A club version of the Kechris-Martin theorem and lightface scales

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Abstract

We answer a question of Woodin on the construction of lightface scales under determinacy assumptions. We outline a general method to construct lightface scales on sets of reals using the notion of stability under AD. The proof is reminiscent of proofs of the Kechris-Martin theorem.

1 Introduction

We begin by recalling the standard notation and objects of descriptive set theory. For any set X, we let $X^{<\omega}$ denote the set of finite sequences of elements of X. If $s \in X^{<\omega}$, then let $l(s)$ denote the length of the sequence s. a tree T of a set X is a set of finite sequence from X which is closed under initial segments.

As usual a tree T is a tree on $\omega \times \kappa$ if T is a set of pairs (s, t) such that:

1. $s \in \omega^{<\omega}$ and $t \in \kappa^{<\omega}$,

2. $l(s) = l(t)$,

3. for all $i < ls(s)$, $(s|i, t|i) \in T$.

Let T be a tree on $\omega \times \kappa$. For $s \in \omega^{<\omega}$, we have the section of the tree T at s:

$$T_s = \{t \in \kappa^{<\omega} : (s, t) \in T\}$$

and for each $x \in \omega^{<\omega}$, let

$$T_x = \bigcup \{T_{x|k} : k \in \omega\}$$

The body of the tree is defined as

$$[T] = \{(x, f) : x \in \omega^\omega, f \in \kappa^\omega \text{ and for all } k \in \omega, (x|k, f|k) \in T\}$$

$[T]$ is then the set of infinite branches of the tree T. The projection of $[T]$ onto the first coordinate is defined as

$$p[T] = \{x \in \omega^\omega : (x, f) \in [T] \text{ for some } f \in \kappa^\omega\}$$

A set of reals is κ-Suslin if there is some tree T on $\omega \times \kappa$ such that $A = p[T]$. This notion is of course interesting only in a choiceless context: in the constructible universe L, every set of reals is $\aleph_1$-Suslin.

If T is an illfounded tree on a set wellordered set $(X, <)$, then the leftmost branch of T, $l(T) = (l(0), l(1), ...)$ is defined as follows: $l(0)$ is the $<$-least element of X such that there exists an $x \in X^\omega$ with $x|1 = l(0)$ and $x \in [T]$. In general, we define inductively $l(n)$ to be the $<$-least element of X such that there exists a branch $f \in X^\omega$ with the property that $f|n = l|n$ and $f \in [T]$. The branch l is leftmost in the sense that whenever $g \in [T]$ and $g \neq l$, then considering the least $n \in \omega$ such that $l(n) \neq g(n)$, we have $l(n) < g(n)$.

A pointclass Γ is a collection of sets of reals closed under continuous inverse images, that is:

$$\text{if } f : \omega^\omega \to \omega^\omega \text{ is continuous and } A \subseteq \omega^\omega \text{ is in } \Gamma \text{ then } B = f^{-1}[A] \in \Gamma$$

The scale property is a central notion in descriptive set theory. It turns out to be equivalent to being Suslin. We recall the following basic definition:

Definition 1.1 (The scale property) 1. *A semiscale is a sequence of norms $\langle \phi_n \rangle$ on a set A such that whenever we have a sequence $\{x_n\} \subseteq A$ converging to some x and for every $n, \phi_n(x_i)$ is eventually constant then $x \in A$. If in addition we have the lower semi-continuity property, $\phi_n(x) \leq \lim \phi_n(x_i)$ then the sequence of norms $\langle \phi_n \rangle$ is a scale.*

2. *A scale $\langle \phi_n \rangle$ is a $\underset{\sim}{\Gamma}$-scale if for every n, ϕ_n is a $\underset{\sim}{\Gamma}$-norm. The pointclass Γ has the scale property if every $\underset{\sim}{\Gamma}$ set has a $\underset{\sim}{\Gamma}$-scale.*

3. *A scale $\langle \phi_n \rangle$ on a set A is good if whenever $\{x_n\} \subseteq A$ and for all $n \in \omega$, $\varphi_n(x_m)$ is eventually constant, then $x = \lim x_m$ exists and $x \in A$.*

4. *A scale $\langle \phi_n \rangle$ on a set A is very-good if $\langle \phi_n \rangle$ is good and whenever $x, y \in A$ and $\varphi_n(x) \leq \varphi_n(y)$ then $\varphi_k(x) \leq \varphi_k(y)$ for all $k < n$.*

5. *A scale $\langle \phi_n \rangle$ on a set A is excellent if it is very good and whenever $x, y \in A$ and $\varphi_n(x) = \varphi_n(y)$, then $x \upharpoonright n = y \upharpoonright n$.*

The following theorem is the *second periodicity theorem.* It shows that under suitable determinacy assumption we can propagate the scale property.

Theorem 1 (Moschovakis) *Assume projective determinacy. Then every $\underset{\sim}{\Pi}^1_{2n+1}$ and every $\underset{\sim}{\Sigma}^1_{2n}$ have the scale property.*

Assuming determinacy hypotheses, one central theme of descriptive set theory is finding methods and techniques which allow the propagation of the scale property to sets of reals throughout the Wadge hierarchy. In particular it is useful to know where the next Suslin cardinal appear and how to obtain scale to optimal complexity on sets of reals immediately at the next level of complexity in the Wadge hierarchy.

One answer to this general problem was given by Martin and Solovay, see [3]. We recall this result and the technical set-up behind it.

Definition 1.2 *(homogeneous tree)*

A tree T on $\omega \times \kappa$ is said to be homogeneous of there is a family of measures $\langle \mu_s : s \in \omega^\omega \rangle$ satisfying :

1. *Each μ_s is a measure on T_s and $\mu_s(T_s) = 1$,*

2. *If t extends s then μ_t projects to μ_s,*

3. *For every $x \in \mathbb{R}$, if T_x is illfounded then for any sequence $\{A_n : n \in \omega\}$ of measure one sets with $\mu_{x \upharpoonright n}(A_n) = 1$, there a branch $f \in \kappa^\omega$ such that for all $n, (x \upharpoonright n, f \upharpoonright) \in T$.*

   T is δ-homogeneous if in addition the measures are δ-complete.

The second clause in the above definition is what makes the tower of measures be countably complete. It is a standard fact that a tower of measures is countably complete if and only if the direct limit of the ultrapowers given by the measures μ_s is wellfounded. We say a tree T is κ-homogeneous if the measures μ_s can be taken to be κ-complete. A set $A \subseteq \mathbb{R}$ is κ-homogeneously-Suslin if $A = p[T]$ for T a κ-homogeneous tree.

If X is a set then we let $m(X)$ denote the set of countably complete ultrafilters on the space $\mathcal{P}(X)$. A special case of interest for us is when $X = \kappa^{<\omega}$ for some ordinal κ. If $\mu \in m(\kappa^{<\omega})$, then by countable completeness, μ must concentrate on κ^n for some $n \in \omega$, i.e there must be a $n \in \omega$ such that $\mu(\kappa^n)$.

Definition 1.3 *Suppose T is a tree on $\omega \times \kappa$. The tree T is δ-weakly homogeneous if there is a partial function*

$$\pi : \omega^{<\omega} \times \omega^{<\omega} \to m(\kappa^{<\omega})$$

such that

1. *if $(s,t) \in dom(\pi)$ then $\pi(s,t)(T_s) = 1$ and $\pi(s,t)$ is a δ-complete measure,*
2. *for all $x \in \omega^{<\omega}$, $x \in p[T]$ if and only if there exists $y \in \omega^\omega$ such that*
   - *(a) for every $k \in \omega$, $(x|k, y|k) \in dom(\pi)$,*
   - *(b) for any sequence of measure one sets $\langle A_k : k \in \omega\rangle$, there is a $f \in \kappa^\omega$ such that $f|k \in A_k$, for all $k \in \omega$.*

The notions of homogeneity and weak-homogeneity are connected by the following fact. Let $A \subseteq \omega^\omega$. Then A is δ-weakly homogeneously Suslin if and only if A is the continuous image of a set B which is δ-homogeneously Suslin.

Let A be a weakly-homogeneously Suslin set of reals and fix a weakly-homogeneous tree T on $\omega \times \kappa$ for some ordinal κ such that $A = p[T]$. Then the Martin-Solovay construction gives a homogeneous tree representation for $\neg A$, that is for $x \in \neg A$, we let

$$\varphi_n(x) = [f_{x,t_n}]_{\mu_{x|l(t_n),t_n}}$$

More precisely, the Martin-Solovay tree which gives a Suslin representation to $\neg A$ is defined as follows:

Definition 1.4 (Martin-Solovay tree) *Let T be a weakly-homogeneous tree on $\omega \times \kappa$ with measures $\mu_{s,t}$ each of which concentrates on $\kappa^{lh(s)}$. The Martin-Solovay tree S is defined as follows. Let (s_i, t_i) be a reasonable enumeration of all $(s,t) \in \omega^{<\omega} \times \omega^{<\omega}$ satisfying $l(s) = l(t)$, in the sense that any proper extension of any (s_i, t_i) is enumerated at a later stage. Define $(s,t) \in S$ if and only if there is a function $f : T_s \to \kappa^+$ which is order-preserving with respect to the Brouwer-Kleene ordering $<^s_{BK}$ on T_s such that for every $i < l(s)$,*

$$\alpha_i = [f^i]_{\mu_{s_i,t_i}},$$

where

$$f^i = f|\{\vec{\gamma} : (s_i, \vec{\gamma}) \in T\}$$

A standard reformulation of the above definition is that the Martin-Solovay tree attempts to ill-found the direct limit of all ultrapowers by the homogeneity measures on the tree T. To illustrate the Martin-Solovay construction we consider the following situation. First for any pointclass $\underset{\sim}{\Gamma}$ and any ordinal κ, let $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ be a sequence of sets of reals. Let $\bar{\mathcal{A}}$ be the collection of all sets of reals A such that for all countable $S \subseteq \omega^\omega$, there is an $\alpha < \kappa$ such that $S \cap A = S \cap A_\alpha$. Let then

$$\Lambda(\underset{\sim}{\Gamma}, \kappa) = \bigcup\{\bar{\mathcal{A}} : \mathcal{A} \subseteq \underset{\sim}{\Gamma} \wedge |\mathcal{A}| \leq \kappa\}$$

where $|\mathcal{A}|$ is the cardinality of the collection $\mathcal{A}$. In practice, κ is taken to be $\underset{\sim}{\delta}(\Gamma)$. $\Lambda(\underset{\sim}{\Gamma}, \kappa)$ is the same as $Env(\underset{\sim}{\Gamma})$ as defined in [4].

Next, suppose $\underset{\sim}{\Gamma}$ is closed under $\forall^{\omega^\omega}$ and $\mathbf{pwo}(\underset{\sim}{\Gamma})$. Assume also that the pointclass $\exists^{\omega^\omega}\underset{\sim}{\Gamma}$ has the scale property with all norms onto $\delta(\underset{\sim}{\Gamma}) = \kappa$. Assume there is a Suslin cardinal $\xi > \kappa$. Then by the Martin-Solovay construction the pointclass $\forall^{\omega^\omega}\underset{\sim}{\Gamma}$ admits a semi-scale each of which norms is in the pointclass $\Lambda(\underset{\sim}{\Gamma}, \kappa)$. The leftmost branches of the tree S give a scale on a universal $\forall^{\omega^\omega}\underset{\sim}{\Gamma}$, however this scale has higher complexity in the Wadge hierarchy than the pointclass $\Lambda(\underset{\sim}{\Gamma}, \kappa)$

The solution to the problem of obtaining a scale from the above semi-scale given by the Martin-Solovay construction in certain situations is the notion of a *stable* tree introduced by Jackson and we recall here the notions involved in the the definition of a stable tree.

Let T be a tree on $\omega \times \omega \times \kappa$ be homogeneous via the measures $\mu_{s,t}$ on $\kappa^{<\omega}$. So, if we identify the last two coordinates of the tree into a single coordinate by a bijection between $\omega \times \kappa$ and κ, the resulting tree T' on $\omega \times \kappa$ is weakly homogeneous.

Recall that a sequence $A_{s,t}$ of measure one sets with respect to the $\mu_{s,t}$ is said to stabilize the tree T if for all x such that T_x is wellfounded we have that for any measure one sets $B_{x\restriction n,t}$ and for any $t \in \omega^{<\omega}$ with has length n, we have $[f^{\vec{A}}_{x\restriction n,t}]_{\mu_{x\restriction n,t}} \leq [f^{\vec{B}}_{x\restriction n,t}]_{\mu_{x\restriction n,t}}$. Here $f^{A}_{x\restriction n,t}(\vec{\alpha})$ is the rank of the tuple $(x \restriction n, t, \vec{\alpha})$ in the tree

$$T_x \restriction \vec{A} = \{(u, \vec{\beta})\colon (x \restriction \mathrm{lh}(u), u, \vec{\beta}) \in T \wedge \forall k \leq n \ (\vec{\beta} \restriction k \in A_{x\restriction k, t\restriction k})\}.$$

We similarly define $f^{B}_{x\restriction n,t}(\vec{\alpha})$. That is the functions $f^{A}_{x\restriction n,t}$ are the ranking subfunctions of the canonical ranking function $f_x : T_x \to \mathrm{ORD}$, for x such that T_x is wellfounded, when the tree is restricted to measure one sets. We isolate the notion of stability in the following definition:

Definition 1.5 *T and $\{\mu_{s,t}\}$ is a stability system if there are measure one sets $A_{s,t}$ such that for all measure one sets $B_{s,t}$ and all $x \in \neg B$, we have*

$$[f^{\vec{A}}_{x,t_n}]_{\mu_{s,t}} \leq [f^{\vec{B}}_{x,t_n}]_{\mu_{s,t}}$$

Lemma 1.6 (Jackson) *Let T be a stable homogeneous tree as witnessed by measures $\{\mu_s : s \in \omega^{<\omega}\}$ and measure one sets $\{A_s : s \in \omega^{<\omega}\}$. Let T' be the Martin-Solovay tree with $B = p[T']$ constructed from $T^{\vec{A}}$ and μ_s for $s \in \omega^{<\omega}$. Let $\vec{\varphi}$ be the corresponding semi-scale given by for $x \in B$,*

$$\varphi_n(x) = [f^{\vec{A}}_{x\restriction n}]_{\mu_{x\restriction n}}.$$

Then $\vec{\varphi}$ is a scale.

Theorem 2 (Jackson) *Assume AD. Let ξ be the supremum of the Suslin cardinals if it exists. Then every weakly homogeneous tree T on $\omega \times \kappa$, as witnessed by a sequence of measures $\{\mu_{s,t}\}$, is stable, where $\kappa < \xi$.*

Proofs of the above lemma and theorem are in [1]. In particular it then easily follows that the semi-scale given by the Martin-Solovay construction is a scale when the weakly-homogeneous tree T is restricted to a specific sequence $\vec{A}$. As a corollary, one thus obtains a complete boldface scale analysis under AD, see chapter 3 of [2].

Question 1 (Woodin) *Is there an effective procedure which refines the boldface scale analysis? Can the scale analysis under AD be refined to obtain lightface scales?*

The goal of this paper is to provide a positive answer to this question. We show a specific procedure which gives lightface scales under AD.

2 Stabilizing $\mathbf{\Delta}^1_1$ homogeneous trees

We will show our main lemma below in the context of ω_1. We later outline the necessary changes to fully generalize the main lemma in $L(\mathbb{R})$. Let U on $\omega \times \omega_1$ be the Kunen tree. It then follows that for every function $f : \omega_1 \to \omega_1$, there is an $x \in \omega^\omega$ such that U_x is wellfounded and $\forall^* \alpha f(\alpha) < |U_x \restriction \alpha|$.

The Kunen tree U may also be used to code club sets of ω_1 in the following way. Let $x \in \omega^\omega$ be such that U_x is wellfounded. Then define

$$C_x = \{\alpha < \omega_1 : \alpha > \omega \wedge \forall \beta < \alpha |U_x \restriction \beta| < \alpha\}$$

It now follows that for every closed unbounded set $C \subseteq \omega_1$, there is an $x \in \omega^\omega$ such that U_x is wellfounded and $C_x \subseteq C$.

Suppose we are given a fixed coding of ordinal less than some ordinal κ (say using a norm on a set of reals or a more complex coding as in chapter 4 of [2]). Letting a real x be a code of an ordinal $\alpha < \kappa$, let $|x|$ be the ordinal α. We then say a set $A \subseteq \kappa$ is Γ in the codes if the set

$$A' = \{x : |x| = \alpha \in A\}$$

is a Γ set.

Theorem 3 *Let T be a tree on $\omega \times \omega \times \omega_1$ which is homogeneous with measures W^n_1 (i.e., the n-fold products of the normal measure on ω_1). Assume also that T is Δ^1_1 in the codes. Then there is a c.u.b. $C \subseteq \omega_1$ which stabilizes T and such that C is Δ^1_3 in the codes.*

Proof. Let $U \subseteq \omega \times \omega_1$ be the Kunen tree. If U_x is wellfounded, then let $f_x : \omega_1 \to \omega_1$ be the function $f_x(\alpha) = |U_x \restriction \alpha|$. In this case, let

$$C_x = \{\alpha < \omega_1 : \forall \beta < \alpha \ f_x(\beta) < \alpha\}$$

be the c.u.b. set coded by x. For every c.u.b. $C \subseteq \omega_1$ there is an x with U_x wellfounded and $C_x \subseteq C$.

For $w \in \omega^\omega$, and $\alpha < \omega_1$, we say w is weakly α-good if for all $\beta \leq \alpha$ either $U_w \restriction \beta$ is wellfounded of rank $< \alpha$ or α is in the wellfounded part of $U_w \restriction \beta$. We say w is strongly α-good if for all $\beta \leq \alpha$ we have that $U_w \restriction \beta$ is wellfounded. We say w is $< \alpha$ weakly (strongly) good if for all $\alpha' < \alpha$, w is weakly (strongly) α'-good. Let WG_α be the set of w which are α-weakly good, and SG_α the set of w which are strongly α-good. Likewise define $\mathrm{WG}_{<\alpha}$ and

$\mathrm{SG}_{<\alpha}$. These sets are defined with respect to the tree U, and so we also write WG^U_α, SG^U_α. We can also speak of good with respect to the tree T, and so write WG^T_α, SG^T_α. Note that WG^U_α, $\mathrm{WG}^U_{<\alpha}$ are $\utilde{\Delta}^1_1$ (SG_α is $\utilde{\Pi}^1_1$).

Consider now the game G where I plays out w_1, y, and II plays out w_2. II wins the run iff there is an $\eta < \omega_1$ such that one of the following holds:

1. $w_1 \in \mathrm{WG}^U_{<\eta}$, $y \in \mathrm{WG}^T_{<\eta}$, $w_2 \in \mathrm{SG}^U_\eta$, with either $w_1 \notin \mathrm{WG}^U_\eta$ or $y \notin \mathrm{WG}^T_\eta$, and $w_2 \in \mathrm{SG}^T_\eta$.
2. $w_1 \in \mathrm{WG}^U_\eta$, $y \in \mathrm{WG}^T_\eta$, $w_2 \in \mathrm{SG}^T_\eta$, and there is a $\gamma \leq \eta$ such that (i) $\forall \beta < \gamma |U_{w_1} \restriction \beta| < \gamma$, (ii) $\forall \beta < \gamma |U_{w_2} \restriction \beta| < \gamma$, (iii) $P_\gamma(w_1, y, w_2)$.

Here $P_\gamma(w_1, y, w_2)$ are, uniformly in γ, $\utilde{\Pi}^1_1$ relations such that if $T_y \restriction \gamma$ is wellfounded and w_1, w_2 satisfy (1) and (2), then $P_\gamma(w_1, y, w_2)$ holds iff $|T_y \restriction (C_{w_2} \cap \gamma)| \leq |T_y \restriction (C_{w_1} \cap \gamma)|$.

Note that this is a Σ^1_2 game for II. So, if II wins G, then II has a Δ^1_3 winning strategy.

Claim 1 *II has a winning strategy for G.*

Proof.

Let $C \subseteq \omega_1$ be c.u.b. and stabilize T. Let w_2 code a c.u.b. set and such that $C_{w_2} \subseteq C$. Let II play w_2 in G. Suppose I plays w_1, y. If either w_1 or y is not α-weakly good for some $\alpha < \omega_1$, then II wins by clause (1) as w_2 is α-strongly good for all α. So assume w_1, y are α-weakly good for all α. Thus, U_{w_1} and T_y are wellfounded. So, C_{w_1} and C_{w_2} are defined. As C_{w_2} still stabilizes T we have that

$$[F_y^{C_{w_2}}]_{W^1_1} \leq [F_y^{C_{w_1}}]_{W^1_1}.$$

It follows that there is an $\alpha < \omega_1$ (in fact, a c.u.b. set) with $\alpha \in C_{w_1} \cap C_{w_2}$ and such that

$$|T_y \restriction C_{w_2} \cap \alpha| \leq |T_y \restriction C_{w_1} \cap \alpha|.$$

Thus II has won by clause (2).

Let τ be a Δ^1_3 winning strategy for II. We define a c.u.b. set C^τ which stabilizes T. To do this, we first define inductively a function $b\colon \omega_1 \to \omega_1$. Assume $b(\beta)$ is defined for all $\beta < \alpha$. Let

$$(w_1, y) \in W_\alpha \leftrightarrow [w_1 \in \mathrm{WG}^U_\alpha \wedge y \in \mathrm{WG}^T_\alpha \wedge \neg \exists \gamma \leq \alpha \text{ (II wins by clause (2) at } \gamma)]$$

So, $W_\alpha \in \mathbf{\Sigma}^1_1$. We also easily have that $W_\alpha \neq \emptyset$. If $(w_1, y) \in W_\alpha$ and $w_2 = \tau(w_1, y)$, then w_2 is α-strongly good, that is, $U_{w_2} \restriction \alpha$ is wellfounded. That is, $f_{w_2}(\alpha) = |U_{w_2} \restriction \alpha|$ is defined. By boundedness we then have that

$$b(\alpha) = \sup\{f_{\tau(w_1,y)}(\alpha)\colon (w_1, y) \in W_\alpha\} < \omega_1.$$

This completes the definition of the b function. Let C_b be the set of closure points of b. We claim that C_b stabilizes T. Suppose not, and let C_1, y be such that T_y is wellfounded and $[F_y^{C_1}]_{W^1_1} < [F_y^{C_b}]_{W^1_1}$. Let C_2 be c.u.b. such that

$$F_y^{C_1}(\alpha) < F_y^{C_b}(\alpha)$$

for all $\alpha \in C_2$. Let w_1 code a c.u.b. set such that $C_{w_1} \subseteq C_1 \cap C_2$. Let I play w_1, y against τ. Let $w_2 = \tau(w_1, y)$. We have that U_{w_1}, U_{w_2}, and T_y are wellfounded.

We claim that for all $\alpha < \omega_1$ that $b(\alpha) \geq f_{w_2}(\alpha) = |U_{w_2} \restriction \alpha|$. We show this inductively on α. Assuming this holds below α, we have that $C_b \cap \alpha \subseteq C_{w_2} \cap \alpha$. From the definitions of C_1 and C_2, there cannot be an $\eta \in C_{w_1}$ such that

$$F_y^{C_b}(\eta) \leq F_y^{C_{w_1}}(\alpha).$$

In particular, there cannot be an $\eta \leq \alpha$ in $C_{w_1} \cap C_{w_2}$ for which $F_y^{C_{w_2}}(\eta) \leq F_y^{C_{w_1}}(\alpha)$. That is, there cannot be an $\eta \leq \alpha$ such that II wins by clause (2) at η. Thus, $(w_1, y) \in W_\alpha$. From the definition of the b function we now have that $b(\alpha) \geq f_{w_2}(\alpha)$.

Since $b(\alpha) \geq f_{w_2}(\alpha)$ for all α, we now have that $C_b \subseteq C_{w_2}$. Again from the definitions of C_1 and C_2 we have that there cannot be an $\eta \in C_{w_1}$ such that

$$F_y^{C_b}(\eta) \leq F_y^{C_{w_1}}(\alpha).$$

So, there cannot be an $\eta \in C_{w_1}$ such that

$$F_y^{C_{w_2}}(\eta) \leq F_y^{C_{w_1}}(\alpha).$$

This shows that II has not won by clause (2), and since all the reals are fully good, I has won the run, a contradiction.

So, C_b is a c.u.b. subset of ω_1 which stabilizes T. Since τ is Δ^1_3, it follows that b is Δ^1_3 in the codes, and hence that C_b is Δ^1_3.

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Finally we show that the relation

$$R(z_1, z_2) \longleftrightarrow z_1, z_2 \in WO \wedge b(|z_1|) = |z_2|$$

is Δ^1_3. We have $R(z_1, z_2)$ holds iff the following holds:

1. $z_1, z_2 \in WO$,
2. $\exists y \in \mathbb{R}$ and $z \in WO$ with $|z| = |z_1| + 1$ and $|0|_{\prec_z} = |z_1|$ satisfying:
   (a) $\forall n, y_n \in WO$
   (b) the map $n \longmapsto |y_n|$ defines an order preserving map from $\prec_z$ to ω_1,
   (c) $\forall n \in dom(\prec_z)$,
   $$|y_n| = \{f_{\tau(w_1,y)}(|n|_{\prec_z}) : \forall m \prec_z n[(w_1, y) \text{ is } |m|_{\prec_z}\text{–good} \wedge \text{ II doesn't win by the second clause.}\}$$
   (d) $|y_0| = |z_2|$.

So R is $\Sigma^1_2(\tau)$, so it is Δ^1_3, so $rng(b) = C$ is Δ^1_3. This concludes the proof of the lemma.

3 General argument under AD

In this section we generalize the above notions and outline how to obtain lightface on the next Suslin cardinal in $L(\mathbb{R})$ under **AD**. The proof of the theorem below is similar to the proof of the corresponding situation at the level of ω_1 above.

Definition 3.1 *A pointclass $\underset{\sim}{\Gamma}$ is said to be $\underset{\sim}{\Pi}^1_1$-like in case $\underset{\sim}{\Gamma}$ has the scale property, $\forall^{\omega^\omega}\underset{\sim}{\Gamma} \subseteq \Gamma$ and there is a pointclass $\underset{\sim}{\Gamma}_1 \subseteq \underset{\sim}{\Gamma}$, such that $\exists^{\omega^\omega}\underset{\sim}{\Gamma}_1 \subseteq \underset{\sim}{\Gamma}_1$ and $\underset{\sim}{\Sigma}^1_1 \subseteq \underset{\sim}{\Gamma}_1$ and such that*

$$\underset{\sim}{\Gamma} = \forall^{\omega^\omega}\underset{\sim}{\Gamma}_1,$$

We fix a $\underset{\sim}{\Pi}^1_1$-like pointclass $\underset{\sim}{\Gamma}$ for the rest of the paper. Let A be a $\underset{\sim}{\Gamma}$ universal set of reals and fix $\{\varphi_n\}$ a $\underset{\sim}{\Gamma}$ scale on A. Let $\kappa = \underset{\sim}{\delta}(\underset{\sim}{\Gamma})$. In practice, κ is a regular cardinal.

Definition 3.2 *Let κ be as above. Let $y \in \omega^\omega$, $S \subseteq \alpha$ and $\alpha < \kappa$. Then a local assignment is a map $\pi : \omega^\omega \times \mathcal{P}(\alpha) \times \kappa \to \kappa$ such that*

$$S_1 \subseteq S_2 \to \pi(y, S_1, \alpha) \leq \pi(y, S_2, \alpha)$$

Still working with κ as above, let μ be the ω-cofinal measure on κ. Let U be the Kunen tree on $\omega \times \kappa$. By Kunen it then follows that for every function $f : \kappa \to \kappa$ there is an $x \in \omega^\omega$ such that U_x is wellfounded and such that $\forall^*_\mu \alpha f(\alpha) < |U_x \restriction \alpha|$. As above the Kunen tree is used to code closed unbounded sets of κ and for $x \in \omega^\omega$ we let $C_x \subseteq \kappa$ be the closed unbounded set coded by x.

Definition 3.3 *Let $\pi(y, S, \alpha)$ be a local assignement. Then we say $\pi(y, S, \alpha)$ is Δ in the codes if there are $\underset{\sim}{\Gamma}$ and $\check{\underset{\sim}{\Gamma}}$ relations P and Q such that for any $x \in A$ and for any $y, w_1, w_2 \in \omega^\omega$ we have:*

$$\pi(y, C_{w_1} \cap \varphi_0(x), \varphi_0(x)) \leq \pi(y, C_{w_2} \cap \varphi_0(x), \varphi_0(x)) \leftrightarrow P(x, y, w_1, w_2) \leftrightarrow Q(x, y, w_1, w_2)$$

Theorem 4 *Let $\underset{\sim}{\Gamma}$ be a $\underset{\sim}{\Pi}^1_1$-like pointclass and let κ be the Wadge ordinal associated to $\underset{\sim}{\Gamma}$. Let $\pi(y, S, \alpha)$ be a local assignment which is Δ in the codes. Suppose there is a closed unbounded set $C \subseteq \kappa$ which minimizes π in the following sense. For any $y \in \omega^\omega$ and for any closed unbounded set $D \subseteq \kappa$, for μ almost all $\alpha < \kappa$, we have*

$$\pi(y, C \cap \alpha, \alpha) \leq \pi(y, D \cap \alpha, \alpha).$$

Let Λ be the pointclass $\Delta(\forall^{\omega^\omega}\exists^{\omega^\omega}\Gamma)$. Thee is then a Λ real z such that U_z is wellfounded and such that C_z minimizes the local assignment π.

Proof.(sketch)

The theorem is proved by a straightforward adaptation of the proof of the theorem at the level of ω_1 in the second section. We outline here the necessary changes. Recall as above that the tree U on $\omega \times \kappa$ is the Kunen tree which analyzes function $f : \kappa \to \kappa$. We define the following two sets:

1. $z \in WG^U_\alpha \leftrightarrow \forall \beta \leq \alpha(|U_z \restriction \beta| \leq \alpha \vee \alpha \in wfp(U_z \restriction \beta))$
2. $z \in WG^U_{<\alpha} \leftrightarrow \forall \beta < \alpha(|U_z \restriction \beta| < \alpha \vee \alpha \in wfp(U_z \restriction \beta))$
3. $z \in SG^U_\alpha \leftrightarrow \forall \beta \leq \alpha(U_z \restriction \beta \text{ is wellfounded})$
4. $z \in SG^U_{<\alpha} \leftrightarrow \forall \beta < \alpha(U_z \restriction \beta \text{ is wellfounded})$

Notice that WG^U_α is a $\mathbf{\Delta}$ set and SG^U_α is $\mathbf{\Gamma}$ set. Consider the game G where player I plays y, w_1 and player II plays w_2. Player II wins if and only if there is an $\eta < \kappa$ such that one of the following holds:

1. $w_1 \in WG^U_{<\eta}$, $w_2 \in SG^U_\eta$, and $w_1 \notin WG^U_\eta$,
2. $w_1 \in WG^U_\eta$, $w_2 \in SG^U_\eta$, and there is an $\gamma < \eta$ such that:
   (a) $\forall \beta < \gamma |U_{w_1} \restriction \beta|$ and $|U_{w_1} \restriction \beta|$ are both $< \gamma$
   (b) $B(y, w_2, w_1, \gamma)$

where $B((y, w_2, w_1, \gamma)$ is the $\mathbf{\Delta}$ relation witnessing that the local assignment map if $\mathbf{\Delta}$ in the codes. The game G is $\exists^{\mathbb{R}}\Gamma$ for player II, so if II wins, then II has a $\Delta_1 = \Delta(\forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma)$ winning strategy by the third periodicity theorem.

Claim 2 *II has a winning strategy for the game G.*

Proof.

We can find a club set $C \subset\subseteq \kappa$ which minimizes the local assignment map π. Let player II play w_2 such that $C_{w_2} \subseteq C$. Suppose that player I plays y and w_1. We must have $w_1 \in WG^U_\alpha$ for all $\alpha < \kappa$. it then follows that U_{w_1} is wellfounded. In addition we have that C_{w_1} and C_{w_2} are club in κ.

Since player II played a code w_2 such that $C_{w_2} \subseteq C$ it then follows from above that C_{w_2} also minimizes the assignment map π:

$$\pi(y, C_{w_2} \cap \alpha, \alpha) \leq \pi(y, C_{w_1} \cap \alpha, \alpha)$$

for all $\alpha < \kappa$.

We may then find an $\alpha < \kappa$ with $\alpha \in C_{w_1} \cap C_{w_2}$ such that $B(y, w_1, w_2, \alpha)$ holds. This then implies that player II has won by clause 2.

□

Fix for the rest of this proof a Δ_1 winning strategy τ for player II in the game G. We must now define a club set $C(\tau) \subseteq \kappa$ which minimizes the assignment map π. Define the function $b : \kappa \to \kappa$ by induction on κ as follows. Assume $b(\beta)$ is defined for all $\beta < \alpha$. Let then

$$(y, w_1) \in W_\alpha \leftrightarrow (w_1 \in WG^U_\alpha \wedge \neg\exists\eta \leq \alpha(\text{II wins by clause2 at } \eta))$$

Notice by the above that W_α is nonempty and $W_\alpha \in \check{\mathbf{\Gamma}}$. If $(y, w_1) \in W_\alpha$ then we have $\tau(y, w_1) = w_2$. It then follows that $U_{w_2} \restriction \alpha$ is wellfounded. Define

$$b(\alpha) = \sup\{|U_{\tau(y,w_1)} \restriction \alpha| : (y, w_1) \in W_\alpha\} < \kappa$$

As above let $C(\tau)$ be the set of closure point of the function b and we next show that $C(\tau)$ must minimize the local assignment π.

Suppose that $C(\tau)$ does not minimize π and let C_1 and y be a counterexample. We then have

$$\forall^* \alpha \pi(y, C_1, \alpha) < \pi(y, C(\tau), \alpha)$$

where again the quantification is with respect to the ω-cofinal measure on κ. Let $C_2 \subseteq \kappa$ be a club set witnessing this statement. Let $C_{w_1} \subseteq C_1 \cap C_2$ and let $\tau(y, w_1) = w_2$.

The next claim establishes that U_{w_2} must be wellfounded and this directly implies by definition that C_{w_2} is a club.

Claim 3 *For all $\alpha < \kappa$, $|U_{w_2} \restriction \alpha| \leq b(\alpha)$*

Proof.

Assume this is true below α. We then have that $C(\tau) \cap \alpha \subseteq C_{w_2} \cap \alpha$. By definition of C_2 there cannot be any $\eta \in C_{w_1}$ such that

$$\pi(y, C(\tau), \eta) \leq \pi(y, C_{w_1}, \eta).$$

Therefore there cannot be any $\eta \leq \alpha$ with $\eta \in C_{w_1} \cap C_{w_2}$ such that

$$\pi(y, C_{w_2}, \eta) \leq \pi(y, C_{w_1}, \eta).$$

This in turn implies that $(y, w_1) \in W_\alpha$. So we must have $|U_{w_2} \restriction \alpha| \leq b(\alpha)$.

□

Now the claim implies that $C(\tau) \subseteq C_{w_2}$. From the definition of C_2, there cannot be any $\eta \in C_{w_1}$ such that

$$\pi(y, C(\tau), \eta) \leq \pi(y, C_{w_1}, \eta)$$

and this implies that there cannot be any $\eta \in C_{w_1} \cap C_{w_2}$ such that

$$\pi(y, C_{w_2}, \eta) \leq \pi(y, C_{w_1}, \eta).$$

But this then implies that τ loses the run of the game G, contradiction!

Therefore $C(\tau)$ minimizes the assignment π and $C(\tau) \in \Delta_1$ since $\tau \in \Delta_1$ and b is Δ_1 in the codes.

□

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