A classification of projective-like hierarchies in $L(\mathbb{R})$

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Abstract

We provide a characterization of projective-like hierarchies in $L(\mathbb{R})$ in the context of AD by combinatorial properties of their associated Wadge ordinal. In the second chapter we concentrate on the type **III** case of the analysis. In the last chapter, we provide a characterization of type **IV** projective-like hierarchies by their associated Wadge ordinal.

1 Introduction

1.1 Outline

The paper is roughly organized as follows. We first review basic definitions of the abstract theory of pointclasses. In the second section we study the type **III** pointclasses and the closure properties of the Steel pointclass generated by a projective algebra $\mathbf{\Lambda}$ in the case where $\mathrm{cof}(o(\mathbf{\Lambda})) > \omega$. In particular we show that Steel's conjecture is false. This leads to isolating a combinatorial property of the Wadge ordinal of the projective algebra generating the

Steel pointclass. This combinatorial property ensures closure of the Steel pointclass under disjunctions. Therefore it is still possible to have κ regular, where κ is the Wadge ordinal for a projective algebra $\mathbf{\Lambda}$ generating the next Steel pointclass and yet the Steel pointclass is not closed under disjunction. In the third section, we look at the type **IV** projective-like hierarchies, that is the pointclass which starts the new hierarchy is closed under quantifiers.

1.2 Review of the abstract theory of pointclasses

We will work in the theory ZF+DC+AD. In some places we may use $\mathrm{AD}^{L(\mathbb{R})}$ so one could think of the work as taking place under ZF+DC+AD^+.

Although we use $\mathbb{R}$ for the set of reals in the paper, it is standard to identify the set of reals $\mathbb{R}$ with the Baire space ω^ω (this can be done by using continued fractions to show that the set of irrational numbers is homeomorphic with ω^ω for example). So whenever we use $\mathbb{R}$, we actually really mean ω^ω. The advantage of this shift is that ω^ω is now homeomorphic with $(\omega^\omega)^2$. Reals simply become ω sequences in ω, instead of Dedekind cuts, which are very complicated objects in themselves.

Any sequence $(x_i : i \leq n)$ with $x_i \in \mathbb{R}$ for every $i \leq n$ can be coded into a single real via a recursive bijection

$$(x_1, ..., x_n) \mapsto \langle x_1, ..., x_n \rangle$$

We will also let $x \mapsto ((x)_0, ..., (x)_n)$ denote the decoding map. We'll often drop the parenthesis and just write x_i instead of $(x)_i$. It is also true that countably many reals can be coded into a single reals and the coding real will be denoted by $\langle x_n \rangle$.

A tree T on a set X is a set of finite sequences $(x_1, ..., x_j)$ from X closed under initial segments, that is,

$$\text{whenever } (x_1, ..., x_j) \in T, (x_1, ..., x_i) \in T, \text{ for any } i \leq j$$

Letting $s = (x_1, ..., x_j)$, it is standard to denote the length of s by $lh(s)$. For $s, t \in T$, we say that t extends s, denoted by $s \lhd t$ if $lh(s) \leq lh(t)$ and $t \restriction lh(s) = s$. A branch through the tree T is an infinite sequence $f = (x_0, x_1, ...)$ such that for every n, $f \restriction n \in T$. If the tree T has a branch then it is said to be illfounded, otherwise it is wellfounded. The set of all branches of a tree T is called the body of T and is denoted by $[T]$. All trees in the paper will be in the descriptive set theoretic sense outlined in this paragraph, that is they will have height ω.

We introduce basic notions of the theory of pointclasses which we need throughout. A pointclass $\mathbf{\Gamma}$ is a collection of sets of reals closed under continuous inverse images, that is:

$$\text{if } f : \mathbb{R} \to \mathbb{R} \text{ is continuous and } A \subseteq \mathbb{R} \text{ is } \in \mathbf{\Gamma} \text{ then } B = f^{-1}[A] \in \mathbf{\Gamma}$$

For example $\mathbf{\Sigma}^0_1$ and $\mathbf{\Sigma}^2_1$ are two examples of pointclasses. Subscripts denote the numbers of quantifiers involved in the syntactic formula defining the set belonging to the pointclass and superscripts denotes the type of objects which fall on the scope of the quantification.

Wadge reduction is a central concept in descriptive set theory. Wadge reduction provides a measure of the complexity of sets of reals. For two sets $A, B \subseteq \mathbb{R}$, we say A is Wadge reducible to B and write $A \leq_W B$ if and only if there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $B = f^{-1}[A]$, i.e computing membership in A should be no more complicated than computing membership in B. In other words, $A \leq_W B$ if and only if there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for all x,

$$x \in A \leftrightarrow f(x) \in B.$$

So a pointclass $\mathbf{\Gamma} \subseteq \mathcal{P}(\mathbb{R})$ is a collection of sets of reals closed under Wadge reduction. One basic consequence of AD is Wadge's Lemma with says that any two sets of reals can be compared simply by the continuous substitution and taking complements. In particular

$$A \leq_w B \leftrightarrow A = f^{-1}[B].$$

It is a very useful fact in descriptive set theory that the relation $\leq_W$ is wellfounded, and this is due to Martin and Monk. Given a pointclass $\mathbf{\Gamma}$, we have the dual pointclass

$$\check{\mathbf{\Gamma}} = \{A : A^c \in \mathbf{\Gamma}\}.$$

If $\mathbf{\Gamma}$ is a pointclass, we say $U \subseteq \mathbb{R}^2$ is a universal set for $\mathbf{\Gamma}$ if and only if for every $B \in \mathbf{\Gamma}$, there is a $y \in \mathbb{R}$ such that $U_y = B = \{x : (y, x) \in U\}$.

A pointclass is non-selfdual if and only if it is not closed under complements and a pointclass is called selfdual if it is closed under complements. Under AD, Wadge's lemma implies that every nonselfdual pointclass has a universal set. Selfdual pointclasses do not have universal sets by a diagonal argument. It is standard to denote selfdual pointclasses by $\mathbf{\Delta}$ and we'll write

$$\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$$

The closure of $\mathbf{\Gamma}$ under existential quantification is given by

$$\exists^{\mathbb{R}}\mathbf{\Gamma} = \{A : \exists B \in \mathbf{\Gamma} \forall x(A(x) \leftrightarrow \exists y B(x, y)\}$$

Notice that this is the same as taking continuous images by continuous functions $f : \mathbb{R} \to \mathbb{R}$. For instance, considering $\mathbf{\Pi}^0_1$ the pointclass of closed sets then one has $\exists^{\mathbb{R}}\mathbf{\Pi}^0_1 = \mathbf{\Sigma}^1_1$, namely a continuous image of a closed set is an analytic set. One can also define $\forall^{\mathbb{R}}\mathbf{\Gamma}$, which is just $\exists^{\mathbb{R}}\check{\mathbf{\Gamma}}$. The projective hierarchy is defined in analogous fashion: $\mathbf{\Sigma}^1_{n+1} = \exists \mathbf{\Pi}^1_n$ and $\mathbf{\Pi}^1_n = \neg\mathbf{\Sigma}^1_n$. Another way to generate to the projective hierarchy is to look at $J(\mathbb{R})$, the Jensen constructible universe containing all the reals and ordinals. We have that $\mathbf{\Sigma}_1(J_1(\mathbb{R})) = \mathbf{\Sigma}^1_1$ and so $\mathbf{\Pi}_1(J_1(\mathbb{R})) = \mathbf{\Pi}^1_1$. Similarly, $\mathbf{\Sigma}_2(J_1(\mathbb{R})) = \mathbf{\Sigma}^1_2$, $\mathbf{\Sigma}_3(J_1(\mathbb{R})) = \mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}_n(J_1(\mathbb{R})) = \mathbf{\Pi}^1_n$, etc... So the projective hierarchy is entirely contained in $J_2(\mathbb{R})$. At the higher up levels, the pointclass of the inductive sets is given by $\mathbf{\Sigma}_1(J_{\kappa^{\mathbb{R}}}(\mathbb{R}))$, where $\kappa^{\mathbb{R}}$ is the least $\mathbb{R}$-admissible ordinal. Also $\mathbf{\Sigma}_1^{L(\mathbb{R})} = \mathbf{\Sigma}^2_1 = \mathbf{\Sigma}_1(J_{\boldsymbol{\delta}^2_1}(\mathbb{R}))$, where $\boldsymbol{\delta}^2_1$ is the least stable cardinal of $L(\mathbb{R})$. The least stable ordinal [1] in $L(\mathbb{R})$ is the least ordinal δ for which we have

$$L_\delta(\mathbb{R}) \preceq^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R})$$

[1]see [7] for a proof of this fact

Definition 1.1 (Levy pointclass) *A Levy pointclass* $\mathbf{\Gamma}$ *is a nonselfdual pointclass which is closed under either* $\exists^{\mathbb{R}}$ *or* $\forall^{\mathbb{R}}$ *or possibly under both.*

Definition 1.2 $\mathbf{\Gamma}$ *has the* ***reduction property*** *if for all* $A, B \in \mathbf{\Gamma}$ *there are* $A', B' \in \mathbf{\Gamma}$ *such that* $A' \subseteq A, B' \subseteq B, A' \cap B' = \emptyset, A' \cup B' = A \cup B$. $\mathbf{\Gamma}$ *has the* ***separation property*** *if for every* $A, B \in \mathbf{\Gamma}$ *such that* $A \cap B = \emptyset$ *there exists a set* $C \in \mathbf{\Delta}$ *such that* $A \subseteq C$ *and* $C \cap B = \emptyset$.

One of the central properties a pointclass can have is the **prewellordering property**: $\mathbf{\Gamma}$ has the prewellordering property if every $\mathbf{\Gamma}$ set admits a $\mathbf{\Gamma}$ norm, where a norm on a set of reals A is a map ϕ such that $\phi : A \to ORD$. The norm is regular if it is into an ordinal κ.

Definition 1.3 *A norm* ϕ *is called a* $\mathbf{\Gamma}$ *norm if the following norm relations are in* $\mathbf{\Gamma}$*:* $\leq^*_\phi, <^*_\phi$ *with:*

$$x \leq^*_\phi y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \phi(x) \leq \phi(y)))$$
$$x <^*_\phi y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \phi(x) < \phi(y)))$$

Notice that the prewellordering property is a way of splitting our $\mathbf{\Gamma}$ set A into $\mathbf{\Delta}$ pieces. Θ is the supremum of the length of the prewellorderings of $\mathbb{R}$, that is:

$$\Theta = \sup\{\alpha : \exists f : \mathbb{R} \twoheadrightarrow \alpha\}.$$

Under AC, Θ is $\mathfrak{c}^+$ but under determinacy Θ can exhibit large cardinal properties. From the point of view of $L(\mathbb{R})$ and assuming determinacy, Θ is already large but it turn out by a result of Woodin that Θ is a Woodin cardinal in the HOD of $L(\mathbb{R})$.

Recall that under ZF, we have the following:

1. if $\mathbf{\Gamma}$ is closed under $\vee$, pwo($\mathbf{\Gamma}$) $\longrightarrow$ Red($\mathbf{\Gamma}$)
2. Red($\mathbf{\Gamma}$) $\longrightarrow$ Sep($\check{\mathbf{\Gamma}}$)
3. if $\mathbf{\Gamma}$ has a universal set then Red($\mathbf{\Gamma}$) $\longrightarrow$ $\neg$Sep($\mathbf{\Gamma}$).
4. (Steel, Van Wesep) Under ZF+AD, if Sep($\check{\mathbf{\Gamma}}$) and for any $A, B \in \mathbf{\Delta}, A \cap B \in \mathbf{\Gamma}$ then Red($\mathbf{\Gamma}$).

It is a classical fact of descriptive set theory that under ZF+AD for any Levy pointclass $\mathbf{\Gamma}$, either pwo($\mathbf{\Gamma}$) or pwo($\check{\mathbf{\Gamma}}$). Under ZF only, if $\mathbf{\Gamma}$ is a pointclass with pwo($\mathbf{\Gamma}$) then every set in $\exists^{\mathbb{R}}\mathbf{\Gamma}$ admits a $\forall^{\mathbb{R}}\exists^{\mathbb{R}}\mathbf{\Gamma}$ norm. What gets us going through the Wadge hierarchy is the *first periodicity theorem*:

Theorem 1 (Moschovakis) *Suppose that* $\mathbf{\Delta}$*-determinacy holds and that* $\mathbf{\Gamma}$ *is a nonselfdual pointclass with pwo(*$\mathbf{\Gamma}$*) then every set in* $\forall^{\mathbb{R}}\mathbf{\Gamma}$ *admits a* $\exists^{\mathbb{R}}\forall^{\mathbb{R}}\mathbf{\Gamma}$ *norm.*

Definition 1.4 (The scale property) *A semiscale is a sequence of norms $\langle \phi_n \rangle$ on a set A such that whenever we have a sequence $\{x_n\} \subseteq A$ converging to some x and for every $n, \phi_n(x_i)$ is eventually constant then $x \in A$. If in addition we have the lower semi-continuity property, $\phi_n(x) \leq \lim \phi_n(x_i)$ then the sequence of norms $\langle \phi_n \rangle$ is a scale. A scale $\langle \phi_n \rangle$ is a $\mathbf{\Gamma}$-scale if for every n, ϕ_n is a $\mathbf{\Gamma}$-norm. The pointclass $\mathbf{\Gamma}$ has the scale property if every $\mathbf{\Gamma}$ set has a $\mathbf{\Gamma}$-scale.*

A scale $\langle \phi_n \rangle$ on a set A is good if whenever $\{x_n\} \subseteq A$ and for all $n \in \omega$, $\varphi_n(x_m)$ is eventually constant, then $x = \lim x_m$ exists and $x \in A$.

A scale $\langle \phi_n \rangle$ on a set A is very-good if $\langle \phi_n \rangle$ is good and whenever $x, y \in A$ and $\varphi_n(x) \leq \varphi_n(y)$ then $\varphi_k(x) \leq \varphi_k(y)$ for all $k < n$.

A scale $\langle \phi_n \rangle$ on a set A is excellent if it is very good and whenever $x, y \in A$ and $\varphi_n(x) = \varphi_n(y)$, then $x \restriction n = y \restriction n$.

Definition 1.5 (Inductive-like pointclass) *A pointclass $\mathbf{\Gamma}$ is inductive like, if it is closed under $\exists^{\mathbb{R}}, \forall^{\mathbb{R}}$ and $\mathbf{\Gamma}$ has the scale property.*

The following theorem is the *second periodicity theorem.* It shows that under suitable determinacy assumption we can propagate the scale property.

Theorem 2 (Moschovakis) *Assume projective determinacy. Then every $\mathbf{\Pi}^1_{2n+1}$ and every $\mathbf{\Sigma}^1_{2n}$ have the scale property.*

Recall that a set $A \subseteq \mathbb{R}$ is κ-Suslin if there is a tree T on $\omega \times \kappa$ such that:

$$A = p[T] = \{x : \exists f \in \kappa^\omega \forall n (x \restriction n, f \restriction n) \in T\}.$$

A cardinal κ is a *Suslin cardinal* if there is a set $A \subseteq \mathbb{R}$ which is κ-Suslin but not γ-Suslin for any $\gamma < \kappa$. The first few Suslin cardinals are $\aleph_0, \aleph_1, \aleph_\omega$ and $\aleph_{\omega+1}$. To draw an analogy with Θ, the supremum of all prewellorderings of the reals, $\aleph_1 = \boldsymbol{\delta}^1_1$ is the supremum of all $\mathbf{\Delta}^1_1$ prewellordering of $\mathbb{R}$. Similarly $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$ [2] is the supremum of all $\mathbf{\Delta}^1_3$ prewellorderings of $\mathbb{R}$ and

$$\boldsymbol{\delta}^2_1 = \sup\{\xi : \xi \text{ is the length of a } \mathbf{\Delta}^2_1 \text{ prewellordering of } \mathbb{R}\}.$$

Basically the problem of the continuum is viewed from the point of view of the Wadge hierarchy. Scales provide sets of reals both with a Suslin representation and a notion of definability associated to that representation.

First, we introduce the pointclass $\mathbf{\Sigma}^1_1(A)$, for some $A \subseteq \mathbb{R}$. We will need this notion below.

Definition 1.6 *Let $A \subseteq \mathbb{R}$. $\mathbf{\Sigma}^1_1(A)$ is the pointclass of all sets B such that:*

$$B(x) \leftrightarrow C(x) \vee \exists y (\forall n (y)_n \in A \wedge D(\langle x, y \rangle)),$$

where C and D are $\mathbf{\Sigma}^1_1$ sets.

[2]this is actually a theorem

Notice that $\mathbf{\Sigma}^1_1(A)$ is a pointclass which contains A, is closed under $\exists^{\mathbb{R}}, \vee, \wedge$. Let $C = \emptyset$, then we have

$$A(x) \leftrightarrow \exists y(\forall n (y)_n \in A \wedge D(\langle x, y\rangle)),$$

where $D(z) \leftrightarrow \forall i, j(((z)_1)_i = (z)_1)_j \wedge x = ((z)_1)_0)$. D is a $\mathbf{\Sigma}^1_1$ set and this shows that $A \in \mathbf{\Sigma}^1_1(A)$. Also notice that $\mathbf{\Sigma}^1_1(A)$ is indeed a pointclass since taking the preimage of a set in $\mathbf{\Sigma}^1_1(A)$ yields another set with complexity $\mathbf{\Sigma}^1_1(A)$. Next we show closure of $\mathbf{\Sigma}^1_1(A)$ under $\vee$. Let $B, B' \in \mathbf{\Sigma}^1_1(A)$ be written as

$$B(x) \leftrightarrow C(x) \vee \exists y(\forall n (y)_n \in A \wedge D(\langle x, y\rangle))$$

and

$$B'(x) \leftrightarrow C'(x) \vee \exists z(\forall n (z)_n \in A \wedge D'(\langle x, z\rangle))$$

where $C, C', D, D' \in \mathbf{\Sigma}^1_1$. Then we have

$$[C(x) \vee \exists y(\forall n (y)_n \in A \wedge D(\langle x, y\rangle))] \vee [C'(x) \vee \exists z(\forall n (z)_n \in A \wedge D'(\langle x, z\rangle))] \leftrightarrow$$
$$F(x) \vee \exists w(\forall n (w)_n \in A \wedge (G(\langle x, y\rangle) \vee G(\langle x, z\rangle)))$$

where $F = C \cup C'$ is a $\mathbf{\Sigma}^1_1$ set since $\mathbf{\Sigma}^1_1$ is closed under arbitrary unions and $G = D' \cup D$ is a $\mathbf{\Sigma}^1_1$ set since $\mathbf{\Sigma}^1_1$ is closed under recursive substitutions. We next show that $\mathbf{\Sigma}^1_1(A)$ is closed under $\exists^{\mathbb{R}}$. Let $B \in \mathbf{\Sigma}^1_1(A)$ be given by

$$B(\langle x, z\rangle) \leftrightarrow C(\langle x, z\rangle) \vee \exists y(\forall n (y)_n \in A \wedge D(\langle\langle x, z\rangle, y\rangle))$$

and let $U(x) \leftrightarrow \exists z B(\langle x, z\rangle)$ with $C, D \in \mathbf{\Sigma}^1_1$. We show that $U \in \mathbf{\Sigma}^1_1(A)$. But notice that

$$\exists z[C(\langle x, z\rangle) \vee \exists y(\forall n (y)_n \in A \wedge D(\langle\langle x, z\rangle, y\rangle))]$$

is logically equivalent to

$$\exists z C(\langle x, z\rangle) \vee \exists y(\forall n (y)_n \in A \wedge \exists z D(\langle\langle x, z\rangle, y\rangle)),$$

using that $\mathbf{\Sigma}^1_1$ is closed under existential quantification. Finally $\mathbf{\Sigma}^1_1(A)$ is closed under $\wedge$. To see this again let $B, B' \in \mathbf{\Sigma}^1_1(A)$ be written as

$$B(x) \leftrightarrow C(x) \vee \exists y(\forall n (y)_n \in A \wedge D(\langle x, y\rangle))$$

and

$$B'(x) \leftrightarrow C'(x) \vee \exists z(\forall n (z)_n \in A \wedge D'(\langle x, z\rangle))$$

where $C, C', D, D' \in \mathbf{\Sigma}^1_1$. We want to see that $B(x) \wedge B'(x) \in \mathbf{\Sigma}^1_1(A)$. Then we consider

$$[C(x) \vee \exists y(\forall n (y)_n \in A \wedge D(\langle x, y\rangle))] \wedge [C'(x) \vee \exists z(\forall n (z)_n \in A \wedge D'(\langle x, z\rangle))].$$

To compute this just notice that when the whole expression is unfolded, the $\mathbf{\Sigma}^1_1$ set C' can be pushed in the second disjunct defining the set B past the quantification over y so that we have

$$C'(x) \wedge \exists y(\forall n (y)_n \in A \wedge D(\langle x, y\rangle)) \leftrightarrow \exists y(\forall n (y)_n \in A \wedge D(\langle x, y\rangle) \wedge C')$$

and $D(\langle x,y\rangle)\wedge C'$ is now a $\mathbf{\Sigma}^1_1$ set. Similarly for C and $\exists z(\forall n(z)_n \in A \wedge D'(\langle x,z\rangle))$. Also when the expression is unfolded one writes

$$\exists y(\forall n(y)_n \in A \wedge D(\langle x,y\rangle)) \wedge \exists z(\forall n(z)_n \in A \wedge D'(\langle x,z\rangle))$$

as

$$\exists w(\forall n(w)_n \in A \wedge \exists \varepsilon_0, \varepsilon_1(D(\langle x,\varepsilon_0\rangle) \wedge D'(\langle x,\varepsilon_1\rangle)) \wedge \forall j((\varepsilon_0)_j = (w)_{2j} \wedge (\varepsilon_1)_j = (w)_{2j+1}).$$

So w is now a single real witnessing the above conjunction in a "zig-zag" way. Notice that

$$\exists \varepsilon_0, \varepsilon_1(D(\langle x,\varepsilon_0\rangle) \wedge D'(\langle x,\varepsilon_1\rangle))$$

is still a $\mathbf{\Sigma}^1_1$ set and $\forall j((\varepsilon_0)_j = (w)_{2j} \wedge (\varepsilon_1)_j = (w)_{2j+1})$ is $\mathbf{\Delta}^1_1$

We will use these closure properties of $\mathbf{\Sigma}^1_1(A)$ below in the analysis of the type **IV** case. The pointclass $\mathbf{\Sigma}^1_1(A)$ also has a universal set which comes from the universal set for $\mathbf{\Sigma}^1_1$ sets in a natural way. Let $U \subseteq \mathbb{R}^2$ be universal for $\mathbf{\Sigma}^1_1$ sets of reals. Then define

$$V(\varepsilon, x) \leftrightarrow U(\varepsilon_0, x) \vee \exists y(\forall n(y)_n \in A \wedge U(\varepsilon_1, \langle x,y\rangle)).$$

Then $V \in \mathbf{\Sigma}^1_1(A)$ and is universal for $\mathbf{\Sigma}^1_1(A)$ sets of reals by letting

$$C(x) \leftrightarrow U_{\varepsilon_0}(x)$$

and

$$D(\langle x,y\rangle) \leftrightarrow U_{\varepsilon_1}(\langle x,y\rangle)$$

be the two $\mathbf{\Sigma}^1_1$ sets coded by ε_0 and ε_1.

We now define the notion of a *projective* hierarchy in the general context. This is will allow us to define the Steel pointclasses which we need for the next section.

Definition 1.7 *A projective algebra is a pointclass* $\mathbf{\Lambda}$ *which is closed under* $\exists^{\mathbb{R}}, \vee, \wedge, \neg$.

A nice additional closure property of $\mathbf{\Lambda}$ is, by Steel-Van Wesep, if $A \in \mathbf{\Lambda}$ and if $\exists B$ which is not ordinal definable from A then $\mathbf{\Lambda}$ is closed under sharps, i.e for any $A \in \mathbf{\Lambda}$, $A^{\#} \in \mathbf{\Lambda}$. This would hold under $\theta_0 < \Theta$ for example, where

$$\theta_0 = \text{ the least ordinal which is not an OD surjective image of } \mathbb{R}.$$

Recall that assuming AD, Wadge's lemma says that for any two sets of reals A, B, either $A \leq_W B$ or $B \leq_W \mathbb{R}\setminus A$. For any set $A \subseteq \mathbb{R}$ there is then a notion of Wadge degree. We say that $A \subseteq \mathbb{R}$ is selfdual if the pointclass $\mathbf{\Gamma}_A = \{B : B \leq_W A\}$ is selfdual. The Wadge degree of A is the equivalence class $[A]_W$ of sets Wadge equivalent to A if A is self-dual, that is $A \leq_W \mathbb{R}\setminus A$ and the pair $([A]_W, [\mathbb{R}\setminus A]_W)$ if A is nonself-dual. Martin and Monk showed that the Wadge degrees are wellfounded under AD. The Wadge degree of a set A is denoted by $o(A)$.

Definition 1.8 $o(\mathbf{\Gamma}) = \sup\{o(A) : A \in \mathbf{\Gamma}\}$, *where* $o(A)$ *is the Wadge degree of* A.

Levy pointclasses are classified into 4 different projective-like hierarchies. Suppose $\mathbf{\Gamma}$ is nonselfdual and closed under either $\exists^{\mathbb{R}}$ or $\forall^{\mathbb{R}}$ or possibly both. First let α be the supremum of the limit ordinals β such that

1. $\mathbf{\Delta}_\beta = \{A : o(A) < \beta\}$ is closed under both $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$ and
2. $\mathbf{\Delta}_\beta \subseteq \mathbf{\Gamma}$.

We then have the following types of projective-like hierarchies:

- Type **I:** If $\mathrm{cof}(\alpha) = \omega$ there is a projective algebra $\mathbf{\Lambda}$ (i.e closed under $\exists^{\mathbb{R}}, \vee \wedge \neg$) of Wadge degree α whose sets are ω-joins of sets of smaller Wadge degree. Letting $\mathbf{\Gamma}_0 = \bigcup_\omega \mathbf{\Lambda}$ then $\mathbf{\Gamma}_0$ is a nonselfdual pointclass at the base of a new projective like hierarchy, $\mathbf{\Lambda} \subseteq \mathbf{\Gamma}_0, \mathbf{\Gamma}_0$ is closed under $\exists^{\mathbb{R}}$ and $\mathrm{pwo}(\mathbf{\Gamma}_0)$. $\mathbf{\Gamma}_0$ is not closed under countable intersections since $\mathbf{\Gamma}_0$ is nonselfdual.

- Type **II/III:** If $\mathrm{cof}(\alpha) > \omega$ then there is a pointclass $\mathbf{\Gamma}_0$ closed under $\forall^{\mathbb{R}}$ with $\mathrm{pwo}(\mathbf{\Gamma}_0)$ of Wadge degree α. $\mathbf{\Gamma}_0$ is not closed under $\exists^{\mathbb{R}}$ in this case. $\mathbf{\Gamma}_0$ is generated from a projective algebra $\mathbf{\Lambda}$: $\mathbf{\Gamma}_0$ is the pointclass of $\mathbf{\Sigma}^1_1$-bounded $\mathrm{cof}(\alpha)$ length unions of $\mathbf{\Lambda}$ sets. If $\mathbf{\Gamma}_0$ is closed under countable unions and disjunction then $\mathbf{\Gamma}_0$ is said to start a type **III** projective-like hierarchy.

- Type **IV:** If $\mathrm{cof}(\alpha) > \omega$ and $\mathbf{\Gamma}_0$ is as above and closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, then $\mathrm{pwo}(\mathbf{\Gamma}_0)$ but this can't be propagated by periodicity as in types **I,II and III**. So define $\mathbf{\Pi}_1 = \mathbf{\Gamma}_0 \wedge \check{\mathbf{\Gamma}}_0$. $\mathbf{\Pi}_1$ is said to be at the base of a type **IV** projective-like hierarchy. $\mathbf{\Pi}_1$ is closed under countable intersections, $\forall^{\mathbb{R}}$ but not under $\vee$ therefore not under $\exists^{\mathbb{R}}$.

We refer the reader to [3] for more facts on the general theory of pointclasses. In the next section, we will look more closely at the type **II** and **III** cases. The goal of the work below will be to characterize the above types of hierarchies of pointclasses with their associated Wadge ordinal.

2 Closure properties of the Steel Pointclass

2.1 Generating the Steel pointclass

We fix a Levy pointclass $\mathbf{\Gamma}$. We let $\mathbf{\Lambda}$ be the pointclass associated to $\mathbf{\Gamma}$ and obtained by taking unions of all sets in $\mathbf{\Delta}$, where $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$, and $\mathbf{\Delta}$ is closed under $\exists^{\mathbb{R}}$, complements and finite intersections. Then we have that $\mathbf{\Lambda} \subseteq \mathbf{\Gamma}$ and $\mathbf{\Lambda}$ is the largest projective algebra contained in $\mathbf{\Gamma}$ since it is closed under $\exists^{\mathbb{R}}$, complements and finite unions and intersections. It can also be shown that $\mathbf{\Lambda}$ is at the base of a projective hierarchy containing $\mathbf{\Gamma}$. Let

$$\alpha = \sup\{o(A) : A \in \mathbf{\Lambda}\}$$

and suppose $\omega < \mathrm{cof}(\alpha)$ (the case $\omega = \mathrm{cof}(\alpha)$ is the case of a type **I** hierarchy). By general theory of the Wadge degrees, we have a nonselfdual pointclass $\mathbf{\Gamma}_0$ such that $o(\mathbf{\Gamma}_0) = \alpha$. One of $\mathbf{\Gamma}_0$ and $\check{\mathbf{\Gamma}}_0$ has the separation property, so let $\check{\mathbf{\Gamma}}_0$ be the side with the separation property. It turns out that $\mathbf{\Gamma}_0$ is closed under $\forall^{\mathbb{R}}$:

Theorem 3 ([6]) *Assume ZF+AD. Let $\mathbf{\Gamma}_0$ be as above and assume that $\check{\mathbf{\Gamma}}_0$ has the separation property. Then $\check{\mathbf{\Gamma}}_0$ is closed under $\exists^{\mathbb{R}}$.*

Proof.
The proof uses a variant of an argument by Addison which was used to show the separation property for the pointclass $\mathbf{\Sigma}^1_3$. Suppose that there is a set $A \in \exists^{\mathbb{R}}\check{\mathbf{\Gamma}}_0 \setminus \check{\mathbf{\Gamma}}_0$. Then by Wadge's lemma, $\mathbf{\Gamma}_0 \subseteq \exists^{\mathbb{R}}\check{\mathbf{\Gamma}}_0$. Let $P, Q \in \mathbf{\Gamma}_0$ such that $P \cap Q = \emptyset$. Since $P, Q \in \exists^{\mathbb{R}}\check{\mathbf{\Gamma}}_0$, then let $A, B \in \check{\mathbf{\Gamma}}_0$ be such that $P(x) \leftrightarrow \exists y A(x, y)$ and $Q(x) \leftrightarrow \exists y B(x, y)$. Define

$$A'(x, y, z) \leftrightarrow A(x, y)$$

and

$$B'(x, y, z) \leftrightarrow B(x, z).$$

Then $A' \cap B' = \emptyset$ and $A', B' \in \check{\mathbf{\Gamma}}_0$. By the separation property of $\check{\mathbf{\Gamma}}_0$, let $D \in \mathbf{\Delta}$ such that $A' \subseteq D$ and $B' \cap D = \emptyset$. But now letting

$$E(x) \leftrightarrow \exists y \forall z D(x, y, z),$$

we have $E \in \mathbf{\Delta}$ since $\mathbf{\Delta}$ is closed under $\exists^{\mathbb{R}}$ and complements and $P \subseteq E$, $E \cap Q = \emptyset$. So $\mathbf{\Gamma}_0$ has the separation property. Contradiction!

□

We call $\mathbf{\Gamma}_0$ as above the Steel pointclass. Notice that there are no reasons why $\mathbf{\Gamma}_0$ should be closed under $\vee$ at this point.

Steel has shown that $\mathbf{\Gamma}_0$ is obtained by taking $\mathrm{cof}(\alpha)$ length $\mathbf{\Sigma}^1_1$ bounded unions of sets in the projective algebra $\mathbf{\Lambda}$. We now show how to generated $\mathbf{\Gamma}_0$ from $\mathbf{\Lambda}$ this way. So let $\omega < \mathrm{cof}(\alpha) = \beta$, where $\alpha = o(\mathbf{\Lambda})$ and let $\mathbf{\Gamma}$ be the Steel pointclass. So we have $\mathrm{Sep}(\check{\mathbf{\Gamma}})$ and there is a set $A \in \mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$ such that $o(A) = \alpha$. By the above theorem $\mathbf{\Gamma}$ is closed under $\forall^{\mathbb{R}}$. We show that $\mathbf{\Lambda}$ is closed under unions of length strictly less than β. We will need this fact to generate the Steel pointclass from $\mathbf{\Lambda}$.

Lemma 2.1 *Assume that $\mathbf{\Lambda} \subsetneq \mathcal{P}(\mathbb{R})$, then β is the least ordinal such that for a sequence of sets $\{A_\gamma\}_{\gamma<\beta}$, with each $A_\gamma \in \mathbf{\Lambda}$ we have that $\bigcup_{\gamma<\beta} A_\gamma \notin \mathbf{\Lambda}$*

Proof.
Let $\preceq$ be a prewellordering of length β in $\mathbf{\Lambda}$. Let δ be the least ordinal such that there is a δ sequence of sets in $\mathbf{\Lambda}$ such that $\bigcup_{\gamma<\delta} A_\gamma \notin \mathbf{\Lambda}$. Then we show that $\delta = \beta$. Notice that δ is a regular cardinal since if not then letting $f : \xi \to \delta$ be a cofinal map for $\xi < \delta$ we could obtain $\bigcup_{\gamma<\xi} A_\gamma \notin \mathbf{\Lambda}$ and then δ is not least. Suppose $\beta < \delta$. Assume $\delta < \alpha$. We can also assume that

there is an $\alpha_0 < \alpha$ such that for each $\gamma < \delta$, we have $|A_\gamma|_W \leq \alpha_0$, since δ is regular. Fix then a nonselfdual pointclass $\mathbf{\Gamma}' \subseteq \mathbf{\Lambda}$ such that $\mathbf{\Gamma}'$ is closed under $\exists^{\mathbb{R}}, \wedge, \vee$, $A_\gamma \in \mathbf{\Gamma}'$ for every $\gamma < \delta$ and such that there is a prewellordering of length δ in $\mathbf{\Gamma}'$. Let $\varphi : \mathbb{R} \to \delta$ be a $\mathbf{\Gamma}'$ norm and for each δ sequence of $\mathbf{\Gamma}'$ sets $\{A_\xi\}_{\xi<\gamma}$ let by the coding lemma $R(w, \varepsilon)$ be a $\mathbf{\Gamma}'$ relation such that

1. $\varphi(w) = \varphi(z) \to (R(w, \varepsilon) \leftrightarrow R(z, \varepsilon))$
2. $R(w, \varepsilon) \to \varepsilon \in C$ where C is the set of codes of the $\mathbf{\Gamma}'$ sets in the sequence $\{A_\gamma\}_{\gamma<\delta}$. C can be defined using a universal $\mathbf{\Gamma}'$ set as follows: let $U \in \mathbf{\Gamma}'$ be a universal set. Then for every $\gamma < \delta$ we let $\varepsilon \in \mathbb{R}$ such that $U_\varepsilon = A_{\varphi(\varepsilon)}$. Then $C \in \mathbf{\Gamma}'$.
3. $\forall w \exists \varepsilon (R(w, \varepsilon) \wedge U_\varepsilon = A_{\varphi(w)})$

Then we have
$$x \in \bigcup_{\gamma<\delta} A_\gamma \leftrightarrow \exists w \exists \varepsilon (R(w, \varepsilon) \wedge x \in U_\varepsilon).$$
So the union is in $\mathbf{\Gamma}'$. Contradiction!

Next, assume $\alpha < \delta$. Let $\mathbf{\Gamma}' \subseteq \mathbf{\Lambda}$ be a pointclass as above. Consider a sequence of $\mathbf{\Gamma}'$ sets $\{A_\gamma\}_{\gamma<\delta}$ and define the natural prewellordering $\leq$ defined by

$$x \leq y \leftrightarrow \exists \gamma_1, \gamma_2 \text{ such that } (\gamma_1 < \gamma_2 \wedge x \in A_{\gamma_1} \setminus A_{<\gamma_1} \wedge y \in A_{\gamma_2} \setminus A_{<\gamma_2})$$

Notice that there is an $\alpha_0 < \alpha$ such that for every $\gamma < \alpha$, we have $| \leq_\gamma |_W \leq \alpha_0$, where $\leq_\gamma$ has length γ. So for each γ, we have $\leq_\gamma \in \mathbf{\Lambda}$. But now $\leq = \bigcup_{\gamma<\alpha} \leq_\gamma$ is a prewellordering of length α in $\mathbf{\Lambda}$, since $\mathbf{\Lambda}$ is closed under unions of length α by minimality of δ. Contradiction!

If $\delta < \beta$ then since $\beta \leq \alpha$ then we still have $\delta < \alpha$ and we would get a contradiction using the coding lemma as above. So we must have $\delta \geq \beta$. In case $\delta = \alpha$, then α is also regular and so $\alpha = \beta$. So $\delta = \beta$.

□

Continuing, we have from the above lemma $\mathbf{\Lambda} \subsetneq \bigcup_\beta \mathbf{\Lambda}$. We cannot have that $\bigcup_\beta \mathbf{\Lambda} = \check{\mathbf{\Gamma}}$. To see this, let $A, B \in \check{\mathbf{\Gamma}}$. Then let $\{A_\gamma\}_{\gamma<\beta}$ be a sequence of sets in $\mathbf{\Lambda}$ such that $A = \bigcup_{\gamma<\beta} A_\gamma$ and let $\{B_\gamma\}_{\gamma<\beta}$ be a sequence of sets in $\mathbf{\Lambda}$ such that $B = \bigcup_{\gamma<\beta} B_\gamma$. We first show that $\check{\mathbf{\Gamma}}$ has the reduction property. Define the set A' by

$$x \in A' \leftrightarrow \exists \gamma_1 (x \in A_{\gamma_1} \wedge x \notin \bigcup_{\gamma<\gamma_1} B_\gamma)$$

and define the set B' by
$$x \in B' \leftrightarrow \exists \gamma_1 (x \in B_{\gamma_1} \wedge x \notin \bigcup_{\gamma\leq\gamma_1} A_\gamma)$$

Then notice both A' and B' are in $\check{\mathbf{\Gamma}}$. Also $A' \subseteq A$ and $B' \subseteq B$ and $A' \cap B' = \emptyset$. So $\mathrm{Red}(\check{\mathbf{\Gamma}})$. But recall that we also have by assumption $\mathrm{Sep}(\check{\mathbf{\Gamma}})$. We quickly justify that the reduction property and the separation property can't both hold for $\check{\mathbf{\Gamma}}$. Let $A, B \in \check{\mathbf{\Gamma}}$. Then by $\mathrm{Red}(\check{\mathbf{\Gamma}})$, let A' and

B' be disjoint sets such that $A' \subseteq A$ and $B' \subseteq B$ and $A' \cup B' = A \cup B$. Let $U \in \check{\mathbf{\Gamma}}$ be a universal set which codes the pair of sets A', B' by

$$A'(x,y) \leftrightarrow U((x)_0, y)$$

and

$$B'(x,y) \leftrightarrow U((x)_1, y).$$

Now let C be a set in $\mathbf{\Delta}$ which separates A' from B', i.e $A' \subseteq C$ and $C \cap B' = \emptyset$. Now let D be an arbitrary $\mathbf{\Delta}$ set. Then there exists a $z \in \mathbb{R}$ such that

$$D(y) \leftrightarrow U_{(z)_0}(y) \leftrightarrow \neg U_{(z)_1}(y).$$

Then we have that

$$D(y) \leftrightarrow A_x(y) \leftrightarrow \neg B_x(y).$$

But then this implies that

$$D(y) \leftrightarrow A'_x(y) \leftrightarrow \neg B'_x(y).$$

So $D(y) \leftrightarrow C_x(y)$, because $C \in \mathbf{\Delta}$ separates A' from B'. So every $\mathbf{\Delta}$ set is coded as a section of a single $\mathbf{\Delta}$ set. But selfdual pointclasses can't have universal sets: if $U \in \mathbf{\Delta}$ is universal for $\mathbf{\Delta}$ sets then $U \in \mathbf{\Gamma}$ and $U \in \check{\mathbf{\Gamma}}$. Define then, $A(x) \leftrightarrow \neg U(x,x)$. Since $\mathbf{\Delta}$ is closed under recursive substitutions, then we have $A \in \mathbf{\Gamma}$. So there exists a $z \in \mathbb{R}$ such that $A = U_z$, but now we have

$$A(z) \leftrightarrow U(z,z) \leftrightarrow \neg A(z),$$

contradiction!

Therefore, by Wadge's lemma we must have that $\mathbf{\Gamma} \subseteq \bigcup_\beta \mathbf{\Lambda}$. Since $\mathbf{\Lambda}$ is a projective hierarchy then $\exists^{\mathbb{R}} \mathbf{\Gamma} \subseteq \bigcup_\beta \mathbf{\Lambda}$.

We say that a union $A = \bigcup_{\alpha<\delta} A_\alpha$ is $\mathbf{\Sigma}^1_1$-bounded if

$$\text{for any } \mathbf{\Sigma}^1_1 \text{ set } S \subseteq A, \text{ there exists a } \gamma < \delta \text{ such that } S \subseteq A_\gamma.$$

Let $\mathbf{\Gamma}_1$ be the pointclass of $\mathbf{\Sigma}^1_1$-bounded β length unions of $\mathbf{\Lambda}$ sets. Using the above set up, it is then shown in [10] and [6] that $\mathbf{\Gamma} = \mathbf{\Gamma}_1$. So the Steel pointclass corresponding to the projective algebra $\mathbf{\Lambda}$ can be characterized as all sets which are $\mathbf{\Sigma}^1_1$-bounded β length unions of sets in $\mathbf{\Lambda}$. We proceed to show that the Steel pointclass has the prewellordering property (see [10]). This will motivate a different characterization of the Steel pointclass which we will adopt in the rest of the section.

Theorem 4 (Steel, [10]) *Let $\mathbf{\Lambda}$ be a projective algebra with $\alpha = o(\mathbf{\Lambda})$ and assume that $\omega <$ $\mathrm{cof}(\alpha)$. let $\mathbf{\Gamma}$ be the Steel pointclass corresponding to $\mathbf{\Lambda}$. Then pwo$(\mathbf{\Gamma})$.*

Proof.

Let $\beta = \mathrm{cof}(\alpha)$ and let $A \subseteq \mathbb{R}$ be a complete $\mathbf{\Gamma}$ set of reals.Let $A = \bigcup_{\gamma<\beta} A_\gamma$ be an increasing $\mathbf{\Sigma}^1_1$ bounded β length union of sets such that for each $\gamma < \beta, A_\gamma \in \mathbf{\Lambda}$. Let φ be the natural norm in A such that for $x \in A$, $\varphi(x) =$ least ξ such that $x \in A_\xi$. The norm $<^*_\varphi$ associated to φ can be written as $\bigcup_{\gamma<\beta} B_\gamma$ where

$$B_\gamma(x,y) \leftrightarrow x \in A_\gamma \wedge y \notin A_\gamma.$$

Then for each $\gamma < \beta$, $B_\gamma \in \mathbf{\Lambda}$. It remains to show that $<^*_\varphi \in \mathbf{\Gamma}$. We proceed to show that $<^*_\varphi$ is $\mathbf{\Sigma}^1_1$ bounded. So let $S \subseteq \mathbb{R} \times \mathbb{R}$ be a $\mathbf{\Sigma}^1_1$ and $S \subseteq <^*_\varphi$. Notice that if $S(x,y)$ holds then $x \in A$. Since by assumption $\bigcup_{\gamma<\beta} A_\gamma$ is a $\mathbf{\Sigma}^1_1$ bounded union, there is a $\gamma_0 < \beta$ such that whenever $S(x,y)$ holds $x \in A_{\gamma_0}$.If $\varphi(x) < \varphi(y)$, then there is a $\gamma < \gamma_0$ such that $x \in A_\gamma$ ad $y \notin A_\gamma$ and $B_\gamma(x,y)$ holds. So $<^*_\varphi \in \mathbf{\Gamma}$. A similar computation shows that $\leq^*_\varphi \in \mathbf{\Gamma}$. So pwo($\mathbf{\Gamma}$).

□

Gathering all the facts above we characterize the Steel pointclass as follows:

Definition 2.2 (Steel pointclass) *If $\mathbf{\Delta}$ is selfdual, closed under real quantifiers, $o(\mathbf{\Delta})$ has uncountable cofinality, $\mathbf{\Delta}$ is not closed under well-ordered unions, then the Steel pointclass is the pointclass $\mathbf{\Gamma}$ such that $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$, $\mathbf{\Gamma}$ is closed under $\forall^{\mathbb{R}}$ and pwo($\mathbf{\Gamma}$).*

Since the Steel pointclass is nonselfdual and closed under $\forall^{\mathbb{R}}$ then it is closed under $\wedge$. A natural question which arises then is whether the Steel pointclass is closed under $\vee$. The following theorem below shows that what prevents closure of the Steel pointclass under $\vee$ is the singularity of $o(\mathbf{\Delta})$.

To introduce the following theorem, recall that if $\mathbf{\Gamma}$ is a nonselfdual pointclass closed under $\forall^{\mathbb{R}}$ and $\vee$, and if $\varphi : A \to \kappa$ is a regular $\mathbf{\Gamma}$-norm on a $\mathbf{\Gamma}$-complete set A, then for every $B \in \check{\mathbf{\Gamma}}$ such that $B \subseteq A$, there is a $\eta < \kappa$ such that $\sup\{\varphi(x) : x \in B\} = \eta$ [3]. In this case we say that φ is $\check{\mathbf{\Gamma}}$-bounded. Similarly say that a norm is κ-Suslin bounded if for every set $B \subseteq A$ which is κ-Suslin, $\sup\{\phi(x) : x \in B\} < \gamma$ for $\phi : A \to \gamma$.

Theorem 5 (Steel, [10]) *Suppose Sep($\check{\mathbf{\Gamma}}$) and suppose $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ is closed under $\exists^{\mathbb{R}}$. Assume $A \in \mathbf{\Delta}$ and that there is a norm $\varphi : A \twoheadrightarrow \lambda$ which is $\mathbf{\Sigma}^1_1$-bounded, where $\lambda = \mathrm{cof}(o(\mathbf{\Delta}))$. Then there is a $B \in \check{\mathbf{\Gamma}}$ such that $A \cap B \notin \check{\mathbf{\Gamma}}$.*

A variation of the proof of the above theorem, shows the following limitation to the closure of the Steel pointclass under $\vee$.

Theorem 6 (Steel, [10]) *Suppose Sep($\check{\mathbf{\Gamma}}$) and suppose $\exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta}$ and $o(\mathbf{\Delta})$ is singular. Then $\check{\mathbf{\Gamma}}$ is not closed under intersections with $\mathbf{\Delta}$ sets.*

Proof.

Let $\alpha = \mathrm{cof}(o(\mathbf{\Delta})) < o(\mathbf{\Delta})$ and let $\{\kappa_\gamma : \gamma < \alpha\}$ be a cofinal sequence in $o(\mathbf{\Delta})$. Let U be a universal $\check{\mathbf{\Gamma}}$ set. Let $A \in \mathbf{\Delta}$ and let $\varphi : A \to \alpha$ be a $\mathbf{\Delta}$ norm of length α. By the coding lemma there is a relation P such that

$$P(x,\varepsilon) \leftrightarrow \forall x \exists \varepsilon (x \in A \to U_{(\varepsilon)_0} = U^c_{(\varepsilon)_1} \wedge |U_{(\varepsilon)_0}|_W \geq \kappa_{\varphi(x)})$$

[3] see [9], $4C.11$ for a proof of this fact

Notice that $P \in \mathbf{\Delta}$. Now define the relation R as follows:

$$R(x, \varepsilon) \leftrightarrow x \in A \wedge (\varepsilon)_0 \notin U_{(\varepsilon)_1}$$

Then $R \in \mathbf{\Gamma}$. But since the set $\{|R_x|_W : x \in A\}$ is cofinal in $o(\mathbf{\Delta})$, then $R \notin \mathbf{\Delta}$ and so $R \notin \check{\mathbf{\Gamma}}$. Also R can be written as:

$$R(x, \varepsilon) \leftrightarrow x \in A \wedge (\varepsilon)_0 \in U_{(\varepsilon)_0}$$

and so R is the intersection of a set in $\mathbf{\Delta}$ and a set in $\check{\mathbf{\Gamma}}$ which is not in $\check{\mathbf{\Gamma}}$.

$\square$

2.2 Steel's conjecture and closure properties of the Steel pointclass

A natural conjecture then, and this what Steel conjectures in [10], is whether the regularity of $o(\mathbf{\Delta})$ would imply closure of $\check{\mathbf{\Gamma}}$ under intersections. However, this conjecture turns out to be false, we will show this later. We precisely state the conjecture:

Conjecture 1 (Steel, [10]) *If $\mathbf{\Gamma}$ is the Steel pointclass such that $o(\mathbf{\Delta})$ is regular and $\exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta}$ then $\mathbf{\Gamma}$ is closed under $\vee$.*

Notice that the conjecture can be rephrased by asking that if $\mathrm{Sep}(\check{\mathbf{\Gamma}})$, $\exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta}$ and $o(\mathbf{\Delta})$ is a regular cardinal, then $\bigcap_2 \check{\mathbf{\Gamma}} \subseteq \check{\mathbf{\Gamma}}$, and this is actually how the conjecture was originally stated.

As in [10], let

$$C \doteq \{o(\mathbf{\Delta}) : \exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta} \wedge \mathbf{\Delta} \text{ is a selfdual pointclass}\}$$

Notice that there are cofinally many in Θ such ordinals $\kappa \in C$, since these are the places where we are at the base of a projective-like hierarchy of type **II**, **III** or **IV**. If $\kappa \in C$ and $\mathrm{cof}(\kappa) > \omega$ then, as noted above, Steel shows in [10] that there is a Steel pointclass $\mathbf{\Gamma}$ such that $o(\mathbf{\Delta}) = \kappa$.

The following is a weaker positive solution to the Steel's conjecture. Essentially it says that the Steel pointclass is closed under unions with κ-Suslin sets for $\kappa < \mathrm{cof}(o(\mathbf{\Delta}))$. In particular, if $o(\mathbf{\Delta})$ is a regular limit of Suslin cardinals then the Steel pointclass is closed under disjunction. The proof uses the Martin-Monk method which exploits the fact that a certain strategy flips membership to construct two disjoint sets which are comeager.

Theorem 7 (Steel, [10]) *Let $\mathbf{\Gamma}$ be nonselfdual, closed under $\forall^{\mathbb{R}}$ and such that $pwo(\mathbf{\Gamma})$.Suppose that $\exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta}$. Then $\mathbf{\Gamma}$ is closed under union with κ-Suslin sets for $\kappa < \mathrm{cof}(o(\mathbf{\Delta}))$.*

This is turn gives the following boundedness principle:

Theorem 8 (Steel, see [3]) *Let $\gamma < \Theta$ be a limit ordinal. Then there is a set $A \subseteq \mathbb{R}$ and a norm $\varphi : A \to \gamma$ which is onto and κ-Suslin bounded for all $\kappa < \mathrm{cof}(\gamma)$.*

Therefore Steel's conjecture is true in the least initial segment of the Wadge hierarchy containing the inductive sets, **IND**, since by a result of Kechris, every $A \subseteq \mathbb{R} \in \mathbf{HYP}$ is κ-Suslin for $\kappa < \kappa^{\mathbb{R}}$ and scales can be localized to smaller pointclass within **HYP**. This implies the following corollary:

Corollary 9 *If $\mathbf{\Gamma}$ is the Steel pointclass and $\boldsymbol{IND} \subseteq \mathbf{\Gamma}$, then for $A \in \boldsymbol{IND}, B \in \mathbf{\Gamma}$, we have that $A \cup B \in \mathbf{\Gamma}$.*

We move to generalize the above boundedness principle to all sets in $\mathbf{\Delta}$ associated to the Steel pointclass $\mathbf{\Gamma}$ and show the equivalence of this generalization with the Steel pointclass being closed under disjunctions. We then show the following theorem, which reduces Steel's conjecture to the question of whether $\mathbf{\Delta}$ sets are *bounded in the norm*. We say that $\mathbf{\Delta}$ sets are bounded in the norm if there is a $\mathbf{\Delta}$-bounded norm, that is a norm $\varphi : P \to \kappa$ for some ordinal κ and a set $P \subseteq \mathbb{R}$ such that for every $\mathbf{\Delta}$ set $S \subseteq P$, $\sup\{\varphi(x) : x \in S\} < \kappa$.

Theorem 10 *Let $\mathbf{\Gamma}$ be the Steel pointclass and let $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ be such that $\exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta}$. Then the following are equivalent:*

1. $\bigcup_2 \mathbf{\Gamma} \subseteq \mathbf{\Gamma}$,
2. $\bigcup_\omega \mathbf{\Gamma} \subseteq \mathbf{\Gamma}$,
3. *$\mathbf{\Gamma}$ is closed under union with $\mathbf{\Delta}$ sets,*
4. *$\mathbf{\Delta}$ sets are bounded in the norm.*

Proof.
Let $\mathbf{\Gamma}$ be a nonselfdual pointclass such that $\exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta}$, pwo($\mathbf{\Gamma}$) and $\mathbf{\Gamma}$ is closed under $\forall^{\mathbb{R}}$. (1) $\longrightarrow$ (2) holds because we have $\neg$Sep($\mathbf{\Gamma}$), this is theorem 2.2 in [10]. (2) $\longrightarrow$ (1) is immediate. That clause (2) implies clause (3) is also immediate. We next show that (3) implies (2). So let $A, B \in \mathbf{\Gamma}$. We show that $A \cup B \in \mathbf{\Gamma}$. Since Red($\mathbf{\Gamma}$) holds, we may assume that $A \cap B = \emptyset$. Let $A = \bigcup_{\beta<\alpha} A_\beta$ and $B = \bigcup_{\beta<\alpha} B_\beta$ where α is the ordinal such that $\bigcup_\alpha \mathbf{\Delta} \nsubseteq \mathbf{\Delta}$. Define

$$\mathbf{\Gamma}^* = \{ \bigcup_{\alpha<o(\mathbf{\Delta})} A_\alpha : \forall\alpha(A_\alpha \in \mathbf{\Delta}) \wedge \bigcup_{\alpha<o(\mathbf{\Delta})} A_\alpha \text{ is } \mathbf{\Delta} \text{ bounded}\}$$

Claim 1 $\mathbf{\Gamma}^* = \mathbf{\Gamma}$

Proof.
We have $\mathbf{\Gamma}^* \subseteq \mathbf{\Gamma}$ since every set on $\mathbf{\Gamma}^*$ is a $\mathbf{\Sigma}^1_1$-bounded union of set $\mathbf{\Delta}$ sets. We next show that $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}^*$. So let $A \in \mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$. Let $A = \bigcup_{\beta<\alpha} A_\beta$ with $A_\beta \in \mathbf{\Delta}$ for every $\beta < \alpha$ and α is least such that $\bigcup_\alpha \mathbf{\Delta} \nsubseteq \mathbf{\Delta}$. We may assume that the union is increasing. Let $\varphi : A \to \alpha$ be a

$\mathbf{\Sigma}^1_1$-bounded $\mathbf{\Gamma}$-norm on A. Let $\{\kappa_\beta : \beta < \alpha\}$ be cofinal in $o(\mathbf{\Delta})$. Let U be a universal $\mathbf{\Gamma}$ set. Define the Solovay game as follows:

$$\begin{array}{ll} \mathrm{I} & x \\ \mathrm{II} & \langle w, y, z\rangle \end{array}$$

The payoff condition is then defined by:

$$\text{Player II wins iff } x \in A \to (U_y = U_z^c = A_{\varphi(w)} \wedge |U_y|_W \geq \kappa_{\varphi(x)}).$$

Since φ is $\mathbf{\Sigma}^1_1$-bounded then Player II has a wining strategy τ for this game. Then let

$$R(x, w, y) \leftrightarrow x \in A \wedge w = \tau(x)_0 \wedge U_{\tau(x)_1} = A_{\varphi(w)} \wedge y \notin U_{\tau(x)_2}.$$

Then we have that $\{|R_x|_W : x \in A\}$ is unbounded in $o(\mathbf{\Delta})$ and so $\{|A_\beta|_W : \beta < \alpha\}$ is unbounded in $o(\mathbf{\Delta})$.

Next for $\beta < \alpha$, let

$$C_\beta = \{(x, y) : y \in A_{\beta+1} \setminus A_\beta \wedge x \text{ codes a continuous function } f_x \text{ s.t } f_x^{-1}(A_\beta) \subseteq A\}.$$

Then for every $\beta < \alpha$, C_β is defined as $(\mathbf{\Delta} \wedge \forall^{\mathbb{R}}(\mathbf{\Delta} \vee \mathbf{\Gamma}))$ and so because we are assuming that $\mathbf{\Gamma}$ is closed under unions with $\mathbf{\Delta}$ sets, we have for every $\beta < \alpha$, $C_\beta \in \mathbf{\Gamma}$. Let $C = \bigcup_{\beta<\alpha} C_\beta$. Then another Solovay game argument as above shows that $C \in \exists^{\mathbb{R}}\mathbf{\Gamma}$. Actually one can show that $C \in \mathbf{\Gamma}$. Notice that because $\exists^{\mathbb{R}}\mathbf{\Delta} = \mathbf{\Delta}$ and because $\mathbf{\Gamma} = \bigcup_\alpha \mathbf{\Delta}$, then $\exists^{\mathbb{R}}\mathbf{\Gamma} \subseteq \bigcup_\alpha \mathbf{\Delta}$. So let $D_\beta \in \mathbf{\Delta}$ for every $\beta < \alpha$ such that $C = \bigcup_{\beta<\alpha} D_\beta$. We may assume that the union is increasing. Define the sets B_β by

$$B_\beta(z) \leftrightarrow \exists(x, y) \in D_\beta \exists\gamma \leq \beta(y \in A_{\gamma+1} \setminus A_\gamma \wedge f_x(z) \in A_\gamma).$$

Then $B_\beta \in \mathbf{\Delta}$. Notice that $A = \bigcup_{\beta<\alpha} B_\beta$ and $\bigcup_{\beta<\alpha} B_\beta$ is $\mathbf{\Delta}$-bounded since every $\mathbf{\Delta}$ set is coded as a set $f_x^{-1}(A_\beta)$ for some $\beta < \alpha$.

□

Now recall that $A = \bigcup_{\beta<\alpha} A_\beta$ and $B = \bigcup_{\beta<\alpha} B_\beta$. These unions are $\mathbf{\Delta}$-bounded and increasing with each A_β and B_β in $\mathbf{\Delta}$. We show that $\bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)$ is $\mathbf{\Delta}$ bounded. Then let $C \subseteq \bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)$ with $C \in \mathbf{\Delta}$. Then $C \cap A \in \mathbf{\Gamma}$ as $\mathbf{\Gamma}$ is closed under intersections. Also $C \cap A = C \cap B^c$ and $C \cap B^c \in \check{\mathbf{\Gamma}}$, since by assumptions $\check{\mathbf{\Gamma}}$ is closed under intersections with $\mathbf{\Delta}$ sets. So $C \cap A \in \mathbf{\Delta}$ and $\exists\gamma_1 < \alpha$ such that $C \cap A \subseteq A_{\gamma_1}$. Similarly, there exists a $\gamma_2 < \alpha$ such that $C \cap B \subseteq B_{\gamma_2}$. Let $\gamma = \max(\gamma_1, \gamma_2)$. Then $C \subseteq A_\gamma \cup B_\gamma$. So $A \cup B \in \mathbf{\Gamma}$ and $\bigcup_2 \mathbf{\Gamma} \subseteq \mathbf{\Gamma}$.

Finally it just remains to show that $\mathbf{\Delta}$ sets are bounded in the norm if and only if $\mathbf{\Gamma}$ is closed under unions with $\mathbf{\Delta}$ sets. Recall that $o(\mathbf{\Delta}) = \kappa$ is regular. We'll make use of this in the proof. Suppose first that $\mathbf{\Delta}$ sets are bounded in the norm. We need to see that $\mathbf{\Gamma}$ is closed under unions with $\mathbf{\Delta}$ sets. So let $A \in \mathbf{\Gamma}$ such that $A = \bigcup_{\beta<\kappa} A_\beta$ with $A_\beta \in \mathbf{\Delta}$ for every $\beta < \kappa$ and let $B \in \mathbf{\Delta}$ such that $B = \bigcup_{\beta<\alpha} B_\beta$ for some $\alpha < \kappa$ with $B_\beta \in \mathbf{\Delta}$ for every $\beta < \alpha$. It suffices to show that $A \cup B$ is $\mathbf{\Delta}$-bounded. We may assume that the unions are increasing and continuous, that is at all limit ordinal $\gamma < \kappa$ we have $A_\gamma = \bigcup_{\beta<\gamma} A_\beta$. So let $C \subseteq A \cup B$ such that $C \in \mathbf{\Delta}$. We also have that

$$A \cup B = \bigcup_{\beta<\kappa} A_\beta \cup \bigcup_{\beta<\alpha} B_\beta = \bigcup_{\beta<\alpha} (A_\beta \cup B_\beta) \cup \bigcup_{\alpha<\xi<\kappa} A_\xi.$$

But notice that we must have $\bigcup_{\beta<\alpha}(A_\beta \cup B_\beta) \in \mathbf{\Delta}$ since κ is a regular cardinal, $\alpha < \kappa$ and since $\mathrm{cof}(\kappa) = \kappa$ is least such that $\bigcup_{\mathrm{cof}(\kappa)} \mathbf{\Delta} \not\subseteq \mathbf{\Delta}$. So let $D = \bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)$. Then $C \cup D \in \mathbf{\Delta}$. So we have $C \cup D \subseteq \bigcup_{\alpha<\xi<\kappa} A_\xi =_{def} A' = A$, since the union is continuous. Let $\varphi : A' \to \kappa$ be the natural norm defined by $\varphi(x) =$ the least $\xi < \kappa$ such that $x \in A_\xi$. Since $\mathbf{\Delta}$ sets are bounded in the norm and since κ is regular, there exists a $\xi_1 < \kappa$ be such that $C \cup D \subseteq A_{\xi_1}$. So the union $A \cup B$ is $\mathbf{\Delta}$ bounded. Next we must show that a union is $\mathbf{\Delta}$-bounded union of $\mathbf{\Delta}$ sets if and only if it is a $\mathbf{\Gamma}$-complete set. This will ensure that $A \cup B$ is in $\mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$. So let $A = \bigcup_{\alpha<\kappa} A_\alpha$ be a $\mathbf{\Delta}$-bounded union of $\mathbf{\Delta}$ sets. We need to see that A is $\mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$. We start first by showing that our assumption implies that if $A \in \mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$ then A is a $\mathbf{\Delta}$ bounded union of $\mathbf{\Delta}$ sets. By pwo($\mathbf{\Gamma}$), let $\varphi : A \to \kappa$ be a $\mathbf{\Gamma}$ norm. Since $\mathbf{\Delta}$ sets are bounded in the norm then for any $\mathbf{\Delta}$ subset of $A_\alpha \subseteq A$, there exists an $\beta < \kappa$ such that elements of A_α are sent before β. In addition every initial segment of the norm φ is a $\mathbf{\Delta}$ set. So A is a union of $\mathbf{\Delta}$ sets which are $\mathbf{\Delta}$ bounded. Now we justify why any $\mathbf{\Delta}$-bounded union of $\mathbf{\Delta}$ sets is in $\mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$. So let $A = \bigcup_{\alpha<\kappa} A_\alpha$ be a $\mathbf{\Delta}$ bounded union of $\mathbf{\Delta}$ sets. We may assume that the union is increasing and continuous. Consider the following game:

$$\begin{array}{ll} \mathrm{I} & x \\ \mathrm{II} & \langle w, y, z \rangle \end{array}$$

The pay off condition is determined by player II wins the run of the game if and only if

$$x \in A \to \exists \alpha (U_w = U_y^c = A_\alpha \wedge x \in U_w \wedge z \in U_w)$$

Then player II has a winning strategy τ. Next notice that

$$x \in \bigcup A_\alpha \leftrightarrow x \in U_{\tau(x)_0} \wedge U_{\tau(x)_0} = U^c_{\tau(x)_1} \wedge \tau(x)_2 \in U_{\tau(x)_0}.$$

Then $\bigcup_{\alpha<\kappa} A_\alpha$ is in $\mathbf{\Gamma} \setminus \mathbf{\Delta}$. Thus $\bigcup_{\alpha<\kappa} A_\alpha \in \mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$.

Finally notice that if $\mathbf{\Gamma}$ is closed under unions with $\mathbf{\Delta}$ sets, then $\mathbf{\Gamma}$ is closed under finite unions by the above and thus Moschovakis argument (see $4.C.11$ in [9]) applies and this implies that $\mathbf{\Delta}$ sets are bounded in the norm. This finishes the proof.

□

2.3 Failure of Steel's conjecture

In the first author's phd, see [1], it was claimed that Steel's conjecture is true. However the "proof" given shows something different. We next show that Steel's conjecture is actually false. Specifically, we show that there are regular cardinals κ such that $\kappa = o(\boldsymbol{\Delta})$ where $\boldsymbol{\Delta}$ is associated to a Steel pointclass $\boldsymbol{\Gamma}$, and yet $\boldsymbol{\Gamma}$ is not closed under disjunctions.

First, recall briefly that Steel's conjecture is the statement that if $\boldsymbol{\Gamma}$ is a pointclass with the prewellordering property, closed under $\forall^{\mathbb{R}}$ and such that for $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}, \exists^{\mathbb{R}}\boldsymbol{\Delta} \subsetneq \boldsymbol{\Delta}$ and $\kappa = o(\boldsymbol{\Delta})$ is regular, then $\boldsymbol{\Gamma}$ is closed under disjunctions.

As usual we let $\boldsymbol{\Gamma}$ be a Steel pointclass, that is $\boldsymbol{\Gamma}$ is defined as all $\boldsymbol{\Sigma}^1_1$ bounded length $o(\boldsymbol{\Delta})$-unions of $\boldsymbol{\Delta}$ sets where $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$, $\exists^{\mathbb{R}}\boldsymbol{\Delta} \subsetneq \boldsymbol{\Delta}$, $\boldsymbol{\Gamma}$ has the prewellordering property and $\forall^{\mathbb{R}}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$. Example 2.6 below shows that if $\omega_1 < \operatorname{cof}(\boldsymbol{\Delta}))$, say $\operatorname{cof}(o(\boldsymbol{\Delta})) = \omega_2$, then the Steel pointclass at $o(\boldsymbol{\Delta})$ is not closed under unions with $\boldsymbol{\Pi}^1_2$ sets. This suggests that even if $\lambda < \operatorname{cof}(o(\boldsymbol{\Delta}))$, Steel's conjecture could fail and this is what we will show below. In other words, what is needed of κ, where $\kappa = o(\boldsymbol{\Delta})$ as above, to obtain closure of the Steel pointclass under disjunctions has to be stronger than mere regularity. Presumably those κ's have to satisfy a property stronger than regularity but weaker than ${}^b\boldsymbol{\Pi}^1_2$-indescribability (see the type **IV** case below).

Definition 2.3 *Let $\boldsymbol{\Lambda}$ be a pointclass. The* spectrum *of $\boldsymbol{\Lambda}$, spec$(\boldsymbol{\Lambda})$, is the set of $\alpha \in On$ such that there is a strictly increasing sequence $E = \bigcup_{\beta<\alpha} E_\alpha$ with $E \in \boldsymbol{\Lambda}$ and with the union $\boldsymbol{\Sigma}^1_1$-bounded (i.e., every $\boldsymbol{\Sigma}^1_1$ $S \subseteq E$ is a subset of some E_β).*

Remark 2.4 *In the definition of spec$(\boldsymbol{\Lambda})$, there is no requirement on the complexity of the E_β sets, only on the union E. Note that $\alpha \in$ spec$(\boldsymbol{\Lambda})$ requires $\operatorname{cof}(\alpha) > \omega$.*

Recall a *projective algebra* is a selfdual pointclass $\boldsymbol{\Delta}$ which is closed under $\exists^{\mathbb{R}}, \vee$ (and so also $\forall^{\mathbb{R}}, \wedge$). For $\boldsymbol{\Delta}$ a projective algebra we have that

$$\begin{aligned} o(\boldsymbol{\delta}) &\doteq \sup\{|A|_W \colon A \in \boldsymbol{\Delta}\} \\ &= \sup\{|\preceq| \colon \prec \text{ is a } \boldsymbol{\Delta} \text{ prewellordering }\} \end{aligned}$$

For $\boldsymbol{\Delta}$ a projective algebra and $\kappa = o(\boldsymbol{\Delta})$, if $\operatorname{cof}(\kappa) > \omega$ then there is a non-selfdual pointclass $\boldsymbol{\Gamma}_\kappa$ of Wadge degree κ, with $\operatorname{pwo}(\boldsymbol{\Gamma}_\kappa)$ and $\forall^{\mathbb{R}}\boldsymbol{\Gamma}_\kappa = \boldsymbol{\Gamma}_\kappa$. We call this pointclass the *Steel pointclass* at κ.

Lemma 2.5 *Let $\kappa = o(\boldsymbol{\Delta})$, where $\boldsymbol{\Delta}$ is a projective algebra with $\operatorname{cof}(\kappa) > \omega$, and let $\boldsymbol{\Gamma}_\kappa$ be the corresponding Steel pointclass. If $\operatorname{cof}(\kappa) \in$ spec$(\boldsymbol{\Lambda})$, then $\check{\boldsymbol{\Gamma}}_\kappa$ is not closed under intersection with $\boldsymbol{\Lambda}$ sets.*

Proof. Let $E = \bigcup_{\beta<\operatorname{cof}(\kappa)} E_\beta$ be a $\boldsymbol{\Sigma}^1_1$-bounded union with $E \in \boldsymbol{\Lambda}$. Let A be $\boldsymbol{\Gamma}_\kappa$ complete, and write $A = \bigcup_{\alpha<\operatorname{cof}(\kappa)} A_\alpha$, an increasing union with each $A_\alpha \in \boldsymbol{\Delta}_\kappa$. Let $U \subseteq \mathbb{R} \times \mathbb{R}$ be a universal $\check{\boldsymbol{\Gamma}}_\kappa$ set. Fix a map $\rho \colon \operatorname{cof}(\kappa) \to \kappa$ increasing and cofinal. Consider the game where I plays x, II plays y, and II wins iff

$$(x \in E) \Rightarrow [\exists\gamma > |x|\ (U_y = A_\gamma)]$$

where $|x|$, for $x \in E$, denotes the least β such that $x \in E_\beta$. By $\boldsymbol{\Sigma}^1_1$-boundedness of the E union, II has a winning strategy τ for this game. We then have

$$z \in A \leftrightarrow \exists x\ [(x \in E) \wedge z \in U_{\tau(x)}].$$

Since $\check{\mathbf{\Gamma}}_\kappa$ is closed under $\exists^{\mathbb{R}}$, and $A \notin \check{\mathbf{\Gamma}}_\kappa$, we must have that the expression inside the square brackets is not in $\check{\mathbf{\Gamma}}_\kappa$. This expression is the intersection of a $\check{\mathbf{\Gamma}}_\kappa$ set with E, a $\mathbf{\Lambda}$ set. $\square$

Example 2.6 *Let $\kappa = o(\mathbf{\Delta})$ where $\mathbf{\Delta}$ is a projective algebra and $\operatorname{cof}(\kappa) = \omega_2$. Let $\mathbf{\Gamma}_\kappa$ be the Steel pointclass. Then $\check{\mathbf{\Gamma}}_\kappa$ is not closed under intersections with $\mathbf{\Pi}^1_2$.*

Proof. Let $\mathbf{\Lambda} = \mathbf{\Pi}^1_2$. Then $\omega_2 \in \operatorname{spec}(\mathbf{\Pi}^1_2)$. For example, we can let E be the set of x such that T_x is wellfounded, where $T \subseteq \omega \times \omega_1$ is the Kunen tree. Then $E = \bigcup_{\beta<\omega_2} E_\beta$, where

$$E_\beta = \{x \in E \colon [\gamma \mapsto |T_x| \restriction \gamma]_{W^1_1} = \beta\}.$$

This is a $\mathbf{\Sigma}^1_1$-bounded union, and $E \in \mathbf{\Pi}^1_2$ (here W^1_1 is the normal measure on ω_1). $\square$

Remark 2.7 *Every $\mathbf{\Pi}^1_2$ set is an ω_1 intersection of $\mathbf{\Delta}^1_1$ sets, and the class $\check{\mathbf{\Gamma}}_\kappa$ of the example is closed under intersections with $\mathbf{\Delta}^1_1$ sets (in fact with $\mathbf{\Sigma}^1_2$ sets) by Steel's theorem. This shows that having $B = \bigcap_{\beta<\lambda} B_\beta$ with $\lambda < \operatorname{cof}(\kappa)$, and $\check{\mathbf{\Gamma}}_\kappa$ closed under intersections with a pointclass containing all the B_β is not sufficient to guarantee that $\check{\mathbf{\Gamma}}_\kappa$ is closed under intersections with B. In contrast, the corresponding statement for unions is true by an easy argument.*

Before proceeding any further, we first recall the polarized partition property and the statement of the uniform coding lemma.

Definition 2.8 *Let $f : \boldsymbol{\delta}^2_1 \to \boldsymbol{\delta}^2_1$ be an everywhere discontinuous function, i.e $f(\alpha) > \sup\{f(\beta) : \beta < \alpha\}$. We say that $F : \boldsymbol{\delta}^2_1 \to \boldsymbol{\delta}^2_1$ is a block function if for every $\alpha < \boldsymbol{\delta}^2_1$, for every $\gamma \in [\sup_{\alpha'<\alpha} f(\alpha'), f(\alpha))$ we have $F(\gamma) \in [\sup_{\alpha'<\alpha} f(\alpha'), f(\alpha))$. We then say $C \subseteq \boldsymbol{\delta}^2_1$ is a block c.u.b set if for every $\alpha < \boldsymbol{\delta}^2_1$ we have $C \cap f(\alpha)$ is a c.u.b set in $f(\alpha)$. We then say $\boldsymbol{\delta}^2_1$ has the strong polarized partition relation property if for every partition $\mathcal{P}$ of the block functions on $\boldsymbol{\delta}^2_1$ into two pieces, there is an homogeneous block c.u.b set $H \subseteq \boldsymbol{\delta}^2_1$ such that for every $\alpha < \boldsymbol{\delta}^2_1$ and for every block function $F : \boldsymbol{\delta}^2_1 \to H$, we have an $i \in \{0, 1\}$ such that $\mathcal{P}(F) = i$.*

Theorem 11 (Uniform Coding Lemma for wellfounded relations) *Let U be universal for the class $\Sigma_1(Q)$ where Q is a binary predicate symbol. Let $\mathbf{\Gamma}$ be a any pointclass such that $\Delta_1(Q) \subseteq \mathbf{\Gamma}$ and $\exists^{\mathbb{R}}\mathbf{\Gamma} \subseteq \mathbf{\Gamma}$. Let $\preceq$ be a $\mathbf{\Gamma}$ wellfounded relation of length $o(\mathbf{\Delta})$. Then for every relation $R \subseteq \mathbb{R}^2$ such that $R = dom(\preceq)$, there exists $\varepsilon \in \mathbb{R}$ which codes, via U, a $\Sigma_1(\preceq_\alpha)$ choice set $C_\alpha \subseteq \mathbb{R}^2$ for $R_\alpha \subseteq \preceq_\alpha \times \mathbb{R}$ uniformly in $\alpha < o(\mathbf{\Delta})$.*

We next attempt generalize example 2.6 to a counterexample to Steel's conjecture, assuming $\mathrm{AD} + V = L(\mathbb{R})$. We will construct an inaccessible $\kappa \in (\boldsymbol{\delta}^2_1, \Theta)$ such that $\kappa = o(\mathbf{\Delta}_\kappa)$ for a projective algebra $\mathbf{\Delta}_\kappa$, but such that $\check{\mathbf{\Gamma}}_\kappa \wedge \mathbf{\Pi}^2_1 \nsubseteq \check{\mathbf{\Gamma}}_\kappa$. To do this we will need to assume the existence of a normal measure μ on $\boldsymbol{\delta}^2_1$ with a certain property. In the next section we attempt to remove this extra assumption.

We first give a general result about normal measures on $\boldsymbol{\delta}^2_1$. Let $C \subseteq \Theta$ be the c.u.b. subset of κ such that $\kappa = o(\mathbf{\Delta})$ for some projective algebra $\mathbf{\Delta} = \mathbf{\Delta}_\kappa$. C is c.u.b. in both Θ and $\boldsymbol{\delta}^2_1$.

Theorem 12 $(\mathbf{AD} + V = L(\mathbb{R}))$ *Let μ be a normal measure on $\boldsymbol{\delta}^2_1$. If $\kappa \doteq j_\mu(\boldsymbol{\delta}^2_1) \in C'$, then κ is a counterexample to Steel's conjecture. In fact, in this case $\check{\boldsymbol{\Gamma}}_\kappa \wedge \boldsymbol{\Pi}^2_1 \not\subseteq \check{\boldsymbol{\Gamma}}_\kappa$.*

Proof. Let μ be a normal measure μ on $\boldsymbol{\delta}^2_1$.

Since $\boldsymbol{\delta}^2_1$ has the strong partition property, there is a c.u.b. $D \subseteq C \cap \boldsymbol{\delta}^2_1$ such that $j_\mu(D) \subseteq C$ To see this, consider the partition of pairs of functions (f, g) with $f(\alpha) < g(\alpha) < f(\alpha + 1)$ for all $\alpha < \boldsymbol{\delta}^2_1$, with f, g of the correct type, according to whether $([f]_\mu, [g]_\mu) \cap C \neq \emptyset$. Using the semi-normality of μ (every c.u.b. set gets measure one), a simple sliding argument shows that on the homogeneous side the stated property holds. We use here the fact that $j_\mu(\boldsymbol{\delta}^2_1)$ is a limit point of C. If D_0 is homogeneous for the partition and $D = (D_0)'$, then D is as desired.

Fix a function $f : \boldsymbol{\delta}^2_1 \to \boldsymbol{\delta}^2_1$ which is strictly increasing, everywhere discontinuous, and such that for all α we have that $f(\alpha)$ is an regular limit Suslin cardinal which is a limit point of D. From Steel's theorem, the corresponding Steel pointclass $\boldsymbol{\Gamma}_{f(\alpha)}$ is closed under $\wedge, \vee$. Let $\kappa = [f]_\mu$. So, κ is a limit point of C, and so $\kappa = o(\boldsymbol{\Delta}_\kappa)$, with $\boldsymbol{\Delta}_\kappa$ a projective algebra. κ is also regular, which follows from the polarized finite exponent partition property for functions into the blocks $[sup_{\beta<\alpha} f(\beta), f(\alpha))$ [in fact, the polarized strong partition property holds].

We may further assume that $\boldsymbol{\Gamma}_{f(\alpha)}$ is type-4, that is, $\boldsymbol{\Gamma}_{f(\alpha)}$ is closed under real quantifiers. By picking f appropriately, we nay also assume that uniformly in $\alpha < \boldsymbol{\delta}^2_1$ there is a $f(\alpha)$ length sequence $(C^\alpha_\gamma)_{\gamma < f(\alpha)}$ of sets in $\boldsymbol{\Delta}_{f(\alpha)}$ which union to a set C^α which is $\exists^\mathbb{R} \boldsymbol{\Gamma}_{f(\alpha)} = \boldsymbol{\Gamma}_{f(\alpha)}$-complete. [There are several ways to see this. For example, we can fix a $\boldsymbol{\Sigma}^2_1$-complete set A and write $A = \bigcup_{\alpha < \boldsymbol{\delta}^2_1} A_\alpha$ with each $A_\alpha \in \boldsymbol{\Delta}^2_1$. Take $f(\alpha) > \sup_{\beta<\alpha} f(\beta)$ to be such that $f(\alpha)$ is regular, for all $\beta < f(\alpha)$ we have $|A_\beta|_W < f(\alpha)$, and $f(\alpha)$ is the Wadge degree of a type-4 pointclass. The sequence $C^\alpha_\beta = A_\beta$, for $\beta < f(\alpha)$, will union to a set C^α which is necessarily $\boldsymbol{\Gamma}_{f(\alpha)}$-complete, and this union will define a $\boldsymbol{\Gamma}_{f(\alpha)}$ prewellordering on C^α. We are using that $\boldsymbol{\Gamma}_{f(\alpha)}$ is type-4 to get that $\boldsymbol{\Gamma}_{f(\alpha)}$ is the class of $f(\alpha)$ unions of $\boldsymbol{\Delta}_{f(\alpha)}$ sets.]

We show the Steel conjecture fails at κ, in fact $\check{\boldsymbol{\Gamma}}_\kappa \wedge \boldsymbol{\Pi}^2_1 \not\subseteq \check{\boldsymbol{\Gamma}}_\kappa$. It suffices to show that $\kappa \in \mathrm{spec}(\boldsymbol{\Pi}^2_1)$. We will define $E = \bigcup_{\beta<\kappa} E_\beta$ a $\boldsymbol{\Sigma}^1_1$-bounded union with $E \in \boldsymbol{\Pi}^2_1$.

E will be the set of codes x for functions $f_x : \boldsymbol{\delta}^2_1 \to \boldsymbol{\delta}^2_1$ with

$$f_x(\alpha) \in [sup_{\beta<\alpha} f(\beta), f(\alpha)).$$

E_β with be the x in E with $[f_x]_\mu = \beta$.

If $y \in C^\alpha$ we say y is an α-code, and let $|y|_\alpha$ denote the least $\gamma < f(\alpha)$ such that $y \in C^\alpha_\gamma$. Let $U(x, \leq)$ be the universal syntactic $\boldsymbol{\Sigma}^1_1(\leq)$ set, as in the uniform coding lemma. Let $\leq$ also denote a prewellordering of length $\boldsymbol{\delta}^2_1$ such that all initial segments $\leq_\alpha = \leq \restriction \alpha$ are in $\boldsymbol{\Delta}_{f(\alpha)}$ (there is no loss of generality in assuming this). We say x is α-good if:

1. There is a $y \in U(x, \leq_\alpha)$ which is an α-code.

2. If $y, z \in U(x, \leq_\alpha)$, then y, z are α-codes and $|y|_\alpha = |z|_\alpha$.

We say x is a code if x is an α-code for all $\alpha < \boldsymbol{\delta}^2_1$. From the uniform coding lemma, for every $g : \boldsymbol{\delta}^2_1 \to \boldsymbol{\delta}^2_1$ with $g(\alpha) < f(\alpha)$ for all $\alpha < \boldsymbol{\delta}^2_1$, there is a code x with $f_x = g$. Also, the set

E of codes is $\mathbf{\Pi}^2_1$ since it a $\boldsymbol{\delta}^2_1$ intersection of $\mathbf{\Delta}^2_1$ sets (to say x is α-good is $\mathbf{\Gamma}_{f(\alpha)}$). Finally, the union $E = \bigcup_{\beta<\kappa} E_\beta$, where

$$E_\beta = \{x \in E \colon [f_x]_\mu = \beta\}$$

is easily $\mathbf{\Sigma}^1_1$-bounded. For let $S \subseteq E$ be $\mathbf{\Sigma}^1_1$. Fix $\alpha < \boldsymbol{\delta}^2_1$. Then the set $B \subseteq C^\alpha$ defined by

$$y \in B \leftrightarrow \exists x\ [(x \in S) \wedge (y \in U(x, \leq_\alpha))]$$

is in $\mathbf{\Delta}_{f(\alpha)}$. Since the $(C^\alpha_\gamma)_{\gamma<f(\alpha)}$ are $\mathbf{\Delta}_{f(\alpha)}$-bounded (this is because $\mathbf{\Gamma}_{f(\alpha)}$ was Type-4), $\{f_x(\alpha) \colon x \in S\}$ is bounded below $f(\alpha)$. $\square$

Remark 2.9 *We can give an easier proof of a version of Theorem 12, which has a slightly bigger pointclass than $\mathbf{\Pi}^2_1$. Namely, let E be the set of x which are good at α for all $\alpha < \boldsymbol{\delta}^2_1$. By good at α we mean that $U_x(\leq\restriction \alpha, <\restriction \alpha)$ is a non-empty subset of P (complete $\mathbf{\Sigma}^2_1$ set) and the reals in this set have the same norm (using a $\mathbf{\Sigma}^2_1$ norm on P). This gives a $\mathbf{\Sigma}^1_1$ bounded union of length $j_\mu(\kappa)$ whose union is E. However, we can only say that E is a $\boldsymbol{\delta}^2_1$ intersection of sets each of which is in $\mathbf{\Sigma}^2_1 \wedge \mathbf{\Pi}^2_1$. This shows $E \in \forall^{\mathbb{R}}(\mathbf{\Sigma}^2_1 \wedge \mathbf{\Pi}^2_1)$. This is enough, though, to conclude that either $j_\mu(\boldsymbol{\delta}^2_1) \notin C'$ or else we have a counteraxample to Steel's conjecture.*

Theorem 12 provides a counterexample to Steel's conjecture provided there is a normal measure μ on $\boldsymbol{\delta}^2_1$ such that $j_\mu(\boldsymbol{\delta}^2_1)$ is closed under the canonical c.u.b. set C (defining the projective algebras). Since the $j_\mu(\boldsymbol{\delta}^2_1)$ are cofinal in Θ by Woodin's results on $\boldsymbol{\delta}^2_1$ being strong in HOD, this seems reasonable, but we do not have a proof. So we state the following question.

Question 1 *Is there a normal measure μ on $\boldsymbol{\delta}^2_1$ such that $j_\mu(\boldsymbol{\delta}^2_1)$ is a limit point of C?*

2.4 Extending the argument

In this section we extend the argument to avoid having to answer Question 1. The argument, at the moment however, seems to leads to a contradiction. We will use some facts from Woodin's proof that $\boldsymbol{\delta}^2_1$ is strong to Θ in $L(\mathbb{R})$. In particular, we will use the existence of strongly normal measures on $\boldsymbol{\delta}^2_1$. We refer to [7] for a presentation of these results.

We fix a $\kappa \in (\boldsymbol{\delta}^2_1, \Theta)$ such that $\kappa \in C'$ and $\mathbf{\Gamma}_\kappa$ is type-4, that is, $\mathbf{\Gamma}_\kappa$ is closed under real quantifiers (so also $\wedge$, $\vee$). We fix an OD prewellordering $\preceq$ of length κ. We let $F \colon \boldsymbol{\delta}^2_1 \to \mathrm{HOD} \cap V_{\boldsymbol{\delta}^2_1}$ be the $\diamond$-like sequence from Woodin's proof (see [7]). Let $\mu = \mu_X$ denote the normal measure on $\boldsymbol{\delta}^2_1$ defined in Woodin's proof corresponding to the OD set $X = (\preceq, \kappa, \boldsymbol{\delta}^2_1)$. We use the following two properties of μ in the following argument:

1. μ respects the reflection filter corresponding to X. That is, if $\varphi(x, X)$ is an $\Sigma_1(x, X)$ statement with $x \in \mathbb{R}$, then there is a μ measure one set of $\alpha < \boldsymbol{\delta}^2_1$ such that $\varphi(x, F(\alpha))$ holds (these formulas are interpreted in $L(\mathbb{R})$).

2. μ is strongly normal (see the remark below).

Let $S_0 \subseteq \boldsymbol{\delta}^2_1$ be the measure one set of α for which $F(\alpha)$ is a triple $(\preceq_\alpha, \kappa_\alpha, \delta_\alpha)$ such that $\preceq_\alpha$ is a prewellordering of $\mathbb{R}$ of length κ_α. For $\alpha \in S_0$ we let $\preceq_\alpha$ denote $F(\alpha)$. Let $\kappa_\alpha = |\preceq_\alpha|$, for $\alpha \in S_0$.

Remark 2.10 *The only consequence of strong normality we need is that $[\alpha \mapsto \kappa_\alpha]_\mu = \kappa$.*

Lemma 2.11 *For μ almost all α, κ_α is in C and is the Wadge ordinal of a type-4 pointclass $\boldsymbol{\Gamma}_\alpha$.*

Proof. Let $\varphi(\leq)$ be the statement

$$\exists\alpha\ [L_\alpha(\mathbb{R}) \models \mathrm{ZF}_N + \mathrm{AD} + V = L(\mathbb{R}) + \exists\beta\ (|\preceq| = \beta \wedge \text{ there is a non-selfdual pointclass of Wadge degree } \beta \text{ which is of type-4})].$$

This is a $\Sigma_1(\preceq)$ statement about $\preceq$ which holds in $L(\mathbb{R})$ since $|\preceq| = \kappa$ and $\boldsymbol{\Gamma}_\kappa$ is of type-4. Thus, there is a μ measure one set of α for which there is an $L_\gamma(\mathbb{R})$ with $|\preceq_\alpha| < \gamma$ and for which $L_\gamma(\mathbb{R}) \models$ there is a type-4 pointclass of Wadge degree $|\preceq_\alpha| = \kappa_\alpha$. This is absolute to $L(\mathbb{R})$, so the pointclass $\boldsymbol{\Gamma}_\alpha$ of Wadge degree κ_α (with $\mathrm{pwo}(\boldsymbol{\Gamma}_\alpha)$) is of type-4 for almost all α. $\square$

Lemma 2.12 *There is μ measure one set $S \subseteq \boldsymbol{\delta}^2_1$ such that uniformly for $\alpha \in S$ there is $\boldsymbol{\Gamma}_\alpha$-complete set A^α and sets $A^\alpha_\beta \in \boldsymbol{\Delta}_\alpha = \boldsymbol{\Delta}(\boldsymbol{\Gamma}_\alpha)$ with $A^\alpha = \bigcup_{\beta<\kappa_\alpha} A^\alpha_\beta$. Moreover, the prewellordering on A^α defined by this union is a $\boldsymbol{\Gamma}_\alpha$-prewellordering.*

Proof. Fix a real x such that there is an $\mathrm{OD}(x)$ set A which is $\boldsymbol{\Gamma}_\kappa$-complete and there is an $\mathrm{OD}(x)$ sequence of sets $\{A_\alpha\}_{\alpha<\kappa}$ such that $A_\alpha \in \boldsymbol{\Delta}_\kappa$ and $A = \bigcup_{\beta<\kappa} A_\beta$.

The following is a $\Sigma_1(x, X)$ statement that is true in $L(\mathbb{R})$:

1. $\exists\gamma\ (\mathrm{L}_\gamma(\mathbb{R}) \models \mathrm{ZF}_N + \mathrm{AD} + V = L(\mathbb{R})$ and $\preceq \in \mathrm{L}_\gamma(\mathbb{R}))$.
2. $\exists\vec{s}\,\exists n(\vec{s} \in \gamma^{<\omega}$, $n \in \omega$, and $\psi_n(\vec{s}, x)$ defines in $L_\gamma(\mathbb{R})$ a $\boldsymbol{\Gamma}_\kappa$-complete set A and a sequence $\{A_\beta\}_{\beta<\kappa}$ of sets in $\boldsymbol{\Delta}_\kappa$ with $A = \bigcup_{\beta<\kappa} A_\beta$ and the prewellordering defined by this union is a $\boldsymbol{\Gamma}_\kappa$-prewellordering.)

Let $\phi(x, X)$ denote the preceding statement.

By the properties of μ there is a μ measure one set S such that $\forall\alpha \in S\ (\phi(x, F(\alpha))$. That gives that for all $\alpha \in S$ there are $\mathrm{OD}(x)$ sets A^α, $\{A^\alpha_\beta\}_{\beta<\kappa_\alpha}$ such that the conclusion of the theorem holds. It is now clear that we can pick such A^α, $\{A^\alpha_\beta\}_{\beta<\kappa_\alpha}$ uniformly in α by, for each α, picking the least $\eta_\alpha \in ON$ such that A^α, $\{A^\alpha_\beta\}_{\beta<\kappa_\alpha}$ are (x, η_α)-definable. $\square$

Fix now a μ measure one set $S \subseteq \boldsymbol{\delta}^2_1$ as in lemma 2.12. Fix also a $\boldsymbol{\Delta}^2_1$-prewellordering $\leq$ on a $\boldsymbol{\Sigma}^2_1$ complete set P. Let $U(x, \leq, <)$ be the universal syntactic $\boldsymbol{\Sigma}_1(\leq, <)$ binary formula. For any $\alpha < \boldsymbol{\delta}^2_1$, and any $x \in \mathbb{R}$, $U(x, \leq\restriction\alpha, <\restriction\alpha)$ defines a relation which we also denote $U(x, \leq\restriction\alpha, <\restriction\alpha)$. We use U and the uniform coding lemma to code functions $f\colon S \to \boldsymbol{\delta}^2_1$ with $f(\alpha) < \kappa_\alpha$.

We let $E \subseteq \mathbb{R}$ be the set of x such that for all $\alpha \in S$ we have:

1. $U(x, \leq\restriction \alpha, <\restriction \alpha) \subseteq A^\alpha$ and $U(x, \leq\restriction \alpha, <\restriction \alpha) \neq \emptyset$.
2. If $y, z \in U(x, \leq\restriction \alpha, <\restriction \alpha)$ then $|y|^\alpha = |z|^\alpha$, where $|y|^\alpha$ denotes the rank of $y \in A^\alpha$ in the prewellordering corresponding to the union $A^\alpha = \bigcup_{\beta<\kappa_\alpha} A^\alpha_\beta$.

For $x \in E$ have the corresponding function $f_x \colon S \to \boldsymbol{\delta}^2_1$ defined by $f_x(\alpha) = |y|^\alpha$ for $y \in U(x, \leq\restriction \alpha, <\restriction \alpha)$. By (1) and (2) this is well-defined and $f_x(\alpha) < \kappa_\alpha$ for all $\alpha \in S$.

For $x \in E$ let $\rho(x) = [f_x]_\mu$. So, $\rho(x) < [\alpha \mapsto \kappa_\alpha]_\mu = \kappa$ by strong normality. Also, by the uniform coding lemma the map $x \in E \mapsto \rho(x)$ is onto κ. For $\beta < \kappa$ let $E_\beta = \{x \in E \colon \rho(x) \leq \beta\}$. Thus, $E = \bigcup_{\beta<\kappa} E_\beta$.

Lemma 2.13 $E \in \boldsymbol{\Pi}^2_1$.

Proof. We have that $x \in E$ iff $\forall \alpha \in S\ (x \in B_\alpha)$, where B_α is the set of x that satisfy conditions (1) and (2) above at α. For any fixed α, since A^α and the A^α_β are $\boldsymbol{\Delta}^2_1$ (in fact in $\boldsymbol{\Gamma}_\alpha$), and since $\leq\restriction \alpha, <\restriction \alpha \in \boldsymbol{\Delta}^2_1$, we see that $B_\alpha \in \boldsymbol{\Delta}^2_1$. Since a $\boldsymbol{\delta}^2_1$ intersection of $\boldsymbol{\Delta}^2_1$ sets is $\boldsymbol{\Pi}^2_1$, we have that $E \in \boldsymbol{\Pi}^2_1$. □

Lemma 2.14 *The union $E = \bigcup_{\beta<\kappa} E_\beta$ is $\boldsymbol{\Sigma}^1_1$-bounded.*

Proof. Let $D \subseteq E$ be $\boldsymbol{\Sigma}^1_1$. Fix $\alpha \in S$. It suffices to show that $\sup\{f_x(\alpha) \colon x \in D\} < \kappa_\alpha$. We may assume without loss of generality that for all $\alpha \in S$ that $\leq\restriction \alpha \in \boldsymbol{\Delta}_\alpha$. This is because there is a c.u.b. set of α for which $\forall \beta < \alpha\ (| \leq\restriction \beta|_W < \alpha)$, and thus $\leq\restriction \alpha$ is an α union of sets of Wadge degree $< \alpha$. For such α, $\leq\restriction \alpha$ will be projective over the pointclass of Wadge degree α, and hence lie in $\boldsymbol{\Delta}_\alpha$ ($\boldsymbol{\Delta}_\alpha$ is closed under quantifiers and properly contains the pointclass of Wadge degree α as $f(\alpha) > \alpha$). The set T defined by

$$y \in T \leftrightarrow \exists x\ [x \in D \wedge y \in U(x, \leq\restriction \alpha, <\restriction \alpha)]$$

is therefore also in $\boldsymbol{\Delta}_\alpha$. Also, $T \subseteq A^\alpha$ since $D \subseteq E$. Since the prewellordering corresponding to the union $A^\alpha = \bigcup_{\beta<f(\alpha)} A^\alpha_\beta$ is a $\boldsymbol{\Gamma}_\alpha$-prewellordering, and since $\boldsymbol{\Gamma}_\alpha$ is closed under $\wedge, \vee$ (as $\boldsymbol{\Gamma}_\alpha$ is of type-4), the usual boundedness argument shows that $\sup\{|y|^\alpha \colon y \in T\} < \kappa_\alpha$. Thus, $\sup\{f_x(\alpha) \colon x \in D\} < \kappa_\alpha$. □

From lemmas 2.13, 2.14 we have that $\kappa \in \text{spec}(\boldsymbol{\Pi}^2_1)$. From lemma 2.5 it follows that $\check{\boldsymbol{\Gamma}}_\kappa \wedge \boldsymbol{\Pi}^2_1 \not\subseteq \check{\boldsymbol{\Gamma}}_\kappa$. This, however, contradicts the fact that $\boldsymbol{\Gamma}_\kappa$ is type-4, which implies that $\check{\boldsymbol{\Gamma}}_\kappa$ is closed under $\wedge, \vee$.

2.5 Positive results on closure properties of the Steel pointclass

Below we show several cases in which we have closure properties of the Steel pointclass.

Theorem 13 *Assume ZF+DC+AD. Let κ be a cardinal such that $o(\boldsymbol{\Delta}_\kappa) = \kappa$ where $\boldsymbol{\Delta}_\kappa = \boldsymbol{\Gamma}_\kappa \cap \check{\boldsymbol{\Gamma}}_\kappa$ and $\boldsymbol{\Delta}_\kappa$ is closed under $\exists^{\mathbb{R}}$, $\wedge$ and $\vee$. Assume Sep($\check{\boldsymbol{\Gamma}}_\kappa$). Let $\lambda < cof(\kappa)$ be a cardinal such that $o(\boldsymbol{\Delta}_\lambda) = \lambda$ and $\boldsymbol{\Delta}_\lambda$ is closed under $\exists^{\mathbb{R}}$, $\wedge$ and $\vee$, where $\boldsymbol{\Delta}_\lambda = \boldsymbol{\Gamma}_\lambda \cap \check{\boldsymbol{\Gamma}}_\lambda$. Assume Sep($\check{\boldsymbol{\Gamma}}_\lambda$). Suppose that $\check{\boldsymbol{\Gamma}}_\kappa \cap \boldsymbol{\Delta}_\lambda \subseteq \check{\boldsymbol{\Gamma}}_\kappa$. Then*

1. $\check{\mathbf{\Gamma}}_\kappa \cap \mathbf{\Gamma}_\lambda \subseteq \check{\mathbf{\Gamma}}_\kappa$ *and more generally if* $\mathbf{\Sigma}$ *is the pointclass of* λ *length unions of* $\mathbf{\Delta}_\lambda$ *sets, then* $\check{\mathbf{\Gamma}}_\kappa \cap \mathbf{\Sigma} \subseteq \check{\mathbf{\Gamma}}_\kappa$.

2. $\mathbf{\Gamma}_\lambda$ *is not closed under real quantifiers and* $\check{\mathbf{\Gamma}}_\kappa \cap \check{\mathbf{\Gamma}}_\lambda \subseteq \check{\mathbf{\Gamma}}_\kappa$.

3. *Suppose* $cof(\lambda) = \omega$ *and let* $\mathbf{\Lambda}$ *be the pointclass of all countable intersections of* $\mathbf{\Delta}_\lambda$ *sets, i.e* $\mathbf{\Lambda} = \bigcap_\omega \mathbf{\Delta}_\lambda$ *then* $\check{\mathbf{\Gamma}}_\kappa \cap \mathbf{\Lambda} \subseteq \check{\mathbf{\Gamma}}_\kappa$.

4. *Suppose* $cof(\lambda) = \omega_1$ *and let* $\mathbf{\Lambda}$ *be the pointclass of all length* ω_1 *intersections* $\mathbf{\Delta}_\lambda$ *sets, i.e* $\mathbf{\Lambda} = \bigcap_{\alpha<\omega_1} \mathbf{\Delta}_\lambda$ *then* $\check{\mathbf{\Gamma}}_\kappa \cap \mathbf{\Lambda} \subseteq \check{\mathbf{\Gamma}}_\kappa$. *In general if* $\lambda < \kappa$ *is any cardinal cardinal, then* $\check{\mathbf{\Gamma}}_\kappa \cap \mathbf{\Lambda} \subseteq \check{\mathbf{\Gamma}}_\kappa$ *where* $\mathbf{\Lambda}$ *is the pointclass of all intersections of* $\mathbf{\Delta}_\lambda$ *sets of length* $\mathrm{cof}(\lambda)$.

Proof.

We begin by showing $\check{\mathbf{\Gamma}}_\kappa \cap \mathbf{\Gamma}_\lambda \subseteq \check{\mathbf{\Gamma}}_\kappa$. Let then $A \in \mathbf{\Gamma}_\lambda$ and $B \in \check{\mathbf{\Gamma}}_\kappa$. Let $A = \bigcup_{\alpha<\lambda} A_\alpha$ where for every $\alpha < \lambda$, $A_\alpha \in \mathbf{\Delta}_\lambda$.

Let σ be a winning strategy for player I in the Wadge game $G_{A\cap B,B}$, that is:

$$x \notin B \rightarrow \sigma(x) \in A \cap B$$
$$x \in B \rightarrow \sigma(x) \notin A \cap B$$

As in Steel [10], we define a sequence of winning strategies $\langle \sigma_n : n \in \omega \rangle$ for I in the game $G_{A\cap B,B}$. Suppose σ_k is defined for all $k < n$. We also let τ be the copying strategy for II. For any $x \in \mathbb{R}$ we let

$$\tau_n = \begin{cases} \sigma_n & \text{if } x(n) = 0 \\ \tau & \text{if } x(n) = 1 \end{cases}$$

....	τ_3	τ_2	τ_1	τ_0
....	$x_3(0)$	$x_2(0)$	$x_1(0)$	$x_0(0)$
....	....	$x_2(1)$	$x_1(1)$	$x_0(1)$
....	....	....	$x_1(2)$	$x_0(2)$
....	....	....	....	$x_0(3)$
....	....	....	....	....
....	x_3	x_2	x_1	x_0

Table 1: Diagram of Martin-Monk games

At stage n we have a pair of $\mathbf{\Delta}_\kappa$ inseparable sets C and D such that $D \in \check{\mathbf{\Gamma}}_\kappa$. That is we have $C \subseteq B^c$ and $D \subseteq B$ with $D \in \check{\mathbf{\Gamma}}_\kappa$ and B as above. Let $E_\alpha = \{x : \sigma(x) \in A_\alpha\}$. Then we have $E_\alpha \in \mathbf{\Delta}_\lambda$. Now by assumption we have that $D \cap E_\alpha = D_\alpha \in \check{\mathbf{\Gamma}}_\kappa$. We show the following claim:

Claim 2 *For some $\alpha < \lambda$, $C \cap E_\alpha$ is $\mathbf{\Delta}_\kappa$-inseparable from $D \cap E_\alpha$.*

Proof.
Notice that since $\lambda < \mathrm{cof}(\kappa)$ and by the Coding lemma applied to $\check{\mathbf{\Gamma}}_\kappa$, for some $\alpha < \lambda$, $C_\alpha = C \cap E_\alpha$ must be $\mathbf{\Delta}_\kappa$-inseparable from D (otherwise C and D would not be $\mathbf{\Delta}_\kappa$ inseparable, since $C = \bigcup_{\alpha<\lambda} C_\alpha$. This then implies that C_α is $\mathbf{\Delta}_\kappa$ inseparable from $D_\alpha = D \cap E_\alpha$ since if not then let $F \in \mathbf{\Delta}_\kappa$ separate C_α from D_α, that is we have $C_\alpha \subseteq F$ and $F \cap D_\alpha = \emptyset$. This would then imply that $F \cap E_\alpha$ separates C_α from D.

□

Next, consider the game in which player I plays x and player II plays y and player I wins iff

$$x \notin B \to y \in C_\alpha$$
$$x \in B \to y \in D_\alpha$$

Notice that player II cannot have a winning strategy τ in this game since if τ is a winning strategy then we have

$$y \in C_\alpha \to \tau(y) \in B$$

and

$$y \in D_\alpha \to \tau(y) \notin B.$$

But this then implies that $C_\alpha \subseteq \tau^{-1}(B)$ and $\tau^{-1}(B) \cap D_\alpha = \emptyset$. But $\tau^{-1}(B), D_\alpha \in \check{\mathbf{\Gamma}}_\kappa$ so by $\mathrm{Sep}(\check{\mathbf{\Gamma}}_\kappa)$, there is a $\mathbf{\Delta}_\kappa$ set which separates C_α from D_α, contradiction!

So fix a winning strategy ρ for player I in the separation game and let $\sigma_n = \sigma \circ \rho$. Notice then that

$$x \notin B \to \rho(x) \in C_\alpha \subseteq E_\alpha,$$

so we have that $\sigma \circ \rho(x) \subseteq A_\alpha \subseteq A$. Also

$$x \in B \to \rho(x) \in D_\alpha \subseteq E_\alpha$$

so we have that

$$\sigma \circ \rho(x) \in A_\alpha \subseteq A.$$

Therefore the strategies σ_n always give a play which is in A. We also need to see that σ_n flips membership in B for every $n \in \omega$. Notice that

$$x \notin B \to \rho(x) \in C_\alpha \subseteq B^c$$

so $\sigma \circ \rho(x) \in B$. Also $x \in B \to \rho(x) \in D_\alpha$ and $\sigma \circ \rho(x) \in A$. Therefore $\sigma \circ \rho(x) \notin B$. So we have

$$x \notin B \to \sigma \circ \rho(x) \in A \cap B$$

and

$$x \in B \to \sigma \circ \rho(x) \in A \cap B^c.$$

This now allows us to derive a contradiction as in Martin-Monk proof that $\leq_W$ is a prewellorder. Namely, let $I = \{x \in \mathbb{R} : \forall^\infty n x(n) = 0\}$ and let $M = \{x \in I : x_0 \in B\}$. M has the Baire property so there is a cone N_s determined by some $s \in \omega^{<\omega}$ on which M is meager or comeager. Let $i \notin dom(s)$ and let

$$T(x)(k) = \begin{cases} x(k) & \text{if } i \neq k \\ 1 - x(k) & \text{if } i = k \end{cases}$$

T is a homeomorphism and we have $T\text{"}N_s = N_s$. Recall that x_k is the real obtained after filling the diagram of Martin-Monk game. Then if $x \in I$ then $T(x)_k = x_k$ for $i < k$ and $T(x)_k \in B$ if and only if $x_k \notin B$ if $k \leq i$. So we have

$$T\text{"}(M \cap I \cap N_s) = M^c \cap I \cap N_s.$$

But since I was comeager, this is a contradiction. This finishes the proof in the case where $\check{\mathbf{\Gamma}}_\kappa \cap \mathbf{\Gamma}_\lambda \subseteq \check{\mathbf{\Gamma}}_\kappa$.

We now proceed to show the second item in the theorem. So let $A = \bigcap_{\alpha<\lambda} A_\alpha$ with $A_\alpha \in \mathbf{\Delta}_\lambda$, so that $A \in \check{\mathbf{\Gamma}}_\lambda$. That is A is a $\mathbf{\Sigma}^1_1$ bounded intersection of sets in $\mathbf{\Delta}_\lambda$, that is the collection $\{A^c_\alpha\}_{\alpha<\lambda}$ is a $\mathbf{\Sigma}^1_1$ bounded union of sets in $\mathbf{\Delta}_\lambda$. Let $B \in \check{\mathbf{\Gamma}}_\kappa$. Next let φ be a prewellordering on a set $F \subseteq \mathbb{R}$ of length λ such that

1. All initial segments of φ are in $\mathbf{\Delta}_\lambda$.
2. $F \in \mathbf{\Gamma}_\lambda$,that is F is a $\mathbf{\Sigma}^1_1$ bounded union of $\mathbf{\Delta}_\lambda$ sets.

This is always possible since if $\mathbf{\Gamma}_\lambda$ is a Steel pointclass closed under $\forall^\mathbb{R}$ we can define $\varphi \in \mathbf{\Gamma}_\lambda$. We will denote F by F_φ. F_φ is of course in $\mathbf{\Gamma}_\lambda$. We will also let $\{F_\alpha\}_{\alpha<\lambda}$ be a λ sequence of $\mathbf{\Delta}_\lambda$ sets such that $F_\varphi = \bigcup_{\alpha<\lambda} F_\alpha$ is a $\mathbf{\Sigma}^1_1$-bounded union of $\mathbf{\Delta}_\lambda$ sets. For every $\alpha < \lambda$, we then consider the game where player I plays a real x and player II plays a real y and player II wins the run of the game iff

$$x \notin A \rightarrow \exists\alpha\exists\beta(y \in F_\alpha \wedge \varphi(y) = \beta \wedge x \notin A_{\varphi(y)}).$$

Then II has a winning strategy ρ for this game by $\mathbf{\Sigma}^1_1$-boundedness [4]. Let σ be as in the previous case. We want to define a sequence of strategies $\langle\sigma_n : n \in \omega\rangle$. At stage n we have σ_n and a pair of $\mathbf{\Delta}_\kappa$-inseparable sets C_n and D_n, where $C_n \subseteq B^c$ and $D_n \subseteq B$. For $\alpha < \lambda$, let

$$E_\alpha = \{x : \rho \circ \sigma(x) \notin F_\alpha \vee (|\rho \circ \sigma(x)| = \alpha \wedge \sigma(x) \in A_\alpha)\}.$$

Notice that we have $F_\alpha \in \mathbf{\Delta}_\kappa$. We also let as above $C_\alpha = C \cap E_\alpha$ and $D_\alpha = D \cap E_\alpha$. Then again by the coding lemma (and since $\lambda < \mathrm{cof}(\kappa)$). we must have that for some $\alpha < \lambda$, C_α must be $\mathbf{\Delta}_\kappa$-inseparable from D since if not D and C would not be $\mathbf{\Delta}_\kappa$-inseparable. We must then have that C_α must be $\mathbf{\Delta}_\kappa$-inseparable from D_α. Notice also that

$$D_\alpha = D \cap \{x : \rho \circ \sigma(x) \notin F_\alpha\} \cup (D \cap \{x : |\rho \circ \sigma(x)| = \alpha \wedge \sigma(x) \in A_\alpha\}).$$

[4] recall that A^c is a $\mathbf{\Sigma}^1_1$ bounded union of $\mathbf{\Delta}_\lambda$ sets

Then since the set $\{x : |\rho \circ \sigma(x)| = \alpha \wedge \sigma(x) \in A_\alpha\}$ and the set F_α are both in $\mathbf{\Delta}_\lambda$ and since $D \in \check{\mathbf{\Gamma}}_\kappa$ then D_α must be in $\check{\mathbf{\Gamma}}_\kappa$. Now as above we consider the separation game in which player I plays a real x and player II plays a real y and player I wins iff

$$x \notin B \to y \in C_\alpha$$
$$x \in B \to y \in D_\alpha$$

Player II cannot have a winning strategy τ in this game since then if τ is winning for II then we have

$$y \in C_\alpha \to \tau(y) \in B$$

and

$$y \in D_\alpha \to \tau(y) \notin B.$$

This would then imply that $C_\alpha \subseteq \tau^{-1}(B)$ and $\tau^{-1}(B) \cap D_\alpha = \emptyset$. But since both $\tau^{-1}(B)$ and D_α are in $\check{\mathbf{\Gamma}}_\kappa$, then by Sep($\check{\mathbf{\Gamma}}_\kappa$), there is a $\mathbf{\Delta}_\kappa$ set which separates C_α from D_α, contradiction!

So we fix a winning strategy ε for player I and we let $\sigma_n = \sigma \circ \varepsilon$. Notice that ε is winning for I for every $\alpha < \lambda$. Suppose first that $\rho \circ \sigma \circ \varepsilon(x) \in F_\alpha$. Then we have

$$x \notin B \to \varepsilon(x) \in C_\alpha \subseteq E_\alpha$$

and so we have $\sigma \circ \varepsilon(x) \in A_\alpha$ for every $\alpha < \lambda$. Also since

$$x \in B \to \varepsilon(x) \in D_\alpha \subseteq E_\alpha$$

so we have $\sigma \circ \varepsilon(x) \in A_\alpha$ for every $\alpha < \lambda$. In both cases, if $\rho \circ \sigma \circ \varepsilon(x) \notin F_\alpha$, for every $\alpha < \lambda$, then $\sigma \circ \varepsilon(x) \in A$, since ρ is winning for player II in the above game involving F_α.

Now as above this gives a contradiction by the Martin-Monk argument.

We now move to show the third item of the theorem, that is if $\mathrm{cof}(\lambda) = \omega$ and let $\mathbf{\Lambda}$ be the pointclass of all countable intersections of $\mathbf{\Delta}_\lambda$ sets, i.e $\mathbf{\Lambda} = \bigcap_\omega \mathbf{\Delta}_\lambda$, then $\check{\mathbf{\Gamma}}_\kappa \cap \mathbf{\Lambda} \subseteq \check{\mathbf{\Gamma}}_\kappa$. Notice that $\check{\mathbf{\Gamma}}_\lambda \subseteq \bigcap_\omega \mathbf{\Delta}_\lambda$. We let $A \in \bigcap_\omega \mathbf{\Delta}_\lambda$ and $B \in \check{\mathbf{\Gamma}}_\kappa$. We need to see that $A \cap B \in \check{\mathbf{\Gamma}}_\kappa$. Let $A = \bigcap_{n<\omega} A_n$, where for every $n < \omega$, $A_n \in \mathbf{\Delta}_\lambda$. As above suppose not. Then this means that player I wins the following Wadge game:

$$x \notin B \to \sigma(x) \in A \cap B$$
$$x \in B \to \sigma(x) \notin A \cap B$$

σ is a winning strategy for player I in the Wadge game $G_{A\cap B,B}$. We wish to define strategies σ_n as above such that we can fill the diagram of Martin-Monk games and derive a contradiction using the usual Martin-Monk argument. We then define the strategies σ_n inductively. Suppose σ_n has been defined at stage n. We show how to define σ_{n+1} at stage $n+1$. Define the set X_i as follows:

$$X_i = \{x : \sigma(x) \in A \wedge \exists i(\sigma \circ \sigma_n \circ ... \circ \sigma_i(x) \notin A_i)\}.$$

Notice that $X_i \in \mathbf{\Delta}_\kappa$. Then there is an i such that $B^c \cap X_i$ is $\mathbf{\Delta}_\kappa$ inseparable from $B \cap A_i$, since $B^c \cap X_i$ is $\mathbf{\Delta}_\kappa$ inseparable from B. In addition we have $\bigcap_{i<\omega} B \cap A_i = B \cap A$. This means that we can run the separation game argument: player I wins the following game

$$x \notin B \to y \in B^c \cap X_i$$
$$x \in B \to y \in B \cap A_i$$

The Martin-Monk contradiction can be carried out as above now.

□

Finally we show the fourth item of the theorem. Notice that our assumption automatically implies that $\kappa \notin \operatorname{spec}(\Lambda)$. Let $A \in \mathbf{\Lambda}$ be such that $A = \bigcap_{\alpha<\omega_1} A_\alpha$ where $A_\alpha \in \mathbf{\Delta}_\lambda$ for every $\alpha < \omega_1$. Let $B \in \check{\mathbf{\Gamma}}_\kappa$. Suppose again that $A \cap B \notin \check{\mathbf{\Gamma}}_\kappa$. Therefore we can fix a winning strategy σ for player I in the Wadge game $G_{B,A\cap B}$. Again our goal will be to define a sequence of winning strategies $\langle \sigma_n : n < \infty \rangle$ for which we can carry out the Martin-Monk contradiction. Recall that the Wadge game $G_{B,A\cap B}$ is given by:

$$x \notin B \to \sigma(x) \in A \cap B$$
$$x \in B \to \sigma(x) \notin A \cap B$$

Notice that σ flips membership in B if $\sigma(x) \in A$. For every $\alpha < \omega_1$ there are strategies for player I, $\sigma^0_\alpha, \sigma^1_\alpha, \sigma^2_\alpha, ...$ such that the following Martin-Monk diagram of games is filled up, that is for any $z \in 2^\omega$ the strategies σ^n_α are picked. Notice that we cannot pick the strategies σ^n_α in function of α since this would give an ω_1 sequence of distinct reals, which is impossible under AD.

....	τ	τ	τ	τ
....	σ^3_α	σ^2_α	σ^1_α	σ^0_α
....	$x_3(0)$	$x_2(0)$	$x_1(0)$	$x_0(0)$
....	....	$x_2(1)$	$x_1(1)$	$x_0(1)$
....	....	....	$x_1(2)$	$x_0(2)$
....	....	....	....	$x_0(3)$
....	....	....	....	....
....	x_3	x_2	x_1	x_0

Table 2: Diagram of Martin-Monk games in the $\operatorname{cof}(\omega_1)$ case

The diagram is such that for $z \in 2^\omega$, the digits of z determine which strategy is picked: either the copying strategy τ or σ^n_α for a given n. The strategies σ^n_α have the following properties. For every n,

1. If $x_{n+1} \notin B$, then $\sigma^{j,n}_\alpha(x_{n+1}) =_{def} \sigma^j_\alpha \circ ... \circ \sigma^n_\alpha(x_{n+1}) \in A, \forall j \leq n$,

2. If $x_{n+1} \in B$ then $\sigma_\alpha^{j,n}(x_{n+1}) \in A_\alpha, \forall j \leq n$ and

3. If $x_{n+1} \notin B$ then $\sigma_\alpha^n(x_{n+1}) \in B$ and if $x_{n+1} \in B$ and $\sigma_\alpha^n(x_{n+1}) \in A$ then we have $\sigma_\alpha^n(x_{n+1}) \notin B$.

We will refer below to these three properties as $(*)$. We now show the following claim:

Claim 3 *For every $n < \omega$, the strategies σ_α^n exist, for any $\alpha < \omega_1$.*

Proof.
Let $\alpha < \omega_1$ be arbitrary. We define the strategies σ_α^n inductively and start with the case $n = 0$. First notice that if $x \notin B$ then $\sigma(x) \in B \cap A \subseteq B \cap A_\alpha$ and $B \cap A_\alpha \in \check{\mathbf{\Gamma}}_\kappa$. Now B and B^c cannot be separated by a $\mathbf{\Delta}_\kappa$ set, therefore B^c cannot be separated by a $\mathbf{\Delta}_\kappa$ set from $B \cap \{x : \sigma(x) \in A_\alpha\}$, which is in $\check{\mathbf{\Gamma}}_\kappa$. Hence, there is a strategy ρ for player I in the separation game such that if $x \notin B$ then $\rho(x) \notin B$ and if $x \in B$ then $\rho(x) \in B \cap \sigma^{-1}{}'' A_\alpha$. Then let $\sigma_\alpha^0 = \sigma \circ \rho$. σ_α^0 has the above three properties $(*)$ and flips membership in B.

We now show the general successor case. Assume that $\sigma_\alpha^0, ..., \sigma_\alpha^{n-1}$ are defined. We show how to define σ_α^n. As in Steel [10], this is done in 2^n steps, depending on whether $z \in 2^\omega$ chooses τ or σ_α^n at its digits. Let

$$X_{n+1} = \{x_{n+1} : \sigma(x_{n+1}) \in A \wedge \exists i \leq n(x_i \notin A)\}.$$

Notice that $B \cap X_{n+1} = \emptyset$. Then $B^c \setminus X_{n+1}$ and B are $\mathbf{\Delta}_\kappa$ inseparable. This then implies that $B^c \setminus X_{n+1}$ and

$$B \cap \{x_{n+1} : \forall i \leq n \sigma_\alpha^i \circ \sigma_\alpha^{i+1} \circ ... \circ \sigma_\alpha^{n-1} \circ \sigma(x_{n+1}) \in A_\alpha\}$$

are $\mathbf{\Delta}_\kappa$ inseparable. Then by the separation game we have a wining strategy ρ for player I such that if $x_{n+1} \notin B$ then $\rho(x_{n+1}) \in B^c \setminus X_{n+1}$ and if $x \in B$ then we have

$$\rho(x_{n+1}) \in B \cap \{x_{n+1} : \sigma_\alpha^i \circ ... \circ \sigma_\alpha^{n-1} \circ \sigma(x_{n+1}) \in A_\alpha \text{ for all } i \leq n\}.$$

Then let $\sigma_\alpha^n = \sigma \circ \rho$.

□

We must next show how the strategies σ_α^n are explicitly defined for and $n < \omega$ and any $\alpha < \omega_1$. By the Coding lemma and uniformization we have a function $f : x \to \sigma_x^n$ on the set WO such that for $x \in$ WO, the strategies $\{\sigma_x^n\}$ are as in $\{\sigma_\alpha^n\}$ for $\alpha = |x|$. We will use the theory of generic codes of Kechris and Woodin. Fix then a generic coding function $f' : \omega_1^\omega \to \mathbb{R}$. Recall that α^ω equipped with the product of the discrete topology carries all notions of category and by AD we have have additivity of category (i.e an arbitrary union of meager sets is meager). The function $f' : \alpha^\omega \to \mathbb{R}$ is such that

1. $\forall \alpha < \omega_1, \forall \vec{\alpha} \in \alpha^\omega, f'(\alpha^\frown \vec{\alpha}) \in$ WO and

2. $\forall^* \vec{\alpha} \in \alpha^\omega |f'(\alpha^\frown \vec{\alpha})| = x$, where $|x| = \alpha$.

We now define a branch $b \in \omega_1^\omega$ which will be used to witness that we have strategies, via the functions f and f' above, for player I and which we denote by $\tilde{\sigma}_0, \tilde{\sigma}_1, ..., \tilde{\sigma}_n, ...$, from which we obtain the usual Martin-Monk contradiction.

We will define $b = lim_n b_n$, and $b_n \in \omega_1^{<\omega}$. We do this by induction so suppose then that b_{n-1} is defined. We show how to define b_n. In addition, we define a sequence of ordinals θ_n as we define the b_n for all n. The ordinals θ_n serve as witnesses for ordinal moves given by strategies employed in Becker-Kechris games, see [2] where such games first appeared. We also let $b_0 \subseteq b_1 \subseteq ... \subseteq b_n \subseteq ...$ and $b = \bigcup_n b_n$. First extend b_{n-1} to b'_n such that there is a sequence $t_n \subseteq s_n$, where $s_n \in 2^{<\omega}$ and t_n is the n^{th}-sequence in an enumeration of sequence in $2^{<\omega}$, such that

1. $\forall^*_{W^1_1} \alpha < \omega_1 \forall^*_{b'_n} \vec{\alpha} \in \alpha^\omega, \sigma^0_{f(\alpha^\frown \vec{\alpha})} \restriction n, ..., \sigma^n_{f(\alpha^\frown \vec{\alpha})} \restriction n$ are fixed, and
2. $\sigma^i_{f(\alpha^\frown \vec{\alpha})} \restriction n$ means we use $z \in 2^\omega$ to decide whether we use τ or $\sigma^i_{f(\alpha^\frown \vec{\alpha})}$ to fill the Martin-Monk diagram

This fixes the values of $\tilde{\sigma}_0 \restriction n, ..., \tilde{\sigma}_n \restriction n$. Next fix a relation

$$R(x, y) \leftrightarrow x \in \mathrm{WO} \wedge y \in A_{|x|}.$$

Let $\vec{\psi}_n$ be a scale on R. We now define $\theta_n(z)$ for all $z \in N_{t_n}$. By additivity of category, we will obtain a sequence $t_n \subseteq s_n$ such that

$$\forall^*_{s_n} z b_n(z) = b_n \wedge \theta_n(z) = \theta_n.$$

So fix $z \in N_{s_n}$ and define $\theta_n(z)$ as follows. Consider the game $G^z_{\alpha,\vec{\alpha}}$ as in Becker and Kechris: player I plays a real $x_1 \in 2^\omega$ and a sequence of ordinals below α, $\vec{\alpha}_n \in \alpha^\omega$ and player II answers by playing:

1. a real $x_2 \in 2^\omega$,
2. a sequence of ordinals $\vec{\beta}_n \in \alpha^\omega$,
3. finitely many reals $y_0, ..., y_n$,
4. finitely many sequences of ordinals $\vec{\xi}^0, ..., \vec{\xi}^n < \sup\{\vec{\psi}_n\}$,
5. finitely many reals $w_0, ..., w_n$,
6. finitely many sequences of ordinals $\vec{\gamma}^0, ..., \vec{\gamma}^n$ each ordinals of which is below ω_1 and
7. finitely many sequences of integers $\vec{\eta}^0, ..., \vec{\eta}^n$.

We have the following diagram for the game $G^z_{\alpha,\vec{\alpha}}$:

In addition player II must play so that $y_i \restriction n = \sigma^i_{f(\alpha^\frown \vec{\alpha})}(z) \restriction n$. Letting T_{WO} be the tree on $\omega \times \omega_1$ projecting to WO, the payoff is defined as follows: player II wins provided

Player I	$x_1(0)$	$x_1(1)$	$x_1(2)$	$x_1(3)$	...	...	$x_1 \in 2^\omega$
	α_0	α_1	α_2	α_3	...	...	$\vec{\alpha} \in \alpha^\omega$
	$x_2(0)$	$x_2(1)$	$x_2(2)$	$x_2(3)$	...	...	$x_2 \in 2^\omega$
	β_0	β_1	β_2	β_3	...	...	$\vec{\beta} \in \alpha^\omega$
	$y_0(0)$	$y_0(1)$	$y_0(2)$	$y_0(3)$	...	...	$y_0 \in \mathbb{R}$
	...	...	...	...	...	...	...
	$y_n(0)$	$y_n(1)$	$y_n(2)$	$y_n(3)$	...	...	$y_n \in \mathbb{R}$
	ξ^0_0	ξ^0_1	ξ^0_2	ξ^0_3	...	...	$\vec{\xi}^0 < \sup\{\psi_0\}$
	...	...	...	...	...	...	...
	ξ^n_0	ξ^n_1	ξ^n_2	ξ^n_3	...	...	$\vec{\xi}^n < \sup\{\psi_n\}$
Player II	$w_0(0)$	$w_0(1)$	$w_0(2)$	$w_0(3)$	...	...	$w_0 \in \mathbb{R}$
	...	...	...	...	...	...	...
	$w_n(0)$	$w_n(1)$	$w_n(2)$	$w_n(3)$	...	...	$w_n \in \mathbb{R}$
	γ^0_0	γ^0_1	γ^0_2	γ^0_3	...	...	$\vec{\gamma}^0 \in \omega_1^\omega$
	...	...	...	...	...	...	...
	γ^n_0	γ^n_1	γ^n_2	γ^n_3	...	...	$\vec{\gamma}^n \in \omega_1^\omega$
	η^0_0	η^0_1	η^0_2	η^0_3	...	...	$\vec{\eta}^0 \in \omega^{<\omega}$
Only fin. many moves here	...	...	...	...	...	...	...
	η^n_0	η^n_1	η^n_2	η^n_3	...	...	$\vec{\eta}^n \in \omega^{<\omega}$

Table 3: The closed game $G^z_{\alpha,\vec{\alpha}}$

$$(x_1 \restriction n, \alpha^\frown\vec{\alpha} \restriction n) \in T_{\text{wo}} \to ((x_2 \restriction n, \alpha^\frown\vec{\beta} \restriction n) \in T_{\text{wo}} \wedge (x_1 \restriction n, x_2 \restriction n, w_1 \restriction n, ..., w_n \restriction n, \eta^0 \restriction n, ..., \eta^n \restriction n, \gamma^0, ..., \gamma^n) \in S),$$

where S is a tree on ω^6 witnessing that

$$(x_1 \restriction n, |w_1|, ..., |w_n|) \in T_{\text{wo}} \restriction |x_2| \text{ and } (x_2, y_i, \xi^i) \in T_{\vec{\psi}},$$

where $T_{\vec{\psi}}$ is the tree from the scale $\vec{\psi}$. The relation $(x_1 \restriction n, |w_1|, ..., |w_n|) \in T_{\text{wo}} \restriction |x_2|$ is $\mathbf{\Sigma}^1_1$ in the codes for $w_1, ..., w_n, x_1, x_2$. This is closed game for II for if the run of the game is infinite then II wins. For each $z \in N_{s_n}$ and for each $\alpha < \omega_1$ and each $\vec{\alpha} \in \alpha^\omega$, II has a canonical winning strategy in $G^z_{\alpha,\vec{\alpha}}$. We call this canonical wining strategy $\tau^z_{\alpha,\vec{\alpha}}$. We next proceed to define

$$\theta_n(z) = \langle \theta^{\pi_0}_n(z), ..., \theta^{\pi_k}_n(z) \rangle$$

and b_n extending b'_n to satisfy the following. We first extend successively b'_n to $b^{\pi_0}_n, b^{\pi_1}_n, ..., b^{\pi_n}_n$ to obtain $b^{\pi_0}_n \subseteq b^{\pi_1}_n \subseteq ... \subseteq b^{\pi_n}_n$. We will then let $b_n = b^{\pi_n}_n$, so that b_n does not depend on which permutation we consider. This is because we must consider which $\vec{\alpha}$ player I plays. Let $\pi = \pi_i$ be a possible permutations of $n-1$. Let

$$b_n^\pi = [(\alpha_0, ..., \alpha_{n-1}) \to b_n^\pi(\alpha_0, ..., \alpha_{n-1})]_{W_1^{n-1}}.$$

This defines b_n if we define $b_n^\pi(\alpha_0, ..., \alpha_{n-1})$. Now we define $\theta_n^\pi(z)(\alpha_0, ..., \alpha_{n-1}, \alpha)$ and $b_n^\pi(\alpha_0, ..., \alpha_{n-1})$ by the following equation:

$$\forall^*_{W_1^n}(\alpha_0, ..., \alpha_{n-1}, \alpha)\forall^*_{b_n^\pi(\alpha_0,...,\alpha_{n-1})}\vec{\alpha} \in \alpha^\omega \tau^{z,\pi}_{\alpha,\vec{\alpha}} = \tau^z_{\alpha,\vec{\alpha}}(\alpha_0, ..., \alpha_{n-1}, \alpha) = \theta_n^\pi(z)(\alpha_0, ..., \alpha_{n-1}, \alpha),$$

where $x_1 \restriction n \cong \pi$ and $\tau^{z,\pi}_{\alpha,\vec{\alpha}}$ is restricted to sequences $\vec{\alpha}$ order-isomorphic to the permutation π. Notice that on a measure one set the strategies $\tilde{\sigma}_0, ..., \tilde{\sigma}_n$ are defined.

We then have a comeager set $G \subseteq 2^\omega$, which is the intersection of the comeager sets N_{s_n} defined above, where the s_n's are dense in $2^{<\omega}$. By countable additivity of the measures W_1^n we can fix the s_n and by additivity of category, a comeager set for each s_n.

We now show this next claim:

Claim 4 *For any $z \in G$ if we fill the diagram using the strategies $\tilde{\sigma}_n$ if $z(n) = 1$ and τ if $z(n) = 0$ then the resulting $y_0, y_1, ..., y_n, ...$ are in A.*

Proof.

We show that $y_i \in A_{\alpha_0}$ for all α_0 and for all i. Fix a measure one sets A_n with respect to W_1^n, so that we have

$$\forall^*_{W_1^n}(\alpha_0, ..., \alpha_{n-1}, \alpha)\forall^*_{b_n(\alpha_0,...,\alpha_{n-1})}\vec{\alpha} \in \alpha^\omega \tau^{z,\pi}_{\alpha,\vec{\alpha}} = \tau^z_{\alpha,\vec{\alpha}}(\alpha_0, ..., \alpha_{n-1}, \alpha) = \theta_n^\pi(z)(\alpha_0, ..., \alpha_{n-1}, \alpha),$$

for all $(\alpha_0, ..., \alpha_{n-1}, \alpha) \in A_n$. Let $C_n \subseteq \omega_1$ be c.u.b sets generating the A_n and let $C = \bigcap_n C_n$. Let $\alpha' > \alpha_0$ be a closure point of C. Let $x_1 \in \mathrm{WO}$ such that $|x_1| = \alpha'$. Let $(\alpha_0, \alpha_1, ...) \in C^\omega$ be such that $(x_1, \alpha_0, \alpha_1, ...) \in T_{\mathrm{WO}}$ by homogeneity of T_{WO}, where the homogeneity measures are really just the W_1^π which are the natural measures on n-tuples $\vec{\alpha}$ which are order-isomorphic to π. This then defines the sequence $b_0 = b(\alpha_0), b_1 = b(\alpha_0, \alpha_1),...$. Let $\pi_n \cong x_1 \restriction n$. From the equation we have, we can fix $\pi_0, \pi_1, ...$ such that

$$\forall^*_{b_n(\alpha_0,...,\alpha_{n-1})}\vec{\alpha} \in \alpha^\omega \theta_n^{\pi_n}(z)(\alpha_0, ..., \alpha_n, \alpha) = \tau^z_{\alpha,\vec{\alpha}}(\alpha_0, ..., \alpha_{n-1}, \alpha),$$

a run of $G^z_{\alpha,\vec{\alpha}}$ in which II has not yet lost. This then shows that II wins the run of $G^z_{\alpha,\vec{\alpha}}$ where player I plays x_1 and $(\alpha_0, \alpha_1, ...)$ as above. In this run of $G^z_{\alpha,\vec{\alpha}}$ the reals $y_0, y_1, ...$ produced are equal to $\tilde{\sigma}_0(z), \tilde{\sigma}_1(z), ...$ So we have $\tilde{\sigma}_n(z) \in A_\alpha$ for all n.

$\square$

Finally the following claim concludes the proof.

Claim 5 $\forall n, \tilde{\sigma}_n$ *flips membership in B in that if $x \notin B$ then $\tilde{\sigma}_n(x) \in B$ and if $x \in B$ and $\tilde{\sigma}_n(x) \in A$ then $\tilde{\sigma}_n(x) \notin B$.*

Proof.

We just have to modify the above game so that player I has to produce ordinals $\delta_0, \delta_1, ...$ which witness $(\tilde{\sigma}_n \restriction n, \vec{\delta} \restriction n)$ are in the tree witnessing the properties $(*)$. Therefore the $\tilde{\sigma}_n$ have the above three properties. So for $z \in G$, the $\tilde{\sigma}_n$ then give a contradiction in the Martin-Monk argument.

□

We next outline how to extend to the previous argument to work for any $\lambda < \kappa$ with λ a regular cardinal. The set up is basically the same except we need to modify the definition of the generic coding function f. We then start out by fixing a regular cardinal λ and assume that we are within scales. We fix a scale $\vec{\varphi}$ on a universal $\mathbf{\Gamma}_\lambda$ set W. Again for every $\alpha < \lambda$, one can show that the strategies σ^n_α exists. We may pick a $\lambda' > \lambda$ with $\lambda' < \kappa$ such that the scale $\vec{\varphi}$ appears. Notice that the scale $\vec{\varphi}$ may be a lot more complicated than $\mathbf{\Gamma}_{\lambda'}$. We also let T_W be the tree from the scale and assume for notational simplicity that it is a tree on $2 \times \lambda'$.

Once the strategies σ^n_α are shown to exist for every $\alpha < \lambda$ then by the Coding lemma and by uniformization we have a function $f : W \to \{\sigma^n_{|x|}\}$ such that the strategies $\{\sigma^n_{|x|}\}$ are as expected. Next we then define the generic coding function $f : (\lambda')^\omega \to \mathbb{R}$. The only difference is that now we need to take the supercompactness measures on ω_1 into account since these appear in the general definition of the generic coding function. Notice that f has the following two properties:

1. $\forall \alpha < \lambda \forall \vec{\alpha} \in \alpha^\omega f(\alpha, \vec{\alpha}) \in W$
2. $\forall \alpha < \lambda \forall^*_\nu S \in \mathcal{P}_{\omega_1}(\lambda') \forall \vec{\alpha} \in S^\omega |f(\alpha, \vec{\alpha})| = \alpha$, where $f(\alpha, \vec{\alpha}) = x$ and $|x| = \alpha$.

The main points are the following. First we fix homogeneity measures $\langle \mu_u : u \in 2^{<\omega} \rangle$ for the tree T_W. As above we must define a branch b_n and the ordinals $\theta^u_n(\alpha_0, ..., \alpha_n)$ which correspond to canonical strategies in the Becker-Kechris game. We then fix a neighborhood determined by t_n (recall these correspond to $z \in 2^\omega$ which determines which strategies to chose to fill up the Martin-Monk diagram) We then define for sequences $u \in 2^{<\omega}$ such that $lh(u) = n$ the product measure $\mu_n = \prod_{\{u : lh(u) = n\}} \mu_u$. We do this in order to handle all possible sequences u of a specific length in our quantifiers computations. Notice that if $u_0 \subseteq u_1$ then by homogeneity the measure μ_{u_1} naturally projects to μ_{u_0}. However if we have two sequence u_0 and u_1 such that $u_0 \nsubseteq u_1$ and $u_1 \nsubseteq u_0$ then we must go to a more general measure which projects to both μ_{u_0} and μ_{u_1} in order to define the ordinal, θ^u. Notice that the product measure μ_n projects to each μ_{u_i} for $i \leq k$, some $k < \omega$ and need not be normal.

We define θ^u_n as follows:

$$\forall^*_{t_n} z \forall u \in 2^n \forall^*_{\mu_n} (\alpha_0, ..., \alpha_{n-1}) \forall^*_\nu S \in \mathcal{P}_{\omega_1}(\lambda') \forall^*_{b_n(\alpha_0, ..., \alpha_{n-1})} \vec{\alpha} \in S^\omega [\theta^u_n(\pi_u(\alpha_0, ..., \alpha_{n-1}) = \tau^z_{\alpha, \vec{\alpha}}(\pi_u(\alpha_0, ..., \alpha_{n-1}))]$$

and similarly for the definition of b^u_n, where π_u is the projection map from the product measure μ_n to the homogeneity measure μ_u. When extending b_{n-1} to b_n we must use normality of the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\lambda')$ to stabilize the extension of b_{n-1}. The rest of the proof involving the Becker-Kechris game with the appropriate modifications is now as above.

To conclude this section, we show the following lemma of independent interest:

Lemma 2.15 *Let κ be a regular cardinal, then $\mathbf{\Gamma}_\kappa$ is closed under $< \kappa$ intersections.*

Proof.
Suppose not. Then we have that $\check{\mathbf{\Gamma}}_\kappa$ is not closed under $< \kappa$ unions. So let let $\delta < \kappa$ be such that $\{A_\alpha\}_{\alpha<\delta}$ be in $\check{\mathbf{\Gamma}}_\kappa$ and $A = \bigcup_{\alpha<\delta} A_\alpha \notin \check{\mathbf{\Gamma}}_\kappa$. Then by Wadge's lemma we have that $A = \bigcup_{\alpha<\delta} A_\alpha \in \mathbf{\Gamma}_\kappa$. By $\mathrm{Sep}(\check{\mathbf{\Gamma}}_\kappa)$, for every $\alpha < \delta$, there is a $\mathbf{\Delta}_\kappa$ set which separates A_α from A^c. Since κ is a regular cardinal and since $\delta < \kappa$ then there is a $\theta < \kappa$ such that for each $\mathbf{\Delta}_\kappa$ sets separating A_α from A (call them C_α), we have that $|C_\alpha|_W \leq \theta$. Next let $\mathbf{\Gamma}_0$ be a pointclass such that $\theta < o(\mathbf{\Gamma})$ and $\exists^{\mathbb{R}}\mathbf{\Gamma}_0 \subseteq \mathbf{\Gamma}_0$. Then by the coding lemma we have a $\mathbf{\Gamma}_0$ relation R such that R is the set of codes of $\mathbf{\Gamma}_0$ sets which separate A_α from A^c. But then $A \in \mathbf{\Gamma}_0$. Contradiction!
$\square$

In the next section we analyze projective-like hierarchies by means of the ordinal associated to the base of the projective-like hierarchy, $o(\mathbf{\Delta})$.

3 Characterization of type IV Projective-Like Hierarchies by the Associated Ordinals

3.1 Summary

Before we move on, we discuss the situation on the projective-like hierarchies of type **II** and **III** which arises from the above results. We will then introduce a conjecture pertaining to the characterization of type **IV** projective-like hierarchies in terms of the associated ordinal and we will give a proof to the conjecture.

First we briefly recall the situation at the level of type **I** projective-like hierarchies. Let $\mathbf{\Lambda}$ be a projective algebra. Let $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3$... be the projective like hierarchy generated by $\mathbf{\Lambda}$. Let α be the ordinal associated with $\mathbf{\Lambda}$, that is

$$\alpha = o(\mathbf{\Lambda}) = \sup\{|A|_W : A \in \mathbf{\Lambda}\}.$$

Kechris, Solovay and Steel conjectured in [6] that α alone determines which projective-like hierarchy arises. If $\mathrm{cof}(\alpha) = \omega$ then we are in the situation of a projective-like hierarchy of type **I**. We briefly recall the set up. Let $\{A_n\}$ be sets such that for every $n < \omega$, we have $|A_n|_W = \alpha_n < \alpha$. Assume that $|A_n|_W < |A_{n+1}|_W$. We then let $A = \oplus A_n$ be the join of the sets A_n. Then at A we have a selfdual degree, that is $A \equiv_W A^c$. Let $\mathbf{\Sigma}_0 = \bigcup_\omega \mathbf{\Lambda}$ be the pointclass of sets which are countable unions of sets in $\mathbf{\Lambda}$. Then $A \in \mathbf{\Sigma}_0$ and $\mathbf{\Sigma}_0$ is closed under countable unions by definition. $\mathbf{\Sigma}_0$ is closed under $\exists^{\mathbb{R}}$, since if $A(x) \leftrightarrow \exists y B(x,y)$ with $B \in \mathbf{\Sigma}_0$ and $B = \bigcup_\omega B_n$ with $B_n \in \mathbf{\Lambda}$, then we have

$$A(x) \leftrightarrow \exists y B(x,y) \leftrightarrow \exists y \exists n B_n(x,y) \leftrightarrow \exists n \exists y B_n(x,y),$$

and this last set is in $\mathbf{\Sigma}_0$ by definition. In addition $\mathbf{\Sigma}_0$ is nonselfdual pointclass. To see this, assuming all A_n as above are nonselfdual degrees, define universal sets U_n for the intermediate pointclasses $\{B : B \leq_W A_n\}$. If we let

$$U(x, y) \leftrightarrow \exists n U_n((x)_n, y)$$

then U is universal for $\boldsymbol{\Sigma}_0$. Also $\boldsymbol{\Sigma}_0$ cannot be closed under countable intersections since if it were then it would contain $\boldsymbol{\Pi}_0 = \check{\boldsymbol{\Sigma}}_0$ and therefore would not be nonselfdual. Then a type **I** projective-like hierarchy is generated in the usual way starting from $\boldsymbol{\Sigma}_0$. Notice that we have pwo($\boldsymbol{\Sigma}_0$) since we can define the natural norm φ on $A = \bigcup_n A_n$, for $A_n \in \boldsymbol{\Lambda}$ by $\varphi(x) =$ the least n such that $x \in A_n$. Then $\leq_\varphi$ and $<_\varphi$ are both countable unions of sets in $\boldsymbol{\Lambda}$.

Next if $\omega < \mathrm{cof}(\alpha)$ and α is singular then $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3, ...$ is a type **II** projective-like hierarchy. If not then $\boldsymbol{\Lambda} = \boldsymbol{\Gamma}_1 \cap \check{\boldsymbol{\Gamma}}_1$ and we are in a type **III** projective-like hierarchy, so by results of [6], we have pwo($\boldsymbol{\Gamma}_1$). Since $\boldsymbol{\Gamma}_1$ is closed under $\forall^{\mathbb{R}}$, letting

$$\alpha = \sup\{\xi : \xi \text{ is the length of a } \boldsymbol{\Delta}_1 \text{ prewellordering}\}$$

and since $\boldsymbol{\Gamma}_1$ is closed under $\wedge, \vee$, in this case by [9] we must have α is regular, contradiction. Notice that this can be seen directly using the above theorem of Steel which shows that the singularity of α implies the non-closure of $\boldsymbol{\Gamma}$ under $\vee$. Then by the results of the second section, it is true that whenever α is regular and $\alpha \notin \mathrm{Spc}(\boldsymbol{\Lambda})$, then $\boldsymbol{\Lambda}$ generates a projective-like hierarchy of type **III** or **IV**. So there are no projective-like hierarchies of type **II** for which α is regular: if $\beta = \mathrm{cof}(\alpha) < \alpha$, then the Steel pointclass in within a type **II** projective-like hierarchy and if α is regular then the Steel pointclass is at least within a type **III** projective-like hierarchy, whenever $\alpha \notin \mathrm{Spc}(\boldsymbol{\Lambda})$. If $\alpha \in \mathrm{Spc}(\boldsymbol{\Lambda})$, then we are in a special subcase of the type **III** projective-like hierarchy, which we call type $\mathbf{II}^+$. In this case, α is still a regular cardinal, however $\alpha \in \mathrm{Spc}(\boldsymbol{\Lambda})$ which implies non closure of the Steel pointclass under disjunction. In the type **IV** case we speak of an *inductive-like* hierarchy instead of a projective-like hierarchy. We summarize the situation:

1. If $\mathrm{cof}(\alpha) = \omega$, then we start a type **I** projective-like hierarchy,
2. If $\mathrm{cof}(\alpha) > \omega$ and α is singular, then we start a type **II** projective-like hierarchy,
3. If α is regular and $\alpha \in \mathrm{Spc}(\boldsymbol{\Lambda})$ then we start a type $\mathbf{II}^+$ projective-like hierarchy,
4. If α is regular and $\alpha \notin \mathrm{Spc}(\boldsymbol{\Lambda})$ then we start a type **III** projective-like hierarchy.

3.2 Characterizing the type IV case

The last item to study is the type **IV** projective-like hierarchies. We then introduce a conjecture below. To introduce the conjecture which pertains to a characterization of type **IV** projective-like hierarchies in terms of the associated ordinal, we recall some definitions from [4]. For any ordinal α, let

$$B_\alpha = \{x : \exists \gamma < \alpha, x \subseteq L_\gamma\}.$$

Notice that $L_\alpha \subseteq B_\alpha$ and B_α is a transitive set. The set of Δ_0 formulas is the closure under boolean combinations and bounded quantification of atomic formulas. A formula in the language of set theory is $\boldsymbol{\Pi}_2$ if it is of the form $\forall y \exists x \varphi$ where $\varphi \in \Delta_0$.

Definition 3.1 *A cardinal α is ${}^b\mathbf{\Pi}^1_2$-indescribable if for every $X \subseteq L_\alpha$ and for every $\mathbf{\Pi}_2$ formula φ of the language of set theory with parameters from B_α we have*

$$(B_\alpha, \in, X) \vDash \varphi \rightarrow \exists \beta < \alpha \ s.t \ (B_\beta, \in, X \cap L_\beta) \vDash \varphi$$

Given the above picture of the Wadge hierarchy, we then have the following conjecture as in [6]:

Conjecture 2 *Let $\mathbf{\Gamma}$ be any pointclass closed under $\forall^{\mathbb{R}}$ and suppose $pwo(\mathbf{\Gamma})$. Suppose $\exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta}$ and $o(\mathbf{\Delta}) = \kappa$ is ${}^b\mathbf{\Pi}^1_2$-indescribable and Mahlo. Then $\mathbf{\Gamma}$ is closed under $\exists^{\mathbb{R}}$.*

Using the above notion of ${}^b\mathbf{\Pi}^1_2$-indescribability, Kechris has shown that if κ is a Suslin cardinal such that $\omega < \mathrm{cof}(\kappa)$, then $S(\kappa)$ is closed under $\forall^{\mathbb{R}}$ if and only if κ is ${}^b\mathbf{\Pi}^1_2$-indescribable, where $S(\kappa)$ is the pointclass of all κ-Suslin sets. It is standard that $S(\kappa)$ is closed under $\exists^{\mathbb{R}}$ (see [9]). Therefore the conjecture is true if we assume that $\mathbf{\Lambda} \subseteq \mathbf{IND}$, where $\mathbf{IND}$ is the boldface pointclass of the inductive sets and where $\mathbf{\Lambda}$ generates $\mathbf{\Gamma}$, since by a result of Kechris every set in $\mathbf{IND}$ is κ-Suslin for some $\kappa < \kappa^{\mathbb{R}}$. Recall that an interval of ordinals $[\alpha, \beta]$ is a $\mathbf{\Sigma}_1$-gap if and only if

1. $L_\alpha(\mathbb{R}) \prec^{\mathbb{R}}_1 L_\beta(\mathbb{R})$
2. $\forall \xi < \alpha (L_\xi(\mathbb{R}) \not\prec^{\mathbb{R}}_1 L_\alpha(\mathbb{R}))$
3. $\forall \gamma > \beta (L_\beta(\mathbb{R}) \not\prec^{\mathbb{R}}_1 L_\gamma(\mathbb{R}))$

The scale property is depends on whether we are in a $\mathbf{\Sigma}_1$-gap. Basically, new scales appear when new $\mathbf{\Sigma}_1$ facts about the reals are verified in $L(\mathbb{R})$. Kechris has shown that once one is past the pointclass of inductive sets $\mathbf{IND}$ then the scale property no longer holds in a projective-like hierarchy of type $\mathbf{IV}$. For example, consider $\mathbf{\Pi}_1 = \forall^{\mathbb{R}}(\mathbf{IND} \vee \check{\mathbf{IND}})$. Then $\mathbf{\Pi}_1$ does not have the scale property and no $\mathbf{\Pi}_n$ or $\mathbf{\Sigma}_n$ can have the scale property. This is a gap of length ω. Past this gap the scale property resumes, since Moschovakis has shown that the pointclass $\mathbf{\Sigma}_\omega$, the least pointclass closed under $\exists^{\mathbb{R}}$ and containing $\bigcup_n \mathbf{\Sigma}_n$, has the scale property. But then, later on, longer and longer gaps occur. We feel that there are characterizations of the lengths of the $\mathbf{\Sigma}_1$ gaps in terms of the associated ordinal of the pointclass which closes a gap, but we do not know how to precisely show this.

The above conjecture is true below the first nontrivial gap in scales. Past the first $\mathbf{\Sigma}_1$ gap in scales, the conjecture remained unsolved. We show the conjecture below. In the proof we use the notion of ∞-Borel set which we first define:

Definition 3.2 (∞-Borel set) *Let $A \subseteq \mathbb{R}$. Then A is ∞-Borel if and only if there is a set $S \subseteq \gamma$, for some $\gamma \in ORD$ and a formula φ in the language of set theory such that*

$$x \in A \leftrightarrow L[S, x] \vDash \varphi[S, x]$$

$(\varphi, S) \subseteq ORD$ is the code of the ∞-Borel set A and we let $A = A_{\varphi, S}$.

Also, we use a theorem of Woodin which gives a bound on where the code of an ∞-Borel set appears.

Theorem 14 (Woodin) *Let $A \subseteq \mathbb{R}$ be an ∞-Borel set. Then there is a $\gamma < \Theta$ and a prewellorder $\preceq \in \mathbf{\Pi}^1_2(A)$ of length γ such that $S \subseteq \gamma$ and S is the Borel code of A.*

We now show the above conjecture pertaining to inductive-like hierarchies.

Theorem 15 ($\mathbf{AD} + V = L(\mathbb{R})$) *Let $\mathbf{\Gamma}$ be a Steel pointclass, that is $\mathbf{\Gamma}$ is closed under $\forall^{\mathbb{R}}$, pwo($\mathbf{\Gamma}$) and suppose that $\exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta}$. Suppose that $o(\mathbf{\Delta}) = \kappa$. Then the following are equivalent:*

1. *κ is ${}^b\mathbf{\Pi}^1_2$-indescribable and Mahlo.*
2. *$\mathbf{\Gamma}$ is closed under $\exists^{\mathbb{R}}$.*

Proof.
Recall that we are in the situation where we have Sep($\check{\mathbf{\Gamma}}$). Assume first that $\mathbf{\Gamma}$ is closed under $\exists^{\mathbb{R}}$. We need to see that κ is ${}^b\mathbf{\Pi}^1_2$-indescribable. By theorem 3.1 of [5], we must have that for every inductive-like pointclass $\mathbf{\Gamma}$, that κ is Mahlo. Let

$$\delta =_{def} \sup\{\xi : \xi \text{ is the length of a } \mathbf{\Delta} \text{ prewellordering of } \mathbb{R}\}.$$

Then by the companion theorem of Moschovakis (see theorem 9E.1 in [8]), δ is the ordinal of its admissible companion $\mathcal{M}$ above $\mathbb{R}$. So $o(\mathcal{M}) = \delta$. Since every admissible ordinal is $\mathbf{\Pi}_2$-reflecting and every set $A \subseteq L_{\delta+1}$ is Δ_1 over $\mathcal{M}$ by the coding lemma, and $|L_{\delta+1}| = \delta$, we have that δ is ${}^b\mathbf{\Pi}^1_2$-indescribable.

We must now show that $\delta = \kappa$. The result is true for any projective algebra.

Claim 6 *Let $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ be a projective algebra. Then the following ordinals are equal:*

1. *$\delta = \sup\{\xi : \xi$ is the length of a $\mathbf{\Delta}$ prewellordering of $\mathbb{R}\}$*
2. *$o(\mathbf{\Delta}) = \kappa = \sup\{|A|_W : A \in \mathbf{\Delta}\}$*

Proof.
The following argument is due to Jackson. First let $\alpha < o(\mathbf{\Delta})$ such that for some $A \in \mathbf{\Delta}$ we have $|A|_W = \alpha$. Then this initial segment determined by A in the Wadge hierarchy defines a prewellordering $\preceq$ in $\mathbf{\Delta}$ of length α, since $\mathbf{\Delta}$ is closed under quantifiers, $\vee$ and $\wedge$. We define $\preceq$ by $x \preceq y \leftrightarrow f_x^{-1}(A) \leq_w f_y^{-1}(A)$, where f_x, f_y are the Lipschitz continuous functions coded by x and y. Notice that for some $n \in \omega$, $\preceq \in \mathbf{\Sigma}^1_n(A)$ and since $\mathbf{\Delta}$ is closed under quantifiers, $\vee$ and $\wedge$ we have $\mathbf{\Sigma}^1_n(\preceq) \in \mathbf{\Delta}$. So $\alpha < \delta(\mathbf{\Delta})$, hence $o(\mathbf{\Delta}) \leq \delta(\mathbf{\Delta})$.

Next let $\alpha < \delta(\mathbf{\Delta})$. We need to see that $\alpha < o(\mathbf{\Delta})$. We will use the jump function. Let $\preceq$ be a prewellordering in $\mathbf{\Delta}$ of length α. We then construct an increasing sequence of Wadge degrees of length α. There is a function $F : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ such that

$$\text{for all } A \subseteq \mathbb{R}, A <_W F(A).$$

The function F is the *jump of* A, where we let $F(A) = A'$ be defined by

$$A'(x) \leftrightarrow (x(0) = 0 \wedge \tau_{x'}(x) \notin A) \vee (x(0) = 1 \wedge \tau_{x'}(x) \in A),$$

where x' is the shift of x, i.e $x'(n) = x(n+1)$ and $\tau_{x'}$ is the continuous function coded by x'. Notice that $F(A)$ is not Wadge reducible to either A or A^c and it has Wadge degree strictly higher to either A or A^c. For if $\tau_{x'}$ reduced A' to A then we would get $0^\frown x' \in A'$ iff $\tau_{x'}(0^\frown x') \in A$ but since

$$0^\frown x' \in A' \longleftrightarrow \tau_{x'}(0^\frown x') \notin A,$$

by definition, contradiction!

Next we define by induction on $\alpha < |\preceq|$ a $\mathbf{\Delta}$ set A_α. Let $A_0 = \emptyset$ and let $A_{\alpha+1} = A'_\alpha$. If α is a limit ordinal then let $A_\alpha(x) \leftrightarrow (|x_0|_\preceq < \alpha \wedge x_1 \in A_{|x_0|_\preceq})$. Then by definition of the jump function and by induction the A_α are strictly increasing in Wadge degrees. Now we check that each $\mathrm{AD}_\alpha \in \mathbf{\Delta}$. Let $R(x,y) \leftrightarrow x \in dom(\preceq) \wedge y \in A_{|x|_\preceq}$. We show that $R \in \mathbf{\Delta}$. We define a relation W, for $i = 0, 1$ such that if $W(x,y,i,z,w,j)$ holds means that $i = 1$ and (z,w,j) witnesses that $R(x,y)$ holds and $i = 0$ and (z,w,j) witnesses that $\neg R(x,y)$ holds. Then define $W(x,y,i,z,w,j)$ as follows:

1. $i = 1$ and x is an immediate successor of z in $\preceq$ and either $0 < y(0)$, $w = \tau_{y'}(y)$and $j = 0$ or $y(0) = 0$ and $w = \tau_{y'}(y)$ and $j = 1$,

2. $i = 1$ and x has limit rank in $\preceq$, $y_0 \preceq x, y_0 = z, w = y_1$ and $j = 1$,

3. $i = 0$ and either $x \notin dom(\preceq)$ or x is an immediate successor of z in $\preceq$ and either $0 < y(0)$, $w = \tau_{y'}(y)$ and $j = 1$ or $y(0) = 0$, $w = \tau_{y'}(y)$ and $j = 0$,

4. $i = 0$ and either $x \notin dom(\preceq)$ or x has limit rank in $\preceq$ and the following hold: $\neg y_0 \prec x \vee (z = y_0 \wedge w = y_1 \wedge j = 0$,

5. $i = 0$ and either $x \notin dom(\preceq)$ or $|x|_\preceq = 0$.

Then W is in $\mathbf{\Delta}$ as $\preceq \in \mathbf{\Delta}$. We then have:

$$R(x,y) \leftrightarrow \exists z, w, \varepsilon(z_0 = x \wedge w_0 = y \wedge \varepsilon(0) = 1 \wedge \forall i W(z_i, w_i, \varepsilon(i), z_{i+1}, w_{i+1}, \varepsilon(i+1)).$$

So $R \in \mathbf{\Delta}$, and for every $\alpha < |\preceq|$, $A_\alpha \in \mathbf{\Delta}$.

$\square$

This now finishes the proof of (2) $\rightarrow$ (1). Next we must show that whenever κ is ${}^b\mathbf{\Pi}^1_2$-indescribable and Mahlo then $\mathbf{\Gamma}$ is closed under $\exists^{\mathbb{R}}$. Assume that κ is ${}^b\mathbf{\Pi}^1_2$-indescribable. We must show that $\mathbf{\Gamma}$ is closed under $\exists^{\mathbb{R}}$. Specifically we show the following:

Claim 7 *Let $\mathbf{\Gamma}$ be a Steel pointclass such that $\exists^{\mathbb{R}}\mathbf{\Delta} \subseteq \mathbf{\Delta}$ and $\kappa = o(\mathbf{\Delta})$ is ${}^b\mathbf{\Pi}^1_2$-indescribable. Then $\mathbf{\Gamma}$ is closed under $\exists^{\mathbb{R}}$.*

Proof.

We make the general assumption that we are in the context where we do not have the scale property, since by the above remark if $\mathbf{\Gamma} \subseteq \mathbf{IND}$ or $\mathbf{\Gamma}$ is not located in a $\mathbf{\Sigma}_1$-gap, then we can localize scales to $\mathbf{\Gamma}$ or $\mathbf{\Gamma}$ sets are κ Suslin for some κ, and then by the result mentioned above of Kechris, see [4], the conjecture is true. We also work by contradiction below. Assume $\mathbf{\Gamma}$ is either located in a $\mathbf{\Sigma}_1$-gap below the last $\mathbf{\Sigma}_1$-gap $[\boldsymbol{\delta}^2_1, \Theta]$, or that $\mathbf{\Gamma}$ is located in the last $\mathbf{\Sigma}_1$ gap $[\boldsymbol{\delta}^2_1, \Theta]$. Suppose that $o(\mathbf{\Delta})$ is ${}^b\mathbf{\Pi}^1_2$-indescribable. We must see that $\mathbf{\Gamma}$ is closed under $\exists^{\mathbb{R}}$. So let $B \in \mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$ and let $A(x) \leftrightarrow \exists y B(x, y)$. Under $\mathrm{AD} + V = L(\mathbb{R})$, every set of reals is ∞-Borel, so the set B is ∞-Borel, and thus there is a formula φ and a set of ordinals $S \subseteq \gamma$ for some γ such that

$$B(x, y) \leftrightarrow L[S, x, y] \vDash \varphi(x, y),$$

see [6]. By Woodin's theorem, the Borel code S can be taken to be subset of γ, where γ is the length of a $\mathbf{\Pi}^1_2(B)$ prewellordering. So we have that $\gamma < \delta^1_2(B)$, where

$$\delta^1_2(B) = \sup\{\xi : \xi \text{ is the length of a } \mathbf{\Delta}^1_2(B) \text{ p.w.o of } \mathbb{R}\}.$$

Since $\mathbf{\Pi}^1_1(B) \subseteq \mathbf{\Gamma}$, because $\mathbf{\Gamma}$ is closed under $\forall^{\mathbb{R}}$ and by the proof of Steel's conjecture, $\mathbf{\Gamma}$ is also closed under $\vee$ as κ is regular, and since there must be a $\mathbf{\Gamma}$ prewellordering of length $\delta^1_1(B) = o(\mathbf{\Delta}^1_1(B))$ and $\delta^1_2(B) = (\delta^1_1(B))^+$, we may then assume that $S \subseteq \kappa$ and $\gamma \leq \kappa$, because $o(\mathbf{\Gamma}) = \kappa + 1$ and since one can define a $\mathbf{\Pi}^1_1(B)$ prewellordering of length $|B|_W$. We then have

$$A(x) \leftrightarrow \exists y L[S, x, y] \vDash \varphi(x, y).$$

Let (φ, S) be the Borel code of the set B. Thus

$$A(x) \leftrightarrow (B_{\kappa+1}, \in, x, (\varphi, S)) \vDash \text{“}\exists y L[S, x, y] \vDash \varphi(x, y)\text{”}.$$

This implies then that there is a $\kappa' < \kappa$ such that

$$(B_{\kappa'+1}, \in, x, (\varphi, S \restriction \kappa' + 1)) \vDash \text{“}\exists y L[S \restriction \kappa' + 1, x, y] \vDash \varphi(x, y)\text{”},$$

since “$\exists y L[S \restriction \kappa' + 1, x, y] \vDash \varphi(x, y)$” is a $\mathbf{\Pi}_2$ formula, as the satisfaction relation is Δ_1. Hence we have $A(x) \leftrightarrow \exists y L[S_1, x, y] \vDash \varphi(x, y)$ for some $S_1 \subseteq \kappa' + 1 \leq \gamma$. Let then

$$\tilde{\mathbf{\Gamma}} = \{A : A \text{ is an effective } \kappa \text{ union of } < \kappa\text{-Borel codes}\}$$

Notice then that we have $\mathbf{\Delta} \subsetneq \tilde{\mathbf{\Gamma}} \subsetneq \bigcup_\kappa \mathbf{\Delta} \subsetneq \exists^{\mathbb{R}}\mathbf{\Gamma}$. We first show that $\tilde{\mathbf{\Gamma}}$ is closed under the $\forall^{\mathbb{R}}$ quantifier. Let then $B \in \tilde{\mathbf{\Gamma}}$ and consider

$$A(x) \leftrightarrow \forall y B(x, y)$$

Now applying ${}^b\mathbf{\Pi}^1_2$-indescribability again we have that

$$A(x) \leftrightarrow \exists \gamma < \kappa(\forall y L[T, x] \vDash \varphi(x, y)),$$

where T is a Borel code of size $\leq \gamma$. This shows that $A \in \tilde{\mathbf{\Gamma}}$. So A is also in $\exists^{\mathbb{R}}\mathbf{\Gamma}$. Notice that we must then have by Wadge $\tilde{\mathbf{\Gamma}} = \mathbf{\Gamma}$. It is then sufficient to notice that $\tilde{\mathbf{\Gamma}}$ is closed under $\exists^{\mathbb{R}}$ to obtain the desired contradiction. This follows by a general argument using the Vopenka algebra to make any real of $L(\mathbb{R})$ generic over the image of $L[S,x]$ in an ultrapower by supercompactness measures (This is an argument of Caicedo and Ketchersid). This shows the theorem. However we explain briefly that the result follows directly from $\mathrm{AD}^{L(\mathbb{R})}$ using Turing-determinacy (which itself is equivalent to AD in the context of $L(\mathbb{R})$, by a result of Woodin), without having to refer to the Vopenka algebra. Let then $B \in \tilde{\mathbf{\Gamma}}$, we wish to see that $A(x) \leftrightarrow \exists y B(x,y)$ is still in $\tilde{\mathbf{\Gamma}}$. Let $\boldsymbol{d}$ denote a Turing degree. By $\forall^* \boldsymbol{d} A(\mathbf{d})$ we means that $\exists \boldsymbol{e}_0 \forall \boldsymbol{e} \geq \boldsymbol{e}_0 A(\boldsymbol{e})$, where $\leq$ is the Turing degree partial order: $x \leq \boldsymbol{d}$ means that $x \leq_T y$ for any y of Turing degree $\boldsymbol{d}$. The main point is that if we have a set $D \in \tilde{\mathbf{\Gamma}}$, then we may replace all occurrences of $\forall^* \boldsymbol{d} \exists x D(x)$ by $\exists x \forall^* \boldsymbol{d} D(x)$ by Turing determinacy.

□

We next include facts about type **IV** projective-like hierarchies. Suppose that κ is ${}^b\mathbf{\Pi}^1_2$-indescribable. Then $\mathbf{\Gamma}$ is closed under $\exists^{\mathbb{R}}$. Thus $\mathbf{\Gamma}$ is closed under both $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, hence also under countable unions and intersections. Define the pointclass $\mathbf{\Pi}_1 = \mathbf{\Gamma} \wedge \check{\mathbf{\Gamma}}$ and let $\mathbf{\Sigma}_1 = \check{\mathbf{\Pi}}_1$. A typical example of this type of hierarchy is letting $\mathbf{\Gamma} = \mathrm{IND}$, the pointclass of inductive sets. In this case, since IND is closed under continuous substitutions,$\wedge, \vee$, we define

$$\mathbf{\Sigma}^*_1(\mathbf{\Gamma}) = \{A \subseteq \mathbb{R} : \exists B \in \mathbf{\Gamma}, C \in \check{\mathbf{\Gamma}} \text{ such that } x \in A \leftrightarrow \exists y (B(x,y) \wedge C(x,y))\}.$$

Then we let

$$\mathbf{\Pi}^*_n(\mathbf{\Gamma}) = \{A^c : A \in \mathbf{\Sigma}^*_n(\mathbf{\Gamma})\}$$

and

$$\mathbf{\Sigma}^*_{n+1} = \{\exists y A(x,y) : A \in \mathbf{\Pi}^*_n(\mathbf{\Gamma})\}.$$

Notice that $\mathbf{\Pi}_1$ is closed under $\forall^{\mathbb{R}}$ since both $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$ are closed under $\forall^{\mathbb{R}}$ and $\exists^{\mathbb{R}}$. Assume that $\mathbf{\Pi}_1$ can be characterized as the pointclass of all $\mathbf{\Sigma}^1_1$ bounded unions of $\check{\mathbf{\Gamma}}$ sets of length κ, that is

$$\mathbf{\Pi}_1 = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall \alpha < \kappa (A_\alpha \in \check{\mathbf{\Gamma}}) \wedge \bigcup_{\alpha<\kappa} A_\alpha \text{ is } \mathbf{\Sigma}^1_1 \text{ bounded}\}.$$

Let $\mathbf{\Pi}'_1 = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall \alpha < \kappa (A_\alpha \in \check{\mathbf{\Gamma}}) \wedge \bigcup_{\alpha<\kappa} A_\alpha$ is $\check{\mathbf{\Gamma}}$ bounded$\}$. Our goal is to show that $\mathbf{\Pi}_1 = \mathbf{\Pi}'_1$ first and then later we verify that $\mathbf{\Pi}_1$ can indeed be characterized as the pointclass of all sets which can be written as $\mathbf{\Sigma}^1_1$-bounded unions of $\check{\mathbf{\Gamma}}$ sets.

Subclaim 1 $\mathbf{\Pi}_1 = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall \alpha < \kappa (A_\alpha \in \check{\mathbf{\Gamma}}) \wedge \bigcup_{\alpha<\kappa} A_\alpha$ *is* $\check{\mathbf{\Gamma}}$ *bounded*$\} = \mathbf{\Pi}'_1$.

Proof.
Every $\check{\mathbf{\Gamma}}$-bounded union is $\mathbf{\Sigma}^1_1$-bounded. Let $A \in \mathbf{\Pi}_1 \setminus \mathbf{\Sigma}_1$. and let $A = \bigcup_{\alpha<\kappa} A_\alpha$ where each $A_\alpha \in \check{\mathbf{\Gamma}}$, the union is $\mathbf{\Sigma}^1_1$-bounded and $\kappa = o(\mathbf{\Delta})$. We may assume that the A_α's are increasing and that the union is continuous. Then $\langle |A_\alpha|_W : \alpha < o(\mathbf{\Delta})\rangle$ is cofinal in $o(\mathbf{\Delta})$. Now for $\alpha < \kappa$ define the sets C_α by

$$C_\alpha =_{def} \{(x,y) : y \in A_{\alpha+1} \setminus A_\alpha \wedge x \text{ codes a continuous function } f_x \text{ s.t } f_x^{-1}(A_\alpha) \subseteq A\}.$$

Then notice that for each $\alpha < \kappa$, C_α is defined as $\check{\mathbf{\Gamma}} \wedge \forall^{\mathbb{R}}(\mathbf{\Gamma} \vee \mathbf{\Gamma}) = \check{\mathbf{\Gamma}} \wedge \mathbf{\Gamma}$. Then by definition, $C_\alpha \in \mathbf{\Pi}_1$. We have that if $C = \bigcup_{\alpha<\kappa} C_\alpha$, then the proof of subclaim 2.28 also shows that $C \in \exists^{\mathbb{R}}\mathbf{\Pi}_1 = \mathbf{\Sigma}_2$, since κ is regular. So let $C = \bigcup_{\alpha<\kappa} D_\alpha$ where each $D_\alpha \in \check{\mathbf{\Gamma}}$ and the union is increasing. Define the sets B_α as follows

$$z \in B_\alpha \leftrightarrow \exists(x, y) \in D_\alpha \exists\beta \leq \alpha(y \in A_{\beta+1} \setminus A_\beta \wedge f_x(z) \in A_\beta)$$

Then for every $\alpha < \kappa$, we have that $B_\alpha \in \check{\mathbf{\Gamma}}$, since $\check{\mathbf{\Gamma}}$ is closed under $\exists^{\mathbb{R}}, \wedge$ and $\vee$, by the proof of Steel's conjecture. Then we have that $\bigcup_{\alpha<\kappa} B_\alpha = A$. In addition $\bigcup_{\alpha<\kappa} B_\alpha$ is a $\check{\mathbf{\Gamma}}$-bounded union since any $\check{\mathbf{\Gamma}}$ is of the form $f_x^{-1}(A_\beta)$ for some $\beta < \kappa$ and some $x \in \mathbb{R}$. So $A \in \mathbf{\Pi}_1'$.

□

Finally we show that the pointclass $\mathbf{\Pi}_1 = \mathbf{\Gamma} \wedge \check{\mathbf{\Gamma}}$ is the pointclass of all sets which can be written as $\mathbf{\Sigma}^1_1$-bounded unions of $\check{\mathbf{\Gamma}}$ sets.

Subclaim 2 $\mathbf{\Pi}_1 = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall\alpha < \kappa(A_\alpha \in \check{\mathbf{\Gamma}}) \wedge \bigcup_{\alpha<\kappa} A_\alpha$ *is* $\mathbf{\Sigma}^1_1$ *bounded*$\}$.

Proof.
Let $\Omega = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall\alpha < \kappa(A_\alpha \in \check{\mathbf{\Gamma}}) \wedge \bigcup_{\alpha<\kappa} A_\alpha \text{ is } \mathbf{\Sigma}^1_1 \text{ bounded}\}$. We must show that $\mathbf{\Pi}_1 = \Omega$. Suppose that $A \in \mathbf{\Pi}_1$. So let $B \in \mathbf{\Gamma}$ and $C \in \check{\mathbf{\Gamma}}$ such that $A = B \cap C$. Then since $\mathbf{\Gamma}$ is a Steel pointclass, let $B = \bigcup_{\alpha<\kappa} B_\alpha$ and the union is increasing and $\mathbf{\Sigma}^1_1$-bounded and each $B_\alpha \in \mathbf{\Delta}$. Then we have that $A = \bigcup_{\alpha<\kappa} B_\alpha \cap C$. This union is a $\mathbf{\Sigma}^1_1$-bounded union of $\check{\mathbf{\Gamma}}$ sets since $\check{\mathbf{\Gamma}}$ is closed under $\wedge$ so in particular $\check{\mathbf{\Gamma}}$ is closed under intersections with $\mathbf{\Delta}$ sets. So we have $\mathbf{\Pi}_1 \subseteq \Omega$.

Next notice that since $\check{\mathbf{\Gamma}}$ is closed under $\forall^{\mathbb{R}}$ then Ω is also closed under $\forall^{\mathbb{R}}$ by Addison's argument. Let $\preceq$ be a $\mathbf{\Gamma}$ prewellordering of length κ, let φ be the $\mathbf{\Gamma}$ norm associated to $\preceq$ and let U be a universal $\check{\mathbf{\Gamma}}$ set of reals. Apply the coding lemma to obtain a relation $R(w, \varepsilon) \in \mathbf{\Gamma}$ such that

1. $\varphi(w) = \varphi(\varepsilon) \to (R(w, \varepsilon) \leftrightarrow R(z, \varepsilon))$
2. $R(w, \varepsilon) \to \varepsilon \in C$, where C is the set of codes of the sets in some sequence of $\check{\mathbf{\Gamma}}$ sets $\{A_\alpha\}_{\alpha<\kappa}$.
3. $\forall w \exists\varepsilon(R(w, \varepsilon) \wedge U_\varepsilon = A_{\varphi(w)})$.

Then we compute that $x \in \bigcup_{\alpha<\kappa} A_\alpha \to \exists w \exists\varepsilon(R(w, \varepsilon) \wedge x \in U_\varepsilon)$. Thefore we have $\bigcup_\kappa \check{\mathbf{\Gamma}} \subseteq \exists^{\mathbb{R}}(\mathbf{\Gamma} \wedge \check{\mathbf{\Gamma}})$. Now since $\mathbf{\Pi}_1 \subseteq \Omega \subseteq \mathbf{\Sigma}_2$ and since Ω is closed under $\forall^{\mathbb{R}}$ then we must have that $\mathbf{\Pi}_1 = \Omega$, since if not then by Wadge's lemma we have $\Omega \subseteq \mathbf{\Sigma}_1$ and thus $\mathbf{\Pi}_1 \subseteq \mathbf{\Sigma}_1$, contradiction!

□

Now from the above we can show that pwo($\mathbf{\Pi}_1$). The following argument is due to Jackson.

Subclaim 3 *pwo*($\mathbf{\Pi}_1$)

Proof.

Let $A \in \mathbf{\Pi}_1$ be such that $A = B \cap C$ for $B \in \mathbf{\Gamma}$ where $B = \bigcup_{\alpha<\kappa} B_\alpha$ a $\mathbf{\Sigma}^1_1$-bounded union of $\mathbf{\Delta}$ sets and $C \in \check{\mathbf{\Gamma}}$. Then we have $A = \bigcup_{\alpha<\kappa} B_\alpha \cap C$. Let $A_\alpha = B_\alpha \cap C$, so that for every $\alpha < \kappa$, $A_\alpha \in \check{\mathbf{\Gamma}}$ and $A = \bigcup_{\alpha<\kappa}$ is a $\mathbf{\Sigma}^1_1$ bounded union of $\check{\mathbf{\Gamma}}$ sets. Let φ be the natural norm on A coming from the union, i.e $\varphi(x) =$ the least γ such that $x \in A_\gamma$. We must see that φ is a $\mathbf{\Pi}_1$ norm. Since $C \in \check{\mathbf{\Gamma}}$ then let $\mathbb{R} \setminus C = \bigcup_{\xi<\kappa} C_\xi$ where for every $\xi < \kappa$, C_ξ are $\mathbf{\Delta}$ sets and the union is $\mathbf{\Sigma}^1_1$ bounded since $\mathbb{R} \setminus C$ is in $\mathbf{\Gamma}$. Let ψ be the norm coming from the union of the C_ξ, i.e the norm defined by $\psi(x) =$ the least γ such that $x \in C_\gamma$. Then the argument below applied to $\mathbf{\Gamma}$ will show that ψ is a $\mathbf{\Gamma}$ norm, and then since $\mathbf{\Gamma}$ is closed under $\wedge, \vee$ and since by $4C.11$ of [9] $\check{\mathbf{\Gamma}}$ will be bounded in the norm ψ. For every $\alpha < \kappa$, let $A^c_\alpha = C_\gamma \cup B^c_\alpha$. But then the sequence of sets $\{C_\gamma \cup B^c_\alpha\}_{\gamma<\kappa}$ is a $\check{\mathbf{\Gamma}}$ bounded union. Now let

$$x <^*_\varphi y \leftrightarrow \exists \beta < \kappa \exists \gamma \leq \beta (x \in A_\alpha \wedge x \in C_\beta \cup B^c_\alpha).$$

Notice that

$$\exists \gamma \leq \beta (x \in A_\alpha \wedge x \in C_\beta \cup B^c_\alpha)$$

defines a $\check{\mathbf{\Gamma}}$ set, since $\check{\mathbf{\Gamma}}$ is closed under union of lengths less than κ and the union is of length less than $\beta < \kappa$. So let E_β be sets in $\check{\mathbf{\Gamma}}$ such that $<^*_\varphi \bigcup_\beta E_\beta$. We need to see that this union is $\mathbf{\Sigma}^1_1$ bounded. Let $S \subseteq <^*_\varphi$ be a $\mathbf{\Sigma}^1_1$ set. Then $S_1 = \{x : \exists y S(x,y)\}$ is also $\mathbf{\Sigma}^1_1$ and $S_1 \subseteq A$, so there is a $\kappa_0 < \kappa$ such that $S_1 \subseteq A_{\kappa_0}$

□

□

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