal{M}^* \subseteq \mathrm{HOD}$. Let $\eta = o(\mathcal{M}^*)$.

Let now $\mu \in V$ be a normal $\mathbb{R}$-complete measure on Θ. It follows from Lemma 0.3 that $V = L(\wp(\mathbb{R}))[\mu]$. Working in HOD_μ, let π_μ be the ultrapower embedding via $\mu \cap \mathrm{HOD}_\mu$. Let $\lambda_\mu = (\Theta^{++})^{\pi_\mu(\mathcal{M}^*)}$. Notice that the ordinal λ_μ may depend on μ. Let then

$$\mathcal{M}_\mu = \pi_\mu(\mathcal{M}^*)|\lambda_\mu$$

and let E_μ be (Θ, η)-extender derived from $\pi_\mu \restriction \mathcal{M}^*$. More precisely,

$$(a, A) \in E_\mu \leftrightarrow a \in (o(\mathcal{M}^*))^{<\omega}, A \in [\Theta]^{|a|} \cap \mathcal{M}^*, \text{ and } a \in \pi_\mu(A)$$

The following is our main theorem.

Theorem 1.4 (Main Theorem) *Assume V is a minimal model of Θms. Suppose μ is a normal $\mathbb{R}$-complete measure on Θ. Then $\mathrm{HOD} = \mathcal{J}(\mathcal{M}_\mu, E_\mu)$.*

We will present the proof as a sequence of lemmas. In the subsections that follows, we assume that V is a minimal model of Θms. However, we remark that

Remark 1.5 *all the results that follow except the results of Subsection 1.6 can be carried out under the assumption that V is just a weakly minimal model of Θms. We ask the reader to keep this remark in mind while reading the subsequent sections.*

Before we go into the proof of the main theorem, we list some of the complications involved with proving it. First we will show that E_μ is amenable to $\mathcal{M}_\mu$ (see Lemma 1.7). It follows from its definition that it coheres $\mathcal{M}_\mu$. It then follows that $(\mathcal{M}_\mu, E_\mu)$ is a hod premouse.

The next challenge is to show that no level of $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$ projects to or below Θ (see Lemma 1.9). A consequence of this is that $V_\Theta^{\mathrm{HOD}^M} = V_\Theta^{\mathcal{J}(\mathcal{M}_\mu, E_\mu)}$. This then allows us to show that $\wp(\mathbb{R})$ can be symmetrically added to $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$ (which is done as part of proving Lemma 1.9). It then follows that the model $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$ is independent of μ. The final piece of the argument is that there is a unique normal $\mathbb{R}$-complete measure μ on Θ minimizing λ_μ. It then immediately follows that $\mathcal{J}(\mathcal{M}_\mu, E_\mu) \subseteq \mathrm{HOD}$. Combining the aforementioned results it is then not hard to see that V is a symmetric extension of $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$, which then easily implies that in fact $\mathrm{HOD} \subseteq \mathcal{J}(\mathcal{M}_\mu, E_\mu)$.

1.1 Amenability

We start by showing that

Lemma 1.6 $\mathcal{M}^* = \pi_\mu(\mathcal{M}_\infty)|\eta$, $\mathcal{M}^* = \mathcal{M}_\mu|(\Theta^+)^{\mathcal{M}_\mu}$ *and whenever* $\mathcal{M}^* \trianglelefteq \mathcal{N} \trianglelefteq \mathcal{M}_\mu$ *is such that* $\rho_\omega(\mathcal{N}) = \eta$, $\mathcal{N}$ *is good.*

Proof. We start by proving the first equality. We have that $\mathcal{M}^* \subseteq \mathrm{HOD}$ and therefore, $\mathcal{M}^* \in \mathrm{HOD}_\mu$. It is then enough to show that whenever $\mathcal{M}_\infty \trianglelefteq \mathcal{N} \trianglelefteq \mathcal{M}^*$ is such that $\rho_\omega(\mathcal{N}) = \Theta$, then $\mathcal{N} \in Ult(\mathrm{HOD}_\mu, \mu \cap \mathrm{HOD}_\mu)$. Since $\mathcal{N}$ is essentially a subset of Θ and $\mu \cap \mathrm{HOD}_\mu$ is a normal Θ-complete ultrafilter in HOD_μ, we indeed have that $\mathcal{N} \in Ult(\mathrm{HOD}_\mu, \mu \cap \mathrm{HOD}_\mu)$.

For the second equality, it is enough to show that whenever $\mathcal{M}_\infty \trianglelefteq \mathcal{N} \trianglelefteq \mathcal{M}_\mu$ is such that $\rho_\omega(\mathcal{N}) = \Theta$ then $\mathcal{N}$ is good. Let then $f : \Theta \to \mathcal{M}_\infty$ be such that $\pi_\mu(f)(\Theta) = \mathcal{N}$. It then follows that for a μ-measure one set of κ, there is an elementary embedding $\pi_\kappa : f(\kappa) \to \mathcal{N}$. It then also follows that whenever $\pi : \bar{N} \to \mathcal{N}$ is an elementary embedding such that $\bar{\mathcal{N}}$ is countable, there is a μ-measure one set of κ such that there is an embedding $\sigma : \bar{\mathcal{N}} \to f(\kappa)$ with the property that $\pi = \pi_\kappa \circ \sigma$. It then follows that $\bar{\mathcal{N}}$ is Σ^π-good.

The proof of the third equality is just like the proof of the second. □

Next, we show that μ is amenable to $\mathcal{M}_\mu$.

Lemma 1.7 μ *is amenable to* $\mathcal{M}_\mu$*, that is* $\mu \cap \mathcal{N} \in \mathcal{M}_\mu$ *whenever* $\mathcal{N} \triangleleft \mathcal{M}^*$.

Proof. This is a standard argument due to Kunen. Fix such an $\mathcal{N}$ projecting to Θ. We have that $\pi_\mu \upharpoonright \mathcal{N} \in \pi_\mu(\mathcal{M})$. The lemma now follows. □

1.2 A strategy for countable submodels of $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$

Suppose ξ is such that E_μ is a total extender in $\mathcal{J}_\xi(\mathcal{M}_\mu, E_\mu)$. Suppose $\pi : \mathcal{Q} \to \mathcal{J}_\xi(\mathcal{M}_\mu, E_\mu)$ is elementary and $\mathcal{Q}$ is countable. In this section, we show that $\mathcal{Q}$ has a π-realizable iteration strategy. Given any such $\mathcal{Q}$, we let $E^{\mathcal{Q}}$ be the preimage of E_μ. We also let $\mathcal{Q}^- = \mathcal{Q}|((\delta^{\mathcal{Q}})^+)^{\mathcal{Q}}$. Recall from [3] that if $\vec{\mathcal{T}}$ is a stack on some model M and R is a node in $\vec{\mathcal{T}}$ then $\vec{\mathcal{T}}_{\geq R}$ is the portion of $\vec{\mathcal{T}}$ after stage R.

Fix then a $\mathcal{Q}$ as above. Given a stack $\vec{\mathcal{T}}$ on $\mathcal{Q}$, it can easily be partitioned into segments by considering when the image of $E^{\mathcal{Q}}$ is used. Thus, we say $(\mathcal{M}_\alpha^0, E_\alpha, \mathcal{M}_\alpha^1, \vec{\mathcal{T}}_\alpha : \alpha < \eta)$ are the essential components of $\vec{\mathcal{T}}$ if

1. $\mathcal{M}_0^0 = \mathcal{Q}$,

2. E_0, $\mathcal{M}_0^1$ are defined if and only if the first extender used in $\vec{\mathcal{T}}$ is $E^{\mathcal{Q}}$ in which case $\mathcal{M}_0^1 = Ult(\mathcal{M}_0^0, E^{\mathcal{Q}})$.

3. $\vec{\mathcal{T}}_0$ is the largest initial segment of $\vec{\mathcal{T}}$ that is based on $(\mathcal{M}_1^1)^-$.

4. If $\alpha + 1 < \eta$ then $\mathcal{M}_{\alpha+1}^0$ is the last model of $\vec{\mathcal{T}}_\alpha$. Then $E_{\alpha+1} = E^{\mathcal{M}_{\alpha+1}^0}$ and $\mathcal{M}_\alpha^1 = Ult(\mathcal{M}_{\alpha+1}^0, E_{\alpha+1})$. Again, $\vec{\mathcal{T}}_{\alpha+1}$ is the largest portion of $\vec{\mathcal{T}}_{\geq \mathcal{M}_{\alpha+1}^1}$ that is based on $\mathcal{M}_{\alpha+1}^1$.

5. If $\alpha < \eta$ is limit then $\mathcal{M}_\alpha^0$ is the direct limit of $(\mathcal{M}_\beta^0 : \beta < \alpha)$ under the iteration embeddings. The rest of the objects are defined as in the successor case.

Suppose now that $\vec{\mathcal{T}}$ is a stack on $\mathcal{Q}$ with essential components $(\mathcal{M}_\alpha^0, E_\alpha, \mathcal{M}_\alpha^1, \vec{\mathcal{T}}_\alpha : \alpha < \eta)$. Suppose that we also have embeddings $(\pi_\alpha^0, \pi_\alpha^1 : \alpha < \eta)$ such that

1. $\pi_0^0 = \pi$.

2. For $\alpha < \eta$ and $i \in 2$, $\pi_\alpha^i : \mathcal{M}_\alpha^i \to \mathcal{J}_\xi[\mathcal{M}, E_\mu]$.

3. For $\alpha < \beta < \eta$ and $i, j \in 2$, $\pi_\alpha^i = \pi_\beta^j \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{M}_\alpha^i, \mathcal{M}_\beta^j}$ and $\pi_\alpha^1 = \pi_\alpha^1 \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{M}_\alpha^0, \mathcal{M}_\alpha^1}$.

4. For $\alpha < \eta$, $\vec{\mathcal{T}}_\alpha$ is according to π_α^1-pullback of Σ.

We then say $\vec{\mathcal{T}}$ is π-*realizable* if there is $\vec{\pi} =_{def} (\pi_\alpha^0, \pi_\alpha^1 : \alpha < \eta)$ witnessing the above clauses.

Suppose then $\vec{\mathcal{T}} = (\mathcal{M}_\alpha^0, E_\alpha, \mathcal{M}_\alpha^1, \vec{\mathcal{T}}_\alpha : \alpha \leq \eta)$ is such that $(\mathcal{M}_\alpha^0, E_\alpha, \mathcal{M}_\alpha^1, \vec{\mathcal{T}}_\alpha : \alpha < \eta)$ is π-realizable as witnessed by $(\pi_\alpha^0, \pi_\alpha^1 : \alpha < \eta)$. We would like to define embeddings (π_η^0, π_η^1) such that

1. For $i \in 2$, $\pi_\eta^i : \mathcal{M}_\eta^0 \to \mathcal{J}_\xi[\mathcal{M}, E_\mu]$.

2. For $\alpha < \eta$ and $i, j \in 2$, $\pi_\alpha^i = \pi_\eta^j \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{M}_\alpha^i, \mathcal{M}_\eta^j}$ and $\pi_\eta^1 = \pi_\eta^1 \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{M}_\eta^0, \mathcal{M}_\eta^1}$.

We then will have that $\vec{\mathcal{T}}$, without its last stack, is π-realizable as witness by $(\pi_\alpha^0, \pi_\alpha^1 : \alpha \leq \eta)$.

Suppose first $\eta = \alpha + 1$. Let $(\mathcal{S}, \Lambda) \in \mathcal{F}$ be such that

$$\sup(\pi_{\alpha+1}^1[\delta^{\mathcal{M}_{\alpha+1}^1}]) = \delta^{\mathcal{M}_\infty(\mathcal{S}, \Lambda)}.$$

The above equality simply says that the direct limit of all Λ-iterates of $\mathcal{S}$ reaches the ordinal mentioned on the left side. Since $\mathcal{M}_{\alpha+1}^1$ is countable, we can also require that

7

$$\pi^1_{\alpha+1}[\mathcal{M}^1_\alpha] \subseteq rng(\pi^\Lambda_{\mathcal{S},\infty}).$$

Let then $k =_{def} (\pi^\Lambda_{\mathcal{S},\infty})^{-1} \circ \pi^1_{\alpha+1} : \mathcal{M}^1_\alpha \to \mathcal{S}$. We then have that $\vec{\mathcal{T}}_\alpha$ is according to k-pullback of Λ. It follows then that letting $\mathcal{R}$ be the last model of $k\vec{\mathcal{T}}_\alpha$ there is an embedding l such that

(1) $l : \mathcal{M}^0_\eta \to \mathcal{R}$ and $\pi^{k\vec{\mathcal{T}}_\alpha} \circ k = l \circ \pi^{\vec{\mathcal{T}}_\alpha}$.

We then set $\pi^0_\eta = \pi^{\Lambda_\mathcal{R}}_{\mathcal{R},\infty} \circ l$. Notice now that

(2) $\pi^\Lambda_{\mathcal{S},\infty} = \pi^{\Lambda_\mathcal{R}}_{\mathcal{R},\infty} \circ \pi^{k\vec{\mathcal{T}}_\alpha}$.

(1), (2) and our choice of $(\mathcal{S}, \Lambda)$ imply that

(3) $\pi^1_\alpha = \pi^0_\eta \circ \pi^{\vec{\mathcal{T}}}_{\mathcal{M}^1_\alpha, \mathcal{M}^0_\eta}$.

(3) then implies that π^0_η is as desired.

To define π^1_η we use countable completeness of μ. First let $\zeta = o((\mathcal{M}^0_\eta)^-)$. Thus, ζ is the successor of $\delta^{\mathcal{M}^0_\eta}$ in $\mathcal{M}^0_\eta$. For each $a \in \zeta^{<\omega}$, let $A_a = \cap\{\pi^0_\eta(A) : (a, A) \in E^{\mathcal{M}^0_\eta}\}$. Fix now a fiber f for the set $\{(\pi^0_\eta(a), A_a) : a \in \zeta^{<\omega}\}$. We can now define

$$\pi^1_\eta([a, g]_{E^{\mathcal{M}^0_\eta}}) = \pi^0_\eta(g)(f(a)).$$

It is a standard argument to show that π^1_η is as desired. It is now easy to show that

Lemma 1.8 *There is an (ω_1, ω_1) iteration strategy Λ for $\mathcal{Q}$ such that whenever $\vec{\mathcal{T}}$ is a stack according to Λ then $\vec{\mathcal{T}}$ is π-realizable.*

1.3 $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$ is a hod premouse

In this section we show that no level of $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$ projects across or to Θ.

Lemma 1.9 $\mathcal{N} = \mathcal{J}(\mathcal{M}_\mu, E_\mu)$ *is a hod premouse such that for every α, $\rho_\omega(\mathcal{N}|\alpha) > \Theta$.*

Proof. We start with the following claim. Later we will show that in fact for every α, $\rho_\omega(\mathcal{J}_\alpha(\mathcal{M}_\mu, E_\mu)) \neq \Theta$.

Claim 1. For every α, $\rho_\omega(\mathcal{J}_\alpha(\mathcal{M}_\mu, E_\mu)) \geq \Theta$

Proof. Suppose not and fix the least α such that $\rho_\omega(\mathcal{J}_\alpha(\mathcal{M}_\mu, E)) < \Theta$. Now fix a β such that $\beta < \lambda^{\mathcal{M}_\mu}$ and $\theta_\beta \leq \rho_\omega(\mathcal{J}_\alpha(\mathcal{M}_\mu, E)) < \theta_{\beta+1}$. Fix a hod pair $(\mathcal{Q}^*, \Lambda^*)$ such that $\mathcal{M}_\infty(\mathcal{Q}^*, \Lambda^*) = \mathcal{M}_\mu(\beta + 2)$ and an elementary hull $\sigma : H \to L_\xi(\mathcal{P}(\mathbb{R}))[\mu]$ such that:

1. $\xi = \alpha^{+17}$,

2. $|H| = \aleph_0$

3. $(\mathcal{Q}^*, \Lambda^*) \in rng(\sigma)$,

4. $\mathcal{J}_\alpha(\mathcal{M}_\mu, E) \in rng(\sigma)$.

Now let $\bar{\mathcal{Q}} = \sigma^{-1}(\mathcal{J}_\alpha(\mathcal{M}_\mu, E)))$ and let $\gamma = \sigma^{-1}(\beta)$. Then by elementarity we have that $\rho_\omega(\bar{\mathcal{Q}}) < \theta^{\bar{\mathcal{Q}}}_{\gamma+1}$. Let

$$\mathcal{Q} = \mathcal{C}^{\bar{\mathcal{Q}}}_\omega(p, \bar{\mathcal{Q}}(\gamma + 1))$$

where p is the standard parameter of $\bar{\mathcal{Q}}$. Notice that $\mathcal{Q}$ is $\delta^{\mathcal{Q}}_{\gamma+1}$-sound. Let $l : \mathcal{Q} \to \bar{\mathcal{Q}}$ be the core embedding. Let Λ be a $\sigma \circ l$-realizable strategy of $\mathcal{Q}$ (see Lemma 1.8).

It follows from the branch condensation of Λ^* that

(1) $\Lambda_{\mathcal{Q}(\gamma+1)} = \Lambda^*_{\mathcal{Q}(\gamma+1)}$.

Consider the pointclass generated by $(\mathcal{Q}, \Lambda)$:

$$\Gamma(\mathcal{Q}, \Lambda) = \{A : \exists(\vec{\mathcal{U}}, \mathcal{R}) \in B(\mathcal{Q}, \Lambda)(A <_W Code(\Lambda_{\mathcal{R}, \vec{\mathcal{u}}}))\}.$$

Since $\mathcal{Q}$ is $\delta^{\mathcal{Q}}_{\gamma+1}$-sound

(2) $\mathcal{Q}$ is $OD_{\Lambda_{\mathcal{Q}(\gamma+1)}}$.

This is because $\mathcal{Q}$ is the unique $\delta^{\mathcal{Q}}_{\gamma+1}$-sound $\Lambda_{\mathcal{Q}(\gamma+1)}$-hod mouse generating the pointclass $\Gamma(\mathcal{Q}, \Lambda)$[1].

Notice now that

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[1]Suppose $(\mathcal{R}, \Phi)$ is another $\Lambda_{\mathcal{Q}(\gamma+1)}$-hod pair such that $\Gamma(\mathcal{Q}, \Lambda) = \Gamma(\mathcal{R}, \Phi)$ and $\mathcal{R}$ is $\delta^{\mathcal{Q}}_{\gamma+1}$-sound. We can compare $(\mathcal{R}, \Phi)$ with $(\mathcal{Q}, \Lambda)$ to a common pair $(\mathcal{S}, \Psi)$ and obtain embeddings $i : \mathcal{R} \to \mathcal{S}$ and $j : \mathcal{Q} \to \mathcal{S}$ such that $i \upharpoonright \delta^{\mathcal{Q}}_{\gamma+1} = j \upharpoonright \delta^{\mathcal{Q}}_{\gamma+1} = id$. Since both $\mathcal{R}$ and $\mathcal{Q}$ are $\delta^{\mathcal{Q}}_{\gamma+1}$-sound, it follows that both i and j are the core embeddings. It then follows that $i = j$ implying that $\mathcal{R} = \mathcal{Q}$. We then get that $\Psi = \Phi^i$ and $\Lambda = \Phi^j$, implying that $\Psi = \Phi$.

(3) $\bar{\mathcal{Q}}(\gamma + 2)$ is $\Lambda_{\mathcal{Q}(\gamma+1)}$-full.

This is simply because $\bar{\mathcal{Q}}(\gamma + 2)$ is a Λ^*-iterate of $\mathcal{Q}^*$ and (as $\mathcal{Q}(\gamma + 1) = \bar{\mathcal{Q}}(\gamma + 1)$)

$$\Lambda_{\mathcal{Q}(\gamma+1)} = \Lambda_{\bar{\mathcal{Q}}(\gamma+1)} = \Lambda^*_{\mathcal{Q}(\gamma+1)}.$$

It now follows from (2) and (3) that we must have $\mathcal{Q} \in \bar{\mathcal{Q}}(\gamma + 2)$. But $\rho_\omega(\mathcal{Q}) < \delta^{\bar{\mathcal{Q}}}_{\gamma+1}$, contradiction! $\qquad\square$

The following claim is easier and finishes the proof of the lemma.

Claim 2. For every α, $\rho_\omega(\mathcal{J}_\alpha(\mathcal{M}_\mu, E))) > \Theta$

Proof. Suppose not. Let α be the least such that $\rho(\mathcal{J}_\alpha(\mathcal{M}_\mu, E))) = \Theta$. Then let $A \subseteq \Theta$ be definable over $\mathcal{J}_\alpha(\mathcal{M}_\mu, E)$ such that $A \notin \mathcal{J}_\alpha(\mathcal{M}_\mu, E)$. Notice that for every $\kappa < \Theta$, $A \cap \kappa \in \mathcal{M}_\mu$.

Let S be the set of $\kappa < \Theta$ such that $\theta_\kappa = \kappa$. Given $\kappa \in S$ we let $\mathcal{M}_\kappa \trianglelefteq \mathcal{M}_\mu$ be the least such that $A \cap \theta_\kappa \in \mathcal{M}_\kappa$.

Let $j : \mathcal{M}_\mu \to Ult(\mathcal{M}_\mu, \mu \cap \mathcal{M}_\mu)$ be the ultrapower embedding. Let $j((\mathcal{M}_\kappa : \kappa \in S)) = (\mathcal{N}_\kappa : \kappa \in j(S))$. But then $A = j(A) \cap \Theta \in \mathcal{N}_\Theta$. Notice that $\mathcal{N}_\Theta$ is good (see Lemma 1.6) implying that $\mathcal{N}_\Theta \trianglelefteq \mathcal{M}_\mu$. It follows that $A \in \mathcal{M}_\mu$, contradiction! $\qquad\square$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

1.4 $\wp(\mathbb{R})$ is symmetrically generic over $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$

We start by recalling Vopenka algebra. We work in $M = L(\wp(\mathbb{R}))$. Let φ be a formula and $s \in \Theta^{<\omega}$. Given $t \in \Theta^{<\omega}$, define

$$\mathcal{A}^s_{\phi,t} = \{a : dom(\vec{a}) = dom(s), a(i) \subseteq s(i)^\omega, L(\wp(\mathbb{R})) \vDash \phi[a, t]\}.$$

We write

$$(\phi, t) \equiv_s (\psi, v) \text{ if and only if } \mathcal{A}^s_{\phi,t} = \mathcal{A}^s_{\psi,v}.$$

We then let $[\phi, t]_s$ be the $\equiv_s$-equivalence class of (ϕ, t).

Next define

$$\mathbb{Q} = \{(s, [\varphi, t]) : s \in \Theta^{<\omega}, \varphi \text{ is a formula }, t \in \Theta^{<\omega}\}$$

The ordering on $\mathbb{Q}$ is defined as follows:

$$(v, [\phi, s]) \leq (u, [\psi, t]) \text{ if and only if } u \lhd v, \mathcal{A}^u_{\psi,t} \subseteq \mathcal{A}^v_{\varphi,s} \text{ and } \mathcal{A}^v_{\varphi,s} \upharpoonright dom(u) \subseteq \mathcal{A}^u_{\psi,t}$$

10

where $\mathcal{A}_{\varphi,s}^v \restriction dom(u) = \{a \restriction dom(u) : a \in \mathcal{A}_{\varphi,s}^v\}$.

Let

$$\mathbb{P} = \{(s,a) : s \in \Theta^{<\omega}, dom(a) = dom(s), \wedge \forall i \leq n, a(i) \subseteq s(i)^\omega\}.$$

We set $(s,a) \leq_{\mathbb{P}} (t,b)$ if and only if $t \trianglelefteq s$ and $b \trianglelefteq a$. Notice that if (f,h) is $\mathbb{P}$-generic then $f : \omega \to \Theta$ and $h : \omega \to (\Theta^\omega)^V$ are surjections such that for each $i \in \omega$, $f(i) \in (h(i))^\omega$.

The following lemma is standard and is due to Vopenka.

Lemma 1.10 (Vopenka's lemma) *Suppose (f,h) is $\mathbb{P}$-generic. Let*

$$G = \{(f \restriction n, [\varphi, v]) : n < \omega, (f \restriction n, [\varphi, v]) \in \mathbb{Q} \wedge h \restriction n \in \mathcal{A}_{\phi,v}^{f \restriction n}\}.$$

Then G is $\mathbb{Q}/\mathcal{M}_\mu$-generic. Therefore, it is $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$-generic over $\mathbb{Q}$.

Proof. We first show that for $\xi < \Theta$, we can bound the ordinal parameters used to define OD^M subsets of $\wp(\xi)$.

Claim 1. In M, there is a function $F : \Theta^{<\omega} \to \Theta$ such that whenever $t \in \Theta^{<\omega}$ and a are such that $dom(a) = dom(t)$, $a(i) \subseteq t(i)$ and a is OD^M then a is OD in $L(\Gamma_{F(t)})$ where $\Gamma_\beta = \{A \subseteq \mathbb{R} : w(A) < \beta\}$.

Proof. For $A \subseteq \mathbb{R}^2$, we say $(\mathbb{R}, A)$ codes an OD structure if $(\mathbb{R}, A)$ is a well-founded, extensional model of some fragment of ZFC and its transitive collapse is OD. Notice that by a standard Skolem hull argument, in M, if $A \subseteq \mathbb{R}^2$ and $(\mathbb{R}, A)$ codes an ordinal definable structure then for some $\beta < \Theta$, $(\mathbb{R}, A)$ codes an ordinal definable structure in $L(\Gamma_\beta)$. Fix now $\alpha < \Theta$. Then we must have that

$$\sup\{\beta : \exists A \subseteq \mathbb{R}^2 (A \in \Gamma_\alpha \wedge L(\Gamma_\beta) \vDash \text{``}(\mathbb{R}, A) \text{ codes an } OD \text{ structure''})\} < \Theta.$$

This is because otherwise we will have a function $G : \Gamma_\alpha \to \Theta$ which is unbounded. As Θ is regular, this is impossible.

Let then

$$F(t) = \sup\{\beta + \omega : \exists A \subseteq \mathbb{R}^2 (A \in \Gamma_{\sup(rng(t))+\omega} \wedge L(\Gamma_\beta) \vDash \text{``}(\mathbb{R}, A) \text{ codes an } OD$$
$$\text{structure''})\}.$$

11

$\square$

Let then $D \in \mathcal{M}_\mu$ be a dense subset of $\mathbb{Q}$. We want to see that $G \cap D \neq \emptyset$. Let E be the set of (s, a) such that $dom(s) = dom(a)$, $s \in \Theta^{<\omega}$, for every $i < dom(s)$, $a(i) \in s(i)^\omega$ and for some $[\phi, v]$ such that $(s, [\phi, v]) \in D$, $a \in \mathcal{A}^s_{\phi,v}$.

We claim that E is dense in $\mathbb{P}$. Fix then $(s, a) \in \mathbb{P}$.

Claim 2.

$$\cup\{\mathcal{A}^t_{\phi,v} \restriction dom(s) : s \trianglelefteq t, (t, [\phi, v]) \in D\} = \{b : dom(b) = dom(s) \text{ and for all}$$
$$i < dom(s), b(i) \in s(i)^\omega\}.$$

Proof. Notice that $D \notin M$, which makes the claim non-trivial. Consider the set $A = \{\mathcal{A}^t_{\phi,v} \restriction dom(s) : (t, [\phi, v]) \in D\}$. Each member of A is OD^M. We want to see that A itself is OD in M.

To see this let $\mathcal{N} \trianglelefteq \mathcal{M}_\mu$ be such that $\rho_\omega(\mathcal{N}) = \Theta$ and $D \in \mathcal{N}$. Fix $\zeta\gamma\Theta$ and let $\pi : H \to L_\zeta(\wp(\mathbb{R}))[\mu]$ be such that

1. $\mathcal{N}, F, D \in rng(\pi)$,

2. $crit(\pi) = \kappa = \theta_\kappa$.

Let $(\bar{\mathcal{N}}, \bar{F}, \bar{D}) = \pi^{-1}(\mathcal{N}, F, D)$. We have that $\bar{F} = F \restriction \theta_\kappa$ and $\bar{D} = \{(t, ([\phi, v])^{L(\wp(\mathbb{R}))^H}) : (t, [\phi, v]) \in D\}$. Let $\bar{A} = \pi^{-1}(A)$. Then $\mathcal{N} \trianglelefteq \mathcal{M}_\infty = V_\Theta^{\mathrm{HOD}^M}$. We then have that $\bar{D} \in OD^M$. Thus $\bar{A}$ is OD^M.

Notice next that $\bar{A} = A$. Indeed, fix $B \in A$. We have that B is OD^M. It follows that B is $OD^{L(\Gamma_{F(s)})}$ and hence, B is $OD^{L(\Gamma_{F(s)})}$. Let then (ϕ, v) be such that $v \in (\Theta^{L(\Gamma_{F(s)})})^{<\omega}$ and B is definable over $L(\Gamma_{F(s)})$ via (ϕ, v). Since $\bar{D}$ is dense in H, we must have $(t, [\psi, u]) \leq_\mathbb{Q} (s, [\phi, v])$ such that $(t, [\psi, u]) \in \bar{D}$. But then $B = \mathcal{A}^t_{\psi,u} \in \bar{A}$.

It now follows that $A \in OD^M$ implying the claim. $\square$

It follows from Claim 2 that there is $(t, [\phi, v]) \in D$ such that $a \in \mathcal{A}^t_{\phi,v} \restriction dom(s)$. Let then $b \in \mathcal{A}^t_{\phi,v} \restriction dom(s)$ be such that $b \restriction dom(a) = a$. Then $(t, b) \in E$ and $(t, b) \leq_\mathbb{P} (s, a)$.

Since we now have that E is dense, we can fix $n < \omega$ such that $(f \restriction n, h \restriction n) \in E$. Let $[\phi, v]$ be such that $h \restriction n \in \mathcal{A}^{f\restriction n}_{\phi,v}$. We then have that $(f \restriction n, [\phi, v]) \in G \cap D$. We leave it to the reader to verify that G is a filter. $\square$

The following is the main lemma of this section.

Lemma 1.11 V *is a symmetric extension of* $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$*. In fact,* $V = \mathcal{J}(\mathcal{M}_\mu, E_\mu)(\Theta^\omega)$*.*

Proof. Let (f, h) be $\mathbb{P}$-generic. Let G be as in Lemma 1.10. Then $\Theta^\omega \in \mathcal{J}(\mathcal{M}_\mu, E_\mu)[G]$. It is a consequence of AD^+ that every set of reals A is $(OD_t)^M$ for some $t \in \Theta^\omega$. Therefore, we have that

$$\wp(\mathbb{R}) \subseteq \text{HOD}^M(\Theta^\omega) \subseteq \mathcal{J}(\mathcal{M}_\mu, E_\mu)(\Theta^\omega).$$

The reader can find more on the above equality by consulting Section 2 of [1]. It then follows that $\mathcal{J}(\mathcal{M}_\mu, E_\mu)(\wp(\mathbb{R})) = \mathcal{J}(\mathcal{M}_\mu, E_\mu)(\Theta^\omega)$.

Because every set $A \in \wp(\Theta) \cap \mathcal{J}(\mathcal{M}_\mu, E_\mu)(\Theta^\omega)$ is added to $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$ by a small forcing (in fact by the Vopenka algebra at some $\theta_\alpha < \Theta$), we have that E_μ has a canonical extension E_μ^+ to $\mathcal{J}(\mathcal{M}_\mu, E_\mu)(\Theta^\omega)$ (see Theorem 2.4 of [1]). Let $\nu = \{A : (\Theta, A) \in E_\mu^+\}$. We then have that

$$\mathcal{J}(\mathcal{M}_\mu, E_\mu)(\Theta^\omega) \vDash \text{"}\nu \text{ is a normal } \mathbb{R}\text{-complete measure on } \Theta\text{"}.$$

It then follows that $V = \mathcal{J}(\mathcal{M}_\mu, E_\mu)(\Theta^\omega)^2$. $\qquad\square$

1.5 Computation of HOD

We can now easily conclude that the model $\mathcal{J}(\mathcal{M}_\mu, E_\mu)$ is independent of μ. The following corollary is a simple consequence of Lemma 1.11 and the fact that small forcing doesn't create new measures.

Corollary 1.12 *Suppose* ν *is a normal* $\mathbb{R}$*-complete measure on* Θ*. Then* $E_\nu \in \mathcal{J}(\mathcal{M}_\mu, E_\mu) \in \mathcal{J}(\mathcal{M}_\mu, E_\mu)$*.*

Corollary 1.13 *Suppose* μ *and* ν *are two normal* $\mathbb{R}$*-complete measures on* Θ*. Then* $\mathcal{J}(\mathcal{M}_\mu, E_\mu) = \mathcal{J}(\mathcal{M}_\mu, E_\nu)$*.*

Proof. We have that $E_\nu \in \mathcal{J}(\mathcal{M}_\mu, E_\mu)$ and $E_\mu \in \mathcal{J}(\mathcal{M}_\nu, E_\nu)$, implying the conclusion. $\qquad\square$

Borrowing Lemma 1.14 from the next subsection, there is a unique normal $\mathbb{R}$-complete measure μ over Θ minimizing λ_μ. It then follows from Corollary 1.13 that $\mathcal{J}(\mathcal{M}_\mu, E_\mu) \subseteq \text{HOD}$. It follows from Lemma 1.11 that $\text{HOD} \subseteq \mathcal{J}(\mathcal{M}_\mu, E_\mu)$. Putting these two results together, we can now show that

$$\text{HOD} = \mathcal{J}(\mathcal{M}_\mu, E_\mu).$$

---

[2]In fact, $\nu = \mu$ but we do not need this.

1.6 The uniqueness of minimal measures

In this section, we show that there is a unique measure minimizing λ_μ.

Lemma 1.14 *Suppose μ and ν are two normal $\mathbb{R}$-complete measures over Θ such that $\lambda_\mu = \lambda_\nu$. Then $\mu = \nu$.*

We spend most of this subsection proving Lemma 1.14. Suppose that μ and ν are as in the hypothesis of the theorem. Notice that it is enough to show that $E_\mu = E_\nu$. This is because given $E_\mu = E_\nu$, Lemma 1.11 will imply that $\mu = \nu$.

Notice first that $\mathcal{M}_\mu = \mathcal{M}_\nu$. Let then $\mathcal{M} =_{def} \mathcal{M}_\mu$ and $\mathcal{N} =_{def} \mathcal{J}(\mathcal{M}, E_\mu, E_\nu)$. Because $\lambda_\mu = \lambda_\nu$, we have that $\mathcal{N}$ is a bicaphulus.

Let now $\xi > \lambda_\mu$ be such that $\mathcal{N}|\xi \vDash ZFC - Powesret +$ "there exists a largest cardinal" and let $\pi : \mathcal{Q} \to \mathcal{N}|\xi$ be a countable hull of $\mathcal{N}|\xi$. Using the construction introduced in Subsection 1.2, we can build an iteration strategy Λ for $\mathcal{Q}$ which is π-realizable. Let $(E, F) = \pi^{-1}(E_\mu, E_\nu)$. The idea now is simple. If we succeed comparing $(\mathcal{Q}, \Lambda)$ with itself then we will get a contradiction as it will show that $E = F$.

There are several issues with the above idea. The problem is that we do not a priori know that Λ is fullness preserving. It need not be fullness preserving, Γ-fullness preserving would suffice. However, it is not clear how to define Γ. A similar issue arises in [3] where one needs to compare two hod pairs whose strategies are not fullness preserving. There the issue is taken care of as follows.

Suppose $(\mathcal{P}, \Phi)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is a limit ordinal. First recall from [3] that

$$I(\mathcal{P}, \Phi) = \{(\vec{\mathcal{T}}, \mathcal{R}) : \vec{\mathcal{T}} \text{ is a stack on } \mathcal{P} \text{ according to } \Phi \text{ such that } \pi^{\vec{\mathcal{T}}} \text{ exists and } \mathcal{R} \text{ is the last model of } \vec{\mathcal{T}}\}$$
$$B(\mathcal{P}, \Phi) = \{(\vec{\mathcal{T}}, \mathcal{S}) : \exists (\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Phi)(\mathcal{S} \lhd_{hod} \mathcal{R})\}.$$

Also recall that $\mathcal{S} \lhd_{hod} \mathcal{R}$ means that $\mathcal{S}$ is a hod mouse initial segment of $\mathcal{R}$, i.e., there is $\alpha < \lambda^{\mathcal{R}}$ such that $\mathcal{S} = \mathcal{R}(\alpha)$. Following Section 3.2 of [3], we let

$$\Gamma(\mathcal{P}, \Phi) = \{A \subseteq \mathbb{R} : \exists (\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Phi)(A \leq_W Code(\Phi))\}.$$

In the above formula, $\leq_W$ denotes Borel reducibility and $Code(\Phi)$ is the set of reals coding Φ. It is shown in [3] (see Theorem 3.27 of [3]) that under some general conditions, Φ is $\Gamma(\mathcal{P}, \Phi)$ fullness preserving. The conditions needed for this fact are as follows.

1. Φ has hull condensation.

2. For all $(\vec{\mathcal{T}}, \mathcal{R}) \in B(\mathcal{P}, \Phi)$ there is a hod pair $(\mathcal{S}, \Psi)$ an an elementary embedding $\sigma : \mathcal{R} \to \mathcal{S}$ such that $\Phi_{\mathcal{R}, \vec{\mathcal{T}}} = (\sigma$ pullback of $\Psi)$ and Ψ has branch condensation and is fullness preserving.

Investigating the proof of Theorem 3.27 of [3], it is not hard to notice that the hull condensation of Φ is needed to infer that

(1) whenever $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{P}, \Phi)$, $\Gamma(\mathcal{P}, \Phi) = \Gamma(\mathcal{R}, \Phi_{\mathcal{R}, \vec{\mathcal{T}}})$.

Clearly (1) is an easy consequence, modulo copy constructions, of hull condensation of Φ.

We now get back to our case and continue with the pair $(\mathcal{Q}, \Lambda)$. The first few definitions generalize immediately. Thus we let

$$I(\mathcal{Q}, \Lambda) = \{(\vec{\mathcal{T}}, \mathcal{R}) : \vec{\mathcal{T}} \text{ is a stack on } \mathcal{Q} \text{ according to } \Lambda \text{ such that } \pi^{\vec{\mathcal{T}}} \text{ exists and } \mathcal{R} \text{ is}$$
$$\text{the last model of } \vec{\mathcal{T}}\}$$
$$B(\mathcal{Q}, \Lambda) = \{(\vec{\mathcal{T}}, \mathcal{S}) : \exists (\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{Q}, \Lambda)(\mathcal{S} \triangleleft_{hod} \mathcal{R})\}$$
$$\Gamma(\mathcal{Q}, \Lambda) = \{A \subseteq \mathbb{R} : \exists (\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{Q}, \Lambda)(A \leq_W Code(\Lambda))\}.$$

Condition 2 above is also easily seen to be satisfied. This is because we have defined Λ to be a π-realizable iteration strategy. Given then a $(\vec{\mathcal{T}}, \mathcal{R}) \in B(\mathcal{Q}, \Lambda)$, letting $\mathcal{R}^+$ be the last model of $\vec{\mathcal{T}}$, we have an embedding $\sigma : \mathcal{R}^+ \to \mathcal{N}|\xi$ such that $\Lambda_{\mathcal{R}, \vec{\mathcal{T}}}$ is the σ-pullback of $\Sigma_{\mathcal{M}(\alpha)}$ where α is such that $\sigma(\mathcal{R}) = \mathcal{M}(\alpha)$. We can then find a hod pair $(\mathcal{S}, \Phi)$ such that $\mathcal{M}_\infty(\mathcal{S}, \Phi) = \mathcal{M}(\alpha)$ and an embedding $k : \mathcal{R} \to \mathcal{S}$ such that $\Lambda_{\mathcal{R}, \vec{\mathcal{T}}} = (k$-pullback of $\Phi)$.

The only issue is that Λ may not have hull condensation. Here is how to go around this problem. Suppose $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{Q}, \Lambda)$. Let then

$$\Gamma_{\vec{\mathcal{T}}} = \Gamma(\mathcal{R}, \Lambda_{\mathcal{R}, \vec{\mathcal{T}}}).$$

Notice that it follows that if $(\vec{\mathcal{T}}_0, \mathcal{R}_0) \in I(\mathcal{Q}, \Lambda)$ and $(\vec{\mathcal{T}}_1, \mathcal{R}_1) \in I(\mathcal{R}_0, \Lambda_{\mathcal{R}_0, \vec{\mathcal{T}}_0})$ then

$$\Gamma_{\vec{\mathcal{T}}_0 ^\frown \vec{\mathcal{T}}_1} \leq_W \Gamma_{\vec{\mathcal{T}}_0}$$

Let then $(\vec{\mathcal{T}}, \mathcal{R}) \in I(\mathcal{Q}, \Lambda)$ be such that $\Gamma_{\vec{\mathcal{T}}}$ is $\leq_W$-minimal. Thus,

(2) whenever $(\vec{\mathcal{U}}, \mathcal{S}) \in I(\mathcal{R}, \Lambda_{\mathcal{R}, \vec{\mathcal{T}}})$, $\Gamma_{\vec{\mathcal{T}}} = \Gamma_{\vec{\mathcal{T}} ^\frown \vec{\mathcal{U}}}$

(2) now plays the role of (1). We can now repeat the proof of Theorem 3.27 and show that $\Lambda_{\mathcal{R}, \vec{\mathcal{T}}}$ is $\Gamma_{\vec{\mathcal{T}}}$-fullness preserving. We can then compare $(\mathcal{R}, \Lambda_{\mathcal{R}, \vec{\mathcal{T}}})$ with itself, a fact that implies $\pi^{\vec{\mathcal{T}}}(E) = \pi^{\vec{\mathcal{T}}}(F)$. Thus, $E = F$.

15

2 The existence of minimal models and some remarks

We left open the question whether there are minimal models of Θms. Here we show that they indeed exist. As was seen in Lemma 0.5, there are weakly minimal models (assuming the existence of models of Θms).

Theorem 2.1 *Suppose V is a weakly minimal model of Θms. Then there is a minimal model of Θms.*

Proof. First recall Remark 1.5. Let μ be a normal $\mathbb{R}$-complete measure over Θ. We then have that $V = \mathcal{J}(\mathcal{M}_\mu, E_\mu)(\Theta^\omega)$. Suppose $W \subsetneq V$ is another weakly minimal model of Θms such that $W \subseteq V$. It follows that $\wp(\mathbb{R})^W = \wp(\mathbb{R})^V$. Working in W, let ν be a normal $\mathbb{R}$-complete measure over Θ. We then have that $W = \mathcal{J}(\mathcal{M}_\nu^W, E_\nu^W)(\Theta^\omega)$.

Notice now that $\mathcal{M}_\nu^W \trianglelefteq \mathcal{M}_\mu$. Suppose first that $\mathcal{M}_\nu^W = \mathcal{M}_\mu$. It now follows from Lemma 1.11 that $E_\nu^W \in \mathcal{J}(\mathcal{M}_\mu, E_\mu)$ and that $V \vDash$ "ν is a normal $\mathbb{R}$-complete measure over Θ". It then follows from Lemma 1.13 that $\mathcal{J}(\mathcal{M}_\mu, E_\mu) = \mathcal{J}(\mathcal{M}_\nu^W, E_\nu^W)$ implying that $V = W$.

It follows that we must have that $\mathcal{M}_\nu^W \triangleleft \mathcal{M}_\mu$. The above discussion then shows that if $W \subseteq V$ is a weakly minimal model of Θms minimizing the height of all possible $\mathcal{M}_\nu^W$ then W itself is a minimal model of Θms. $\qquad\square$

We finish by asking the following question. The question is whether one can compute the HOD of the minimal model of $AD_\mathbb{R} +$ "Θ is a strong cardinal". First assume AD^+. We say that Θ is a strong cardinal if for every $\lambda \geq \Theta$ there is a set $E = \{(a, A) : a \in \lambda^{<\omega} \wedge A \in [\kappa]^{|a|}\}$ such that

1. for every $a \in \lambda^{<\omega}$, E_a is an $\mathbb{R}$-complete ultrafilter over $\kappa^{|a|}$,

2. for $a \subseteq b \in \lambda^{<\omega}$, letting $\pi_{b,a}$ be the projection map[3], for any $A \in \kappa^{|b|}$, $A \in E_b \leftrightarrow \pi_{b,a}[A] \in E_a$, and

3. for any $((a_i, A_i) : i \in \omega) \subseteq E$ there is $f : \cup_{i<\omega} a_i \to \kappa$ such that for every i, $f[a_i] \in A_i$, and

4. for some $a \in [\lambda]^{<\omega}$, there is a function $f \in HOD_{E,\wp(\mathbb{R})}$ such that $f : \Theta^{|a|} \to L_\Theta(\wp(\mathbb{R}))$ and $[a, f]_E = V_\lambda^{HOD_{E,\wp(\mathbb{R})}}$.

---

[3]Recall that $\pi_{b,a}(t) = (t_{i_0}, t_{i_1}, ..., t_{i_{|a|-1}})$ where letting $b = (b_0, b_1, ..., b_{|b|-1})$, $(i_0, ..., i_{|a|-1})$ are chosen in such a way that $a = (b_{i_0}, b_{i_1}, ..., b_{i_{|a|-1}})$.

We then let "$\Theta strong$" stand for the theory $AD_{\mathbb{R}} +$ "Θ is a strong cardinal" and say that M is *a minimal model of* $\Theta strong$ if M is a transitive model of ZF such that $Ord \cup \mathbb{R} \subseteq M$, $M \vDash \Theta strong$ and for every transitive $N \subsetneq M$ such that $Ord \cup \mathbb{R} \subseteq N$, $N \vDash \neg \Theta strong$. We conjecture that minimal models of $\Theta strong$ exist.

Conjecture 2.2 *Assume AD^+ and that there is a largest Suslin cardinal which is a member of the Solovay sequence. Then there is a minimal model of $\Theta strong$.*

The hypothesis of the conjecture is known as LSA. The reader can find more about it by consulting [2]. There it is shown that the hypothesis is consistent relative to a Woodin cardinal which itself is a limit of Woodin cardinals (see Theorem 13.1 of [2]).

Question 2.3 *Suppose M is a minimal model of $\Theta strong$. What is the fine structural form of* HOD^M?

One may also consider models of $AD_{\mathbb{R}}$ in which Θ is a strong cardinal and the model itself has a distinguished extender sequence above Θ. Such models will perhaps have the form $\mathcal{M} = \mathcal{J}^{\vec{E}}(\wp(\mathbb{R}))$ where $\vec{E}$ is a good extender sequence. We may allow that some of the extenders in $\mathcal{M}$ have critical point Θ. If in addition we require that $\mathcal{M}$ is *minimal* then it should be possible to study the fine structural properties of $\mathrm{HOD}^{\mathcal{M}}$. However, one has to be a bit more careful. While the authors haven't done much towards completing the project mentioned above, at the moment it is hard to perceive a proof from large cardinals that there is an $\mathcal{M}$ as above in which Θ is a strong cardinal and $\mathcal{M} \vDash$ "there is a Woodin cardinal which is a limit of Woodin cardinals". It seems that one would need to wait until the theory of hod mice evolves into that region. Nevertheless, it seems that studying models of LSA will lead towards confirmation of the following conjecture.

Conjecture 2.4 *Assume LSA. Then there is a model of $AD_{\mathbb{R}}$ of the form $\mathcal{M} = \mathcal{J}^{\vec{E}}(\wp(\mathbb{R}))$ in which Θ is a strong cardinal and $\mathcal{M} \vDash$ "there is a Woodin cardinal".*

Another question that authors find interesting is weather there are connections between $\Theta strong$ and super compactness of ω_1. In [7], Nam Trang has shown that Θms is equiconsistent with $AD_{\mathbb{R}} +$ "Θ is regular"$+$"ω_1 is Θ supercompact". It can then be asked weather $\Theta strong$ is equiconsistent with any theory stipulating that ω_1 has high degree of supercompactness.

Question 2.5 *Assume $\Theta strong$. Is there a model of $AD_{\mathbb{R}} +$ "Θ is regular" $+$"ω_1 is Θ^+-supercompact"?*

We suspect that the theory $AD_\mathbb{R} +$ "Θ is regular" $+$"ω_1 is Θ^+-supercompact" has a high consistency strength. A natural attempt to produce a model of $AD_\mathbb{R} +$ "Θ is regular" $+$"ω_1 is Θ^+-supercompact" is as follows. Start with a model of AD^+ and suppose α is such that $\theta_\alpha < \Theta$. Let $\lambda = (\theta_\alpha^+)^{\mathrm{HOD}}$ and let $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. Let μ be the ω_1-supercompactness measure on λ. By a result of Woodin, it is unique. Consider then the model $L(\lambda^\omega, \Gamma, \mu)$. The only problem is that $\wp(\mathbb{R}) \cap L(\lambda^\omega, \Gamma, \mu)$ maybe bigger than Γ. In fact, it is an unpublished theorem of the second author (but see [2]) that in the minimal model of LSA, this intersection is bigger than Γ.

Theorem 2.6 *Assume AD^+ and that for some α, $\theta_\alpha < \Theta$. Let $\lambda = (\theta_\alpha^+)^{\mathrm{HOD}}$ and $\Gamma = \{A \subseteq \mathbb{R} : w(A) < \theta_\alpha\}$. Then if $L(\lambda^\omega, \Gamma) \cap \wp(\mathbb{R}) = \Gamma$ then there is $B \subseteq \mathbb{R}$ such that $L(B, \mathbb{R}) \vDash LSA$.*

However, Woodin, in unpublished work, showed that the theory $AD_\mathbb{R} +$ "ω_1 is supercomact" is consistent relative to the theory $ZFC +$ " there is a proper class of Woodin cardinals that are limit of Woodin cardinals". Based on intuitions coming from the theory of hod mice in the region of LSA we conjecture the following.

Conjecture 2.7 *The following theories are equiconsistent.*

1. *$ZF + AD_\mathbb{R} +$ "Θ is regular"$+$"ω_1 is supercompact".*

2. *$ZFC +$ "there is a Woodin cardinal which is a limit of Woodin cardinals"*

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