CONTRIBUTIONS TO DESCRIPTIVE SET THEORY

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CHAPTER 1

INTRODUCTION

1.1. Main Results and Motivation

In this paper we prove several descriptive set theoretical results. The main theme is that of closure properties of pointclasses, lightface scales on sets of reals and canonical inner models of ZFC which naturally appear in models of determinacy. Scales on sets of reals are a central object of study in descriptive set theory since from scales one obtains *Suslin representations* for sets of reals and Suslin representations are the best way to understand sets of reals. Descriptive set theory is the study of definable sets of reals. The subject essentially got developed as an effective approach to the continuum problem [1]. Under large cardinal hypothesis, it turns out that $L(\mathbb{R})$ is a natural model of determinacy. In this context, the structure of $L(\mathbb{R})$, in particular its cardinal structure, reflects in a central way to properties of sets of reals. In addition, in $L(\mathbb{R})$ and under large cardinal hypothesis, the bound of the complexity of sets of reals, Θ, is very large. Under choice, Θ is just the successor of the continuum, $\mathfrak{c}^+$. Below we outline results which come out of our work and which can be classified as pertaining to the analysis of the structure of $L(\mathbb{R})$. Occasionally, we look at the structure from the point of view of inner model theory, this is deferred to the fourth chapter.

First we investigate general closure properties of pointclasses [2]. We give a solution to a conjecture of Steel on certain pointclasses of the Wadge hierarchy [3]. From it we can reprove a result on strong partition properties for the ordinals associated to the *Steel pointclasses*. Assuming $\mathrm{AD} + V = L(\mathbb{R})$:

Theorem 1.1. *Suppose Γ is a Steel pointclass and let $\Delta = \Gamma \cap \check{\Gamma}$ such that $o(\Delta)$ is a regular*

[1]First with Cantor, then Borel Baire, Lebesgue, Luzin and Suslin

[2]Pointclasses are a measure of the complexity of sets of reals, we define them precisely below

[3]see the section Preliminaries and basic definitions for a definition of the Wadge hierarchy. Roughly, it is a pre-wellordered hierarchy of the complexity of sets of reals

cardinal. Then Γ is closed under $\vee$. Equivalently, Δ sets are bounded in the norm.

The theorem allows us to obtain a very strong form of boundedness which could be useful on its own. The above also allows characterizing type **III** projective-like hierarchies in terms of the associated ordinals. Pushing the analysis further we also characterize **IV** projective-like hierarchies, which solves a conjecture of Kechris, Solovay and Steel.

THEOREM 1.2. *Let Γ be a Steel pointclass and let $\Delta = \Gamma \cap \check{\Gamma}$ and $o(\Delta)$ is ${}^b\Pi^1_2$-indescribable and Mahlo. Then Γ is closed under $\exists^{\mathbb{R}}$.*

Therefore we have the following characterization of projective-like hierarchies: let $\kappa = o(\Delta)$ and Γ starts a projective-like hierarchies. Then if κ has cofinality ω, Γ starts a type **I** projective-like hierarchy. If κ is singular such that $\omega < cof(\kappa)$, then Γ starts a type **II** projective-like hierarchy. If κ is regular then Γ starts a type **III** projective-like hierarchy. If κ is ${}^b\Pi^1_2$-indescribable then Γ starts a type **IV** projective-like hierarchy. The second chapter is devoted to proofs of these theorems.

From the proof of Steel's conjecture, we also obtain a new strong partition property result on some regular Suslin cardinals. It should be noted that it is still open whether every regular Suslin cardinal has the strong partition property. This would require a deep analysis of the structure of $L(\mathbb{R})$.

THEOREM 1.3. *If κ such that $\kappa = o(\Delta)$, Δ is selfdual and $\exists^{\mathbb{R}}\Delta \subseteq \Delta$, then the strong partition property holds at κ, i.e., $\kappa \longrightarrow (\kappa)^{\kappa}$.*

We thank Steve Jackson for introducing us to the above topic and for numerous discussion on the above results. Strong partition relations are important for the structure of $L(\mathbb{R})$, since they imply that all sets of reals are homogeneous. It should be noted that the above closure properties are very general. However finer closure properties of pointclasses and methods to obtain scales on sets of reals are very closely related to the study of canonical inner models of ZFC, containing all the ordinals, which naturally arise in models of determinacy. This is our next topic of investigation. Namely we investigate the $L[T_{2n}]$ models and show their uniqueness, where T_{2n} is a tree on $\omega \times \kappa^1_{2n+1}$, and where κ^1_{2n+1} is the least ordinal such that Σ^1_{2n+1} sets are κ^1_{2n+1}-Suslin. One could think of these models as a very small definable part of a hierarchy of canonical inner models of ZFC, which starts with $L = L[T_1]$, with T_1 being the Schoenfield tree, and which potentially goes to HOD. To put this into perspective, recall that the constructible universe L is obtained by iterated the definable power set operation using first order logic and it is a theorem of Scott and Myhill that HOD is the constructible universe obtained by iterating the definable power set operation using second order logic. So basically the models $L[T_{2n}]$ can be thought of as fragments [4] of HOD corresponding to some levels of determinacy well below AD. Neeman and Woodin has shown that these levels of determinacy below AD correspond to specific large cardinals and we touch on this aspect later on in the paper. In particular, we show that the models $L[T_{2n}]$ are constructible models from direct limits of mice.

To show the uniqueness of the $L[T_{2n}]$, we need to prove a generalization of the Kechris-Martin theorem and a characterization of the sets of reals of $L_\kappa[T_{2n}]$, where κ is the least admissible above κ^1_{2n+3}. The Kechris-Martin theorem states a closure property of the pointclass Π^1_3 under existential quantification over a set of ordinals coded by reals.

THEOREM 1.4. *For every $n \in \omega$, the pointclass Π^1_{2n+3} is closed under existential quantification up to κ^1_{2n+3}, where κ^1_{2n+3} is the $(2n+3)^{rd}$ Suslin cardinal of cofinality ω. In particular every Π^1_{2n+3} subset of κ^1_{2n+3} contains a Δ^1_{2n+3} member.*

In the above statement, a Δ^1_{2n+3} ordinal is simply an ordinal coded by a Δ^1_{2n+3} real. We will make this notion precise below. The generalizations of the Kechris-Martin theorem simplify the complexity of descriptive set theoretical statements. This means that results like the above allow us to obtain better bounds in computing the complexity of objects we encounter in descriptive set theory. In addition to analyzing the structure of the $L[T_{2n}]$ models, we apply the methods used in the proof of the generalizations of the Kechris-Martin theorem to show that certain lightface sets of reals admit lightface scales. The advantage of this method is that it avoids any reference to periodicity phenomena in $L(\mathbb{R})$ under determinacy as in the third periodicity theorem of Moschovakis. From these scales, we

[4]although this is not literally true as we show in chapter 4

construct canonical trees T_{2n} which project to universal Π^1_{2n} sets of reals, in the same vein as the Martin-Solovay tree construction. The main technical lemma which is used in the construction of lightface scales on projective sets of reals is the following:

LEMMA 1.5. *Let T be a tree on $\omega \times \omega \times \delta^1_{2n+1}$ which is homogeneous with measures W^n_{2n+1}, i.e., the n-fold products of the normal measure on δ^1_{2n+1}. Assume also that T is Δ^1_{2n+1} in the codes. Then there is a c.u.b. $C \subseteq \delta^1_{2n+1}$ which stabilizes T and such that C is Δ^1_{2n+3} in the codes.*

As mentioned above, the above generalization of the Kechris-Martin theorem is central in showing that the $L[T_{2n}]$ models do not depend on the choice of universal Π^1_{2n} sets and the choice of scales on the universal sets. In prior work, Hjorth has choice that the model $L[T_2]$ is unique. The proof however depends on the theory of sharps.

THEOREM 1.6. *The models $L[T_{2n}]$ are independent of the choice of the Π^1_{2n} universal set A and of the choice of the scale $\vec{\varphi}$ on A.*

Since the models $L[T_{2n}]$ satisfy AC, these models cannot satisfy significant amount of boldface determinacy. We do not know how much boldface determinacy holds in these models. It turns out the the $L[T_{2n}]$ models can be characterized precisely using inner model theory. Woodin has conjectured that the models $L[T_{2n}]$ satisfy the GCH, for every $n \in \omega$. We give a positive solution to this conjecture.

THEOREM 1.7. *Let $\mathcal{M}^\#_{2n+1,\infty}$ be the HOD limit associated to $\mathcal{M}^\#_{2n+1}$, where $\mathcal{M}^\#_{2n+1}$ is the minimal active mouse with $2n+1$ Woodin cardinals. Then*

$$L[T_{2n}] = L[\mathcal{M}^\#_{2n+1,\infty}]$$

Moreover $L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}}$ is an extender model, satisfies the GCH and thus $L[T_{2n}] \vDash GCH$.

The above result is the counterpart to Steel's result that the H_Γ models, in the case where Γ is a Π^1_1-like pointclass, are extender models. A proof of the above theorem can be found in section 4. We thank Grigor Sargsyan for having introduced us to this topic and for providing invaluable help in showing the above result. We are also thankful to Hugh Woodin

for very helpful discussions on how to show that the GCH holds in the $L[T_{2n}]$ models. The H_Γ models are defined as follows. Let Γ be a pointclass which resembles Π^1_1. For A a set of reals in Γ, let $\rho : A \to \underset{\sim}{\delta}$ be a regular Γ norm onto $\underset{\sim}{\delta}$. By definition, Γ is ω-parametrized, so let $G \subseteq \omega \times \mathbb{R}$ be a good universal set in $\exists^{\mathbb{R}}\Gamma$. Define the set $P_{\rho,G} \subseteq \omega \times \delta$ by

$$P_{\rho,\delta}(n,\alpha) \leftrightarrow \exists x(x \in A \wedge \rho(x) = \alpha \wedge G(n,x))$$

Then if AD holds we let $H_\Gamma = L[P_{\rho,G}]$. Moschovakis has shown that the models H_Γ do not depend on the choice of universal set and norm. Subsequently Becker and Kechris have shown that for $\Gamma = \Pi^1_{2n+1}$, $H_{\Pi^1_{2n+1}} = L[T_{2n+1}]$ where T_{2n+1} is a tree which projects to a universal Π^1_{2n+1} set. In addition, Harrington, Kechris and Solovay have shown that $\mathbb{R} \cap L[T_{2n+1}] = C_{2n+2}$ using descriptive set theoretical methods. Later in the 90's, Steel has shown using the HOD analysis, that the model H_Γ satisfy the GCH for Γ a pointclass which resembles Π^1_1, by showing that they are fully sound extender models. Theorems 1.5 and 1.6 above are thus counterparts to this analysis but for the Π^1_{2n} pointclasses. Part of the difficulty in the analysis is that the Π^1_{2n} do not have the scale property. Furthermore there is a difficulty in directly trying to show that the GCH holds in these models and this requires adapting the HOD analysis to our context. In this same line of investigation, we have the following characterization of the set of reals of $L_\kappa[T_{2n+2}]$ in terms of $\mathcal{Q}$-theory [5] which follows from the generalizations of the Kechris-Martin theorem. We show the following at the end of section 3.

Theorem 1.8. *Assume AD and let κ be the least admissible above κ^1_{2n+3} Then*

$$Q_{2n+3} = L_\kappa[T_{2n+2}] \cap \mathbb{R}.$$

A lot more can of course be said on the interactions between descriptive set theory and inner model theory , but this requires us to go to the context of axioms of determinacy which significantly go beyond AD and which belong to the *Solovay hierarchy.* In particular, beyond $\mathrm{AD}^{L(\mathbb{R})}$, one consider determinacy axioms based on AD^+ and models of the form $L(\mathcal{P}(\mathbb{R}))$

[5]We define the notions of $\mathcal{Q}$-theory in the last section of chapter 3

and $L(\Gamma, \mathbb{R})$. This is the area of modern descriptive inner model theory. We will not touch on this important interplay between inner model theory and descriptive set theory. Instead we limit ourselves to study the structure of $L(\mathbb{R})$ under AD and for this goal we may use pure descriptive set theory or inner model theory. Extending the context of this paper, we believe most of the theorems proved using combinatorial methods in this paper can be proved using inner model theoretic tools. These tools also have deep applications to $\mathcal{Q}$-theory. We leave the aspect of this subject for a different paper.

1.2. Preliminaries and Basic Notions of Descriptive Set Theory

The purpose of this section is to introduce the notions and objects we'll use in the paper. We introduce here the basic notions of descriptive set theory used throughout the paper. We will introduce the inner model theory notions as they come along in section 4.3.

We will work in the theory ZF+DC+AD. In some places we may use $\mathrm{AD}^{L(\mathbb{R})}$ so one could think of the work as taking place under ZF+DC+AD^+.

Although we use $\mathbb{R}$ for the set of reals in the paper, it is standard to identify the set of reals $\mathbb{R}$ with the Baire space ω^ω (this can be done by using continued fractions to show that the set of irrational numbers is homeomorphic with ω^ω for example). So whenever we use $\mathbb{R}$, we actually really mean ω^ω. The advantage of this shift is that ω^ω is now homeomorphic with $(\omega^\omega)^2$. Reals simply become ω sequences in ω, instead of Dedekind cuts, which are very complicated objects in themselves.

Any sequence $(x_i : i \leq n)$ with $x_i \in \mathbb{R}$ for every $i \leq n$ can be coded into a single real via a recursive bijection

$$(x_1, ..., x_n) \mapsto \langle x_1, ..., x_n \rangle$$

We will also let $x \mapsto ((x)_0, ..., (x)_n)$ denote the decoding map. We'll often drop the parenthesis and just write x_i instead of $(x)_i$. It is also true that countably many reals can be coded into a single reals and the coding real will be denoted by $\langle x_n \rangle$.

A tree T on a set X is a set of finite sequences $(x_1, ..., x_j)$ from X closed under initial segments, that is,

$$\text{whenever } (x_1, ..., x_j) \in T, (x_1, ..., x_i) \in T, \text{ for any } i \leq j$$

Letting $s = (x_1, ..., x_j)$, it is standard to denote the length of s by $lh(s)$. For $s, t \in T$, we say that t extends s, denoted by $s \lhd t$ if $lh(s) \leq lh(t)$ and $t \restriction lh(s) = s$. A branch through the tree T is an infinite sequence $f = (x_0, x_1, ...)$ such that for every n, $f \restriction n \in T$. If the tree T has a branch then it is said to be illfounded, otherwise it is wellfounded. The set of all branches of a tree T is called the body of T and is denoted by $[T]$. All trees in the paper will be in the descriptive set theoretic sense outlined in this paragraph, that is they will have height ω.

Although one could define the notion of a tree T on a general perfect product space

$$\mathcal{X} = X_1 \times ... \times X_n, \text{ where } X_i = \mathbb{R} \text{ or } X_i = \omega,$$

we will not need this more general notion and prefer to concentrate on the basic case where T is a tree on $\omega \times \kappa$ where κ is an ordinal. This move is harmless as suggested below.

Definition 1.9 (Γ-measurable function). Let Γ be a pointclass and X, Y two Polish spaces. We say a function $f : X \to Y$ is Γ-measurable if for every open set $U \subseteq Y$, $f^{-1}(U) \in \Gamma$.

Theorem 1.10. *Any Polish space X is a continuous surjective image of $\mathbb{R}$ via a Δ^0_3-measurable function.*

It is standard to identify $(\omega \times \kappa)^{<\omega}$ with $\omega^{<\omega} \times \kappa^{<\omega}$, since they are homeomorphic and when we write the former we always mean the latter.

Let $T \subseteq (\omega \times \kappa)^{<\omega}$. The projection of the tree T is defined as

$$p[T] = \{x : \exists f \in \kappa^\omega ((x \restriction n, f \restriction n) \in T), \text{ for every } n\}.$$

The section of the tree T at $x \in \mathbb{R}$ is

$$T_x = \{s : (x \restriction lh(s), s) \in T\}.$$

The notion of a left-most branch is essential in the context of scales on sets of reals, so we proceed to introduce it. For T on $\omega \times \kappa$ it makes sense to speak of the left-most branch since $\omega \times \kappa$ comes equipped with a natural wellordering $\preceq$ it inherits from the ordinals. The left-most branch l is the lexicographically least branch in the wellorder $\preceq$, that is for all branches $g \in [T]$,

$$f \neq g \longrightarrow \text{for the least n such that } f(n) \neq g(n) \text{ we have } f(n) \preceq g(n).$$

For T be a tree on $\omega \times \kappa$ and for $x \in \mathbb{R}$, the natural wellordering $\preceq$ on κ induces a linear order on T_x called the Brouwer-Kleene order $<_{BK}$. The linear order $<_{BK}$ is defined as follows:

$$s <_{BK} t \leftrightarrow s \lhd t \vee \exists n < min\{lh(s), lh(t)\}$$

$$\text{such that } s(n) \neq t(n) \text{ and for a least such } n, s(n) < t(n).$$

The Brouwer-Kleene, on T_x is a wellordering if and only if T_x is wellfounded, that is $p[T] = \emptyset$. It is standard to use the following notation in computations involving trees and sections of trees: $|T_x(s)|$ is the rank of s in T_x and it is denoted by $|s|_{T_x}$. Also $|T_x \upharpoonright \alpha(s)|$ denotes the rank of s in the tree $T_x \upharpoonright \alpha$, if $T_x \upharpoonright \alpha$ is wellfounded . We define $T_x \upharpoonright \alpha = T_x \cap \alpha^{<\omega}$ as follows:

$$T_x \upharpoonright \alpha = \{s \in T_x : s(i) < \alpha, \forall i \leq lh(s)\}$$

Also, we denoted this by $|s|_{T_x \upharpoonright \alpha}$. Instead of writing $T_x \upharpoonright \alpha(s)$, we will often write $T_x \upharpoonright \alpha(\delta)$, after identifying finite sequences of ordinals, s, with single ordinals (say via Godel's pairing function for example).

AD is the statement that every two player game on $\mathbb{N}$, with perfect information, is determined. This means that given an $A \subseteq \mathbb{R}$, players I and II play integers and a run of the game is an $x \in \mathbb{R}$ and I wins the run of the game if and only if $x \in A$. Equivalently, II wins the run of the game if and only if $x \notin A$. This basic game will be denoted by G_A.

A measure on a set A is a countably complete ultrafilter on A. Recall that under AD every ultrafilter on a set A is countably complete. This follows from the fact that of μ is a non-principal ultrafilter on ω then μ is non-measurable and does not have the property of

Baire [6]. Recall that AD eliminates the pathological sets introduced by AC. In particular, AD implies that every set of reals has the perfect set property, the Baire property and is Lebesgue measurable. Notice that we are not studying AD in the hope that it will be adopted as an axiom to be added to ZF. The situation is a bit more subtle: determinacy is a phenomenon which naturally occurs in symmetric submodels of generic extensions of HOD and as such determinacy can help study the large cardinals hierarchy.

Next we introduce basic notions of the theory of pointclasses which we need throughout. A pointclass Γ is a collection of sets of reals closed under continuous inverse images, that is:

$$\text{if } f : \mathbb{R} \to \mathbb{R} \text{ is continuous and } A \subseteq \mathbb{R} \text{ is } \in \Gamma \text{ then } B = f^{-1}[A] \in \Gamma$$

For example Σ^0_1 and Σ^2_1 are two examples of pointclasses. Subscripts denote the numbers of quantifiers involved in the syntactic formula defining the set belonging to the pointclass and superscripts denotes the type of objects which fall on the scope of the quantification.

Wadge reduction is a central concept in descriptive set theory. Wadge reduction provides a measure of the complexity of sets of reals. For two sets $A, B \subseteq \mathbb{R}$, we say A is Wadge reducible to B and write $A \leq_W B$ if and only if there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $B = f^{-1}[A]$, i.e computing membership in A should be no more complicated than computing membership in B. In other words, $A \leq_W B$ if and only if there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for all x,

$$x \in A \leftrightarrow f(x) \in B.$$

So a pointclass $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a collection of sets of reals closed under Wadge reduction. One basic consequence of AD is Wadge's Lemma with says that any two sets of reals can be compared simply by the continuous substitution and taking complements. In particular

$$A \leq_w B \leftrightarrow A = f^{-1}[B].$$

[6] A set of reals X has the Baire property if it differs from an open set by a meager set

It is a very useful fact in descriptive set theory that the relation $\leq_W$ is wellfounded, and this is due to Martin and Monk. Given a pointclass Γ, we have the dual pointclass

$$\check{\Gamma} = \{A : A^c \in \Gamma\}.$$

Recall that there are two hierarchies of definability: the lightface hierarchy and the boldface hierarchy. Sets of reals are said to be lightface if their definition does not involve reals as parameters in the definitions and they are boldface if reals parameters are mentioned in the definitions. As customary, lightface pointclasses will be denoted by Γ and boldface pointclasses will be denoted by $\utilde{\Gamma}$. The boldface pointclasses can be derived by relativizing the lightface pointclasses:

$$\utilde{\Gamma} = \bigcup_{x \in \mathbb{R}} \Gamma(x).$$

In other words, for $X \subseteq \mathbb{R}$

$$X \in \utilde{\Gamma} \longleftrightarrow \exists X^* \subseteq \mathbb{R}^2, X^* \in \Gamma \text{ and some } x \in \mathbb{R} \text{ such that } X = X^*_x = \{y : X^*(x,y)\}$$

The most robust notion of definability one can have if that of *ordinal definability*. In the lightface case we talk about OD sets of reals and in the boldface case we talk about OD($\mathbb{R}$) sets of reals, that is we are allowed real parameters in the definition of the sets.

If $\utilde{\Gamma}$ is a pointclass, we say $U \subseteq \mathbb{R}^2$ is a universal set for $\utilde{\Gamma}$ if and only if for every $B \in \utilde{\Gamma}$, there is a $y \in \mathbb{R}$ such that $U_y = B = \{x : (y,x) \in U\}$.

A pointclass is non-selfdual if and only if it is not closed under complements and a pointclass is called selfdual if it is closed under complements. Under AD, Wadge's lemma implies that every nonselfdual pointclass has a universal set. Selfdual pointclasses do not have universal sets by a diagonal argument. It is standard to denote selfdual pointclasses by $\utilde{\Delta}$ and we'll write

$$\utilde{\Delta} = \utilde{\Gamma} \cap \utilde{\check{\Gamma}}$$

The closure of $\utilde{\Gamma}$ under existential quantification is given by

$$\exists^{\mathbb{R}}\utilde{\Gamma} = \{A : \exists B \in \utilde{\Gamma} \forall x(A(x) \leftrightarrow \exists y B(x,y))\}$$

Notice that this is the same as taking continuous images by continuous functions $f : \mathbb{R} \to \mathbb{R}$. For instance, considering $\underset{\sim}{\Pi}^0_1$ the pointclass of closed sets then one has $\exists^{\mathbb{R}}\underset{\sim}{\Pi}^0_1 = \underset{\sim}{\Sigma}^1_1$, namely a continuous image of a closed set is an analytic set. One can also define $\forall^{\mathbb{R}}\underset{\sim}{\Gamma}$, which is just $\exists^{\mathbb{R}}\check{\underset{\sim}{\Gamma}}$. The projective hierarchy is defined in analogous fashion: $\underset{\sim}{\Sigma}^1_{n+1} = \exists\underset{\sim}{\Pi}^1_n$ and $\underset{\sim}{\Pi}^1_n = \neg\underset{\sim}{\Sigma}^1_n$. Another way to generate to the projective hierarchy is to look at $J(\mathbb{R})$, the Jensen constructible universe containing all the reals and ordinals. We have that $\Sigma_1(J_1(\mathbb{R})) = \Sigma^1_1$ and so $\Pi_1(J_1(\mathbb{R})) = \Pi^1_1$. Similarly, $\Sigma_2(J_1(\mathbb{R})) = \Sigma^1_2$, $\Sigma_3(J_1(\mathbb{R})) = \Sigma^1_3$ and $\Pi_n(J_1(\mathbb{R})) = \Pi^1_n$, etc... So the projective hierarchy is entirely contained in $J_2(\mathbb{R})$. At the higher up levels, the pointclass of the inductive sets is given by $\Sigma_1(J_{\kappa^{\mathbb{R}}}(\mathbb{R}))$, where $\kappa^{\mathbb{R}}$ is the least $\mathbb{R}$-admissible ordinal. Also $\Sigma_1^{L(\mathbb{R})} = \Sigma^2_1 = \Sigma_1(J_{\delta^2_1}(\mathbb{R}))$, where δ^2_1 is the least stable cardinal of $L(\mathbb{R})$. The least stable ordinal [7] in $L(\mathbb{R})$ is the least ordinal δ for which we have

$$L_\delta(\mathbb{R}) \preceq^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R})$$

Definition 1.11 (Levy pointclass). A Levy pointclass Γ is a nonselfdual pointclass which is closed under either $\exists^{\mathbb{R}}$ or $\forall^{\mathbb{R}}$ or possibly under both.

There are other pointclasses than the Levy pointclasses, for instance the $\Game(\omega n) - \Pi^1_1$ or the $\Game^n(\omega n) - \Pi^1_1$ are pointclasses which we will introduce later. These pointclasses are used in central ways in the sections below for complexity estimates of norms. We remind some basic properties of pointclasses.

Definition 1.12. Γ has the **reduction property** if for all $A, B \in \underset{\sim}{\Gamma}$ there are $A', B' \in \underset{\sim}{\Gamma}$ such that $A' \subseteq A, B' \subseteq B, A' \cap B' = \emptyset, A' \cup B' = A \cup B$. $\underset{\sim}{\Gamma}$ has the **separation property** if for every $A, B \in \Gamma$ such that $A \cap B = \emptyset$ there exists a set $C \in \Delta$ such that $A \subseteq C$ and $C \cap B = \emptyset$.

One of the central properties a pointclass can have is the **prewellordering property**: $\underset{\sim}{\Gamma}$ has the prewellordering property if every $\underset{\sim}{\Gamma}$ set admits a $\underset{\sim}{\Gamma}$ norm, where a norm on a set of reals A is a map ϕ such that $\phi : A \to ORD$. The norm is regular if it is into an ordinal κ.

[7] see [19] for a proof of this fact

Definition 1.13. A norm ϕ is called a $\underset{\sim}{\Gamma}$ norm if the following norm relations are in $\underset{\sim}{\Gamma}$: $\leq^*_\phi, <^*_\phi$ with:

$$x \leq^*_\phi y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \phi(x) \leq \phi(y)))$$

$$x <^*_\phi y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \phi(x) < \phi(y)))$$

Notice that the prewellordering property is a way of splitting our Γ set A into $\underset{\sim}{\Delta}$ pieces. Θ is the supremum of the length of the prewellorderings of $\mathbb{R}$, that is:

$$\Theta = \sup\{\alpha : \exists f : \mathbb{R} \twoheadrightarrow \alpha\}.$$

Under AC, Θ is $\mathfrak{c}^+$ but under determinacy Θ can exhibit large cardinal properties.

Recall that under ZF, we have the following:

(1) if $\underset{\sim}{\Gamma}$ is closed under $\vee$, PWO($\underset{\sim}{\Gamma}$) $\longrightarrow$ Red($\underset{\sim}{\Gamma}$)

(2) Red($\underset{\sim}{\Gamma}$) $\longrightarrow$ Sep($\check{\underset{\sim}{\Gamma}}$)

(3) if $\underset{\sim}{\Gamma}$ has a universal set then Red($\underset{\sim}{\Gamma}$) $\longrightarrow$ $\neg$Sep($\underset{\sim}{\Gamma}$).

(4) (Steel, Van Wesep) Under ZF+AD, if Sep($\check{\underset{\sim}{\Gamma}}$) and for any $A, B \in \underset{\sim}{\Delta}$, $A \cap B \in \underset{\sim}{\Gamma}$ then Red($\underset{\sim}{\Gamma}$).

It is a classical fact of descriptive set theory that under ZF+AD for any Levy pointclass $\underset{\sim}{\Gamma}$, either PWO($\underset{\sim}{\Gamma}$) or PWO($\check{\underset{\sim}{\Gamma}}$). Under ZF only, if $\underset{\sim}{\Gamma}$ is a pointclass with PWO($\underset{\sim}{\Gamma}$) then every set in $\exists^{\mathbb{R}}\Gamma$ admits a $\forall^{\mathbb{R}}\exists^{\mathbb{R}}\Gamma$ norm. What gets us going through the Wadge hierarchy is the *first periodicity theorem*:

Theorem 1.14 (Moschovakis). *Suppose that $\underset{\sim}{\Delta}$-determinacy holds and that $\underset{\sim}{\Gamma}$ is a nonselfdual pointclass with PWO($\underset{\sim}{\Gamma}$) then every set in $\forall^{\mathbb{R}}\Gamma$ admits a $\exists^{\mathbb{R}}\forall^{\mathbb{R}}\Gamma$ norm.*

Definition 1.15 (The scale property). A semiscale is a sequence of norms $\langle \phi_n \rangle$ on a set A such that whenever we have a sequence $\{x_n\} \subseteq A$ converging to some x and for every $n, \phi_n(x_i)$ is eventually constant then $x \in A$. If in addition we have the lower semi-continuity property, $\phi_n(x) \leq \lim \phi_n(x_i)$ then the sequence of norms $\langle \phi_n \rangle$ is a scale. A scale $\langle \phi_n \rangle$ is a $\underset{\sim}{\Gamma}$-scale if for every n, ϕ_n is a $\underset{\sim}{\Gamma}$-norm. The pointclass Γ has the scale property if every $\underset{\sim}{\Gamma}$ set has a $\underset{\sim}{\Gamma}$-scale.

A scale $\langle \phi_n \rangle$ on a set A is good if whenever $\{x_n\} \subseteq A$ and for all $n \in \omega$, $\varphi_n(x_m)$ is eventually constant, then $x = \lim x_m$ exists and $x \in A$.

A scale $\langle \phi_n \rangle$ on a set A is very-good if $\langle \phi_n \rangle$ is good and whenever $x, y \in A$ and $\varphi_n(x) \leq \varphi_n(y)$ then $\varphi_k(x) \leq \varphi_k(y)$ for all $k < n$.

A scale $\langle \phi_n \rangle$ on a set A is excellent if it is very good and whenever $x, y \in A$ and $\varphi_n(x) = \varphi_n(y)$, then $x \restriction n = y \restriction n$.

Definition 1.16 (Inductive-like pointclass). A pointclass $\underset{\sim}{\Gamma}$ is inductive like, if it is closed under $\exists^{\mathbb{R}}, \forall^{\mathbb{R}}$ and $\underset{\sim}{\Gamma}$ has the scale property.

The following theorem is the *second periodicity theorem*. It shows that under suitable determinacy assumption we can propagate the scale property.

Theorem 1.17 (Moschovakis). *Assume projective determinacy. Then every* Π^1_{2n+1} *and every* Σ^1_{2n} *have the scale property.*

Recall that a set $A \subseteq \mathbb{R}$ is κ-Suslin if there is a tree T on $\omega \times \kappa$ such that:

$$A = p[T] = \{x : \exists f \in \kappa^\omega \forall n (x \restriction n, f \restriction n) \in T\}.$$

A cardinal κ is a *Suslin cardinal* if there is a set $A \subseteq \mathbb{R}$ which is κ-Suslin but not γ-Suslin for any $\gamma < \kappa$. The first few Suslin cardinals are $\aleph_0, \aleph_1, \aleph_\omega$ and $\aleph_{\omega+1}$. To draw an analogy with Θ, the supremum of all prewellorderings of the reals, $\aleph_1 = \underset{\sim}{\delta}^1_1$ is the supremum of all $\underset{\sim}{\Delta}^1_1$ prewellordering of $\mathbb{R}$. Similarly $\underset{\sim}{\delta}^1_3 = \aleph_{\omega+1}$ [8] is the supremum of all $\underset{\sim}{\Delta}^1_3$ prewellorderings of $\mathbb{R}$. Basically the problem of the continuum is viewed from the point of view of the Wadge hierarchy. Scales provide sets of reals both with a Suslin representation and a notion of definability associated to that representation. There is a basic relationship between having scales and being Suslin:

Fact 1.18. A set $A \subseteq \mathbb{R}$ is κ-Suslin if and only if it admits a κ-semi-scale if and only if it admits a κ-scale if and only if it admits an excellent κ-scale.

[8] this is actually a theorem

Constructing a scale from a semi-scale turns out to be a fundamental problem in descriptive inner model theory. Part of the work in this paper is to explore methods which allow constructing scales on certain sets of reals.

We now state the *third periodicity theorem*. This is a result on the definability of iteration strategies in integer games. The Third periodicity theorem is a very useful result on lowering the complexity of winning strategies τ. For instance let A be a Σ^1_{2n} set and let τ be a winning strategy for player II in the game G_A. Then the set of all winning strategies for II is computed to be Π^1_{2n+1}:

$$\tau \in \mathcal{W} \leftrightarrow \forall x(\tau * [x] \in A)$$

Assuming AD (Det(Δ^1_{2n}) suffices), the pointclass Π^1_{2n+1} satisfies the Basis theorem (see [22]), so there a winning strategy $\tau \in \Delta^1_{2n+2}$. The third periodicty theorem states that one can find a winning strategy in $\mathcal{W}$ which is Δ^1_{2n+1}. We'll use the results in several places in the paper:

THEOREM 1.19 (Third periodicity theorem). *Suppose* Γ *be an adequate pointclass, Det*($\underset{\sim}{\Gamma}$) *holds and let* $A \subseteq \mathbb{R}$ *be in* Γ *and admits a* Γ *semi-scale. If player I wins the game* G_A *then I has a* $\Game\Gamma$*-recursive winning strategy* σ*.*

We now define the notion of a *projective* hierarchy in the general context. This is will allow us to define the Steel pointclasses which we need for the next section.

DEFINITION 1.20. A projective algebra is a pointclass Λ which is closed under $\exists^{\mathbb{R}}, \vee, \wedge, \neg$.

A nice additional closure property of Λ is, by Steel-Van Wesep, if $A \in \Lambda$ and if $\exists B$ which is not ordinal definable from A then Λ is closed under sharps, i.e for any $A \in \Lambda$, $A^\# \in \Lambda$. This would hold under $\theta_0 < \Theta$ for example, where

$$\theta_0 = \text{ the least ordinal which is not an OD surjective image of } \mathbb{R}.$$

Next we introduce Levy pointclasses, one of the most basic objects in descriptive set theory.

Definition 1.21. A Levy pointclass $\underset{\sim}{\Gamma}$ is a non-selfdual pointclass that is closed under either $\exists^{\mathbb{R}}$ or $\forall^{\mathbb{R}}$ or possibly under both.

Recall that assuming AD, Wadge's lemma says that for any two sets of reals A, B, either $A \leq_W B$ or $B \leq_W \mathbb{R} \setminus A$. For any set $A \subseteq \mathbb{R}$ there is then a notion of Wadge degree. We say that $A \subseteq \mathbb{R}$ is selfdual if the pointclass $\underset{\sim}{\Gamma}_A = \{B : B \leq_W A\}$ is selfdual. The Wadge degree of A is the equivalence class $[A]_W$ of sets Wadge equivalent to A if A is self-dual, that is $A \leq_W \mathbb{R} \setminus A$ and the pair $([A]_W, [\mathbb{R} \setminus A]_W)$ if A is nonself-dual. Martin and Monk showed that the Wadge degrees are wellfounded under AD. The Wadge degree of a set A is denoted by $o(A)$.

Definition 1.22. $o(\underset{\sim}{\Gamma}) = \sup\{o(A) : A \in \underset{\sim}{\Gamma}\}$, where $o(A)$ is the Wadge degree of A.

Levy pointclasses are classified into 4 different projective-like hierarchies. Suppose Γ is nonselfdual and closed under either $\exists^{\mathbb{R}}$ or $\forall^{\mathbb{R}}$ or possibly both. First let α be the supremum of the limit ordinals β such that

(1) $\Delta_\beta = \{A : o(A) < \beta\}$ is closed under both $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$ and
(2) $\Delta_\beta \subseteq \Gamma$.

We then have the following types of projective-like hierarchies:

- Type **I:** If $cof(\alpha) = \omega$ there is a projective algebra Λ (i.e closed under $\exists^{\mathbb{R}}, \vee \wedge \neg$) of Wadge degree α whose sets are ω-joins of sets of smaller Wadge degree. Letting $\Gamma_0 = \bigcup_\omega \Lambda$ then Γ_0 is a nonselfdual pointclass at the base of a new projective like hierarchy, $\Lambda \subseteq \Gamma_0, \Gamma_0$ is closed under $\exists^{\mathbb{R}}$ and PWO(Γ_0). Γ_0 is not closed under countable intersections since Γ_0 is nonselfdual.
- Type **II/III:** If $cof(\alpha) > \omega$ then there is a pointclass Γ_0 closed under $\forall^{\mathbb{R}}$ with PWO(Γ_0) of Wadge degree α. Γ_0 is not closed under $\exists^{\mathbb{R}}$ in this case. Γ_0 is generated from a projective algebra Λ: Γ_0 is the pointclass of $\underset{\sim}{\Sigma}^1_1$-bounded $cof(\alpha)$ length unions of Λ sets. If Γ_0 is closed under countable unions and disjunction then Γ_0 is said to start a type **III** projective-like hierarchy.

- Type **IV:** If $cof(\alpha) > \omega$ and Γ_0 is as above and closed under $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, then $\mathrm{PWO}(\Gamma_0)$ but this can't be propagated by periodicity as in types **I,II and III**. So define $\Pi_1 = \Gamma_0 \wedge \check{\Gamma}_0$. Π_1 is said to be at the base of a type **IV** projective-like hierarchy. Π_1 is closed under countable intersections, $\forall^{\mathbb{R}}$ but not under $\vee$ therefore not under $\exists^{\mathbb{R}}$.

We refer the reader to [6] for more facts on the general theory of pointclasses.

We now introduce partition relations in the context of AD. Partition relations are central in the context of $\mathrm{AD}^{L(\mathbb{R})}$ for the internal structure of $L(\mathbb{R})$, since many properties of sets of reals rely on these relations. An example of such properties is that of *homogeneity* and *weak homogeneity.*

Definition 1.23. Let λ be a cardinal and let κ, γ be cardinals such that $\lambda \leq \kappa, \gamma < \kappa$. Then we say that κ has the weak partition property if for every $\lambda < \kappa$, $\kappa \to (\kappa)^{\lambda}_{\gamma}$, i.e for every partition $F : [\kappa]^{\lambda} \to \gamma$ of the set of increasing λ sequences from κ into γ pieces there is a set $H \subseteq \kappa$ such that $|H| = \kappa$ which is homogeneous for F, i.e $F \upharpoonright [H]^{\lambda}$ is constant. κ has the strong partition partition if $\kappa \to (\kappa)^{\kappa}_{\gamma}$

Notice that in the above definition, if $\gamma > 2$, then $\kappa \to (\kappa)^{\lambda}_{2}$ holds. We will work with partition into 2 pieces and we drop the subscript 2. It should be true that every regular Suslin cardinal satisfies the strong partition property, but this turns out to be a very hard problem. In general it should also be true that if $\undertilde{\Gamma}$ is a nonselfdual pointclass such that $\mathrm{PWO}(\undertilde{\Gamma})$, $\forall^{\mathbb{R}}\undertilde{\Gamma} \subseteq \undertilde{\Gamma}, \cup_{\omega}\undertilde{\Gamma} \subseteq \undertilde{\Gamma}$ and $\cap_{\omega}\undertilde{\Gamma} \subseteq \undertilde{\Gamma}$, then for

$$\undertilde{\delta} =_{def} \text{sup of the length of the } \undertilde{\Delta} \text{ prewellorderings of } \mathbb{R},$$

$\undertilde{\delta}$ should satisfy the strong partition property. In the next chapter, we extend previous results of [14] with regards to which ordinals associated to a pointclass satisfy the strong partition property. In particular it is shown in [14] that AD implies that for every $\kappa < \Theta$, there exists $\lambda > \kappa$ such that λ has the strong partition property. It turns out that the converse is also true:

THEOREM 1.24 (Kechris, Woodin, [18]). *Assume ZF+DC+$V = L(\mathbb{R})$. Then the following are equivalent:*

(1) $L(\mathbb{R}) \models AD$,

(2) $L(\mathbb{R}) \models \forall\lambda < \Theta\exists\kappa\ s.t\ \kappa > \lambda \wedge \kappa \to (\kappa)^\kappa$,

(3) $L(\mathbb{R}) \models \forall\lambda < \Theta\exists\kappa\ s.t\ \kappa > \lambda \wedge \kappa \to (\kappa)^\lambda$

See [18] for more on the equivalence of determinacy with strong partition properties.

CHAPTER 2

A PROOF OF A CONJECTURE OF STEEL ON POINTCLASSES, CHARACTERIZATION OF PROJECTIVE-LIKE HIERARCHIES BY THE ASSOCIATED ORDINALS AND STRONG PARTITION RELATIONS

2.1. Closure property of the Steel Pointclass

In this section, we give a positive answer to a conjecture of Steel in [27]. We introduce the Steel pointclass below and the background needed to show that the conjecture is true.

We fix a Levy pointclass Γ. We let Λ be the pointclass associated to Γ and obtained by taking unions of all sets in Δ, where $\Delta = \Gamma \cap \check{\Gamma}$, and Δ is closed under $\exists^{\mathbb{R}}$, complements and finite intersections. Then we have that $\Lambda \subseteq \Gamma$ and Λ is the largest projective algebra contained in Γ since it is closed under $\exists^{\mathbb{R}}$, complements and finite unions and intersections. It can also be shown that Λ is at the base of a projective hierarchy containing Γ. Let $\alpha = \sup\{o(A) : A \in \Lambda\}$ and suppose $\omega < cof(\alpha)$ (the case $\omega = cof(\alpha)$ is the case of a type **I** hierarchy). By general theory of the Wadge degrees, we have a nonselfdual pointclass Γ_0 such that $o(\Gamma_0) = \alpha$. One of Γ_0 and $\check{\Gamma}_0$ has the separation property, so let $\check{\Gamma}_0$ be the side with the separation property. It turns out that Γ_0 is closed under $\forall^{\mathbb{R}}$:

Theorem 2.1 ([17]). *Assume ZF+AD. Let Γ_0 be as above and assume that $\check{\Gamma}_0$ has the separation property. Then $\check{\Gamma}_0$ is closed under $\exists^{\mathbb{R}}$.*

Proof. The proof uses a variant of an argument by Addison which was used to show the separation property for the pointclass Σ^1_3. Suppose that there is a set $A \in \exists^{\mathbb{R}}\check{\Gamma}_0 \setminus \check{\Gamma}_0$. Then by Wadge's lemma, $\Gamma_0 \subseteq \exists^{\mathbb{R}}\check{\Gamma}_0$. Let $P, Q \in \Gamma_0$ such that $P \cap Q = \emptyset$. Since $P, Q \in \exists^{\mathbb{R}}\check{\Gamma}_0$, then let $A, B \in \check{\Gamma}_0$ be such that $P(x) \leftrightarrow \exists y A(x,y)$ and $Q(x) \leftrightarrow \exists y B(x,y)$. Define $A'(x,y,z) \leftrightarrow A(x,y)$ and $B'(x,y,z) \leftrightarrow B(x,z)$. Then $A' \cap B' = \emptyset$ and $A', B' \in \check{\Gamma}_0$. By the separation property of $\check{\Gamma}_0$, let $D \in \Delta$ such that $A' \subseteq D$ and $B' \cap D = \emptyset$. But now letting $E(x) \leftrightarrow \exists y \forall z D(x,y,z)$, we have $E \in \Delta$ since Δ is closed under $\exists^{\mathbb{R}}$ and complements and $P \subseteq E$, $E \cap Q = \emptyset$. So Γ_0 has the separation property. Contradiction!

$\square$

We call Γ_0 as above the Steel pointclass. Notice that there are no reasons why Γ_0 should be closed under $\vee$ at this point.

Steel has shown that Γ_0 is obtained by taking $cof(\alpha)$ length Σ^1_1 bounded unions of sets in the projective algebra Λ. We now show how to generated Γ_0 from Λ this way. So let $\omega < cof(\alpha) = \beta$, where $\alpha = o(\Lambda)$ and let Γ be the Steel pointclass. So we have $\mathrm{Sep}(\check{\Gamma})$ and there is a set $A \in \Gamma \setminus \check{\Gamma}$ such that $o(A) = \alpha$. By the above theorem Γ is closed under $\forall^{\mathbb{R}}$. We show that Λ is closed under unions of length strictly less than β. We will need this fact to generate the Steel pointclass from Λ.

Lemma 2.2. *Assume that $\Lambda \subsetneq \mathcal{P}(\mathbb{R})$, then β is the least ordinal such that for a sequence of sets $\{A_\gamma\}_{\gamma<\beta}$, with each $A_\gamma \in \Lambda$ we have that $\bigcup_{\gamma<\beta} A_\gamma \notin \Lambda$*

Proof. Let $\preceq$ be a prewellordering of length β in Λ. Let δ be the least ordinal such that there is a δ sequence of sets in Λ such that $\bigcup_{\gamma<\delta} A_\gamma \notin \Lambda$. Then we show that $\delta = \beta$. Notice that δ is a regular cardinal since if not then letting $f : \xi \to \delta$ be a cofinal map for $\xi < \delta$ we could obtain $\bigcup_{\gamma<\xi} A_\gamma \notin \Lambda$ and then δ is not least. Suppose $\beta < \delta$. Assume $\delta < \alpha$. We can also assume that there is an $\alpha_0 < \alpha$ such that for each $\gamma < \delta$, we have $|A_\gamma|_W \leq \alpha_0$, since δ is regular. Fix then a nonselfdual pointclass $\Gamma' \subseteq \Lambda$ such that Γ' is closed under $\exists^{\mathbb{R}}, \wedge, \vee$, $A_\gamma \in \Gamma'$ for every $\gamma < \delta$ and such that there is a prewellordering of length δ in Γ'. Let $\varphi : \mathbb{R} \to \delta$ be a Γ' norm and for each δ sequence of Γ' sets $\{A_\xi\}_{\xi<\gamma}$ let by the coding lemma $R(w, \varepsilon)$ be a Γ' relation such that

(1) $\varphi(w) = \varphi(z) \to (R(w, \varepsilon) \leftrightarrow R(z, \varepsilon))$

(2) $R(w, \varepsilon) \to \varepsilon \in C$ where C is the set of codes of the Γ' sets in the sequence $\{A_\gamma\}_{\gamma<\delta}$. C can be defined using a universal Γ' set as follows: let $U \in \Gamma'$ be a universal set. Then for every $\gamma < \delta$ we let $\varepsilon \in \mathbb{R}$ such that $U_\varepsilon = A_{\varphi(\varepsilon)}$. Then $C \in \Gamma'$.

(3) $\forall w \exists \varepsilon (R(w, \varepsilon) \wedge U_\varepsilon = A_{\varphi(w)})$

Then we have $x \in \bigcup_{\gamma<\delta} A_\gamma \leftrightarrow \exists w \exists \varepsilon (R(w, \varepsilon) \wedge x \in U_\varepsilon)$. So the union is in Γ'. Contradiction!

Next, assume $\alpha < \delta$. Let $\Gamma' \subseteq \Lambda$ be a pointclass as above. Consider a sequence of Γ' sets $\{A_\gamma\}_{\gamma<\delta}$ and define the natural prewellordering $\leq$ defined by

$$x \leq y \leftrightarrow \exists \gamma_1, \gamma_2 \text{ such that } (\gamma_1 < \gamma_2 \wedge x \in A_{\gamma_1} \setminus A_{<\gamma_1} \wedge y \in A_{\gamma_2} \setminus A_{<\gamma_2})$$

Notice that there is an $\alpha_0 < \alpha$ such that for every $\gamma < \alpha$, we have $| \leq_\gamma |_W \leq \alpha_0$, where $\leq_\gamma$ has length γ. So for each γ, we have $\leq_\gamma \in \Lambda$. But now $\leq = \bigcup_{\gamma<\alpha} \leq_\gamma$ is a prewellordering of length α in Λ, since Λ is closed under unions of length α by minimality of δ. Contradiction!

If $\delta < \beta$ then since $\beta \leq \alpha$ then we still have $\delta < \alpha$ and we would get a contradiction using the coding lemma as above. So we must have $\delta \geq \beta$. In case $\delta = \alpha$, then α is also regular and so $\alpha = \beta$. So $\delta = \beta$.

□

Continuing, we have from the above lemma $\Lambda \subsetneq \bigcup_\beta \Lambda$. We cannot have that $\bigcup_\beta \Lambda = \check{\Gamma}$. To see this, let $A, B \in \check{\Gamma}$. Then let $\{A_\gamma\}_{\gamma<\beta}$ be a sequence of sets in Λ such that $A = \bigcup_{\gamma<\beta} A_\gamma$ and let $\{B_\gamma\}_{\gamma<\beta}$ be a sequence of sets in Λ such that $B = \bigcup_{\gamma<\beta} B_\gamma$. We first show that $\check{\Gamma}$ has the reduction property. Define the set A' by

$$x \in A' \leftrightarrow \exists \gamma_1 (x \in A_{\gamma_1} \wedge x \notin \bigcup_{\gamma<\gamma_1} B_\gamma)$$

and define the set B' by

$$x \in B' \leftrightarrow \exists \gamma_1 (x \in B_{\gamma_1} \wedge x \notin \bigcup_{\gamma\leq\gamma_1} A_\gamma)$$

Then notice both A' and B' are in $\check{\Gamma}$. Also $A' \subseteq A$ and $B' \subseteq B$ and $A' \cap B' = \emptyset$. So $\text{Red}(\check{\Gamma})$. But recall that we also have by assumption $\text{Sep}(\check{\Gamma})$. We quickly justify that the reduction property and the separation property can't both hold for $\check{\Gamma}$. Let $A, B \in \check{\Gamma}$. Then by $\text{Red}(\check{\Gamma})$, let A' and B' be disjoint sets such that $A' \subseteq A$ and $B' \subseteq B$ and $A' \cup B' = A \cup B$. Let $U \in \check{\Gamma}$ be a universal set which codes the pair of sets A', B' by $A'(x, y) \leftrightarrow U((x)_0, y)$ and $B'(x, y) \leftrightarrow U((x)_1, y)$. Now let C be a set in Δ which separates A' from B', i.e $A' \subseteq C$ and $C \cap B' = \emptyset$. Now let D be an arbitrary Δ set. Then there exists a $z \in \mathbb{R}$ such that

$$D(y) \leftrightarrow U_{(z)_0}(y) \leftrightarrow \neg U_{(z)_1}(y).$$

Then we have that $D(y) \leftrightarrow A_x(y) \leftrightarrow \neg B_x(y)$. But then $D(y) \leftrightarrow A'_x(y) \leftrightarrow \neg B'_x(y)$. So $D(y) \leftrightarrow C_x(y)$, because $C \in \Delta$ separates A' from B'. So every Δ set is coded as a section of a single Δ set. But selfdual pointclasses can't have universal sets: if $U \in \Delta$ is universal for Δ sets then $U \in \Gamma$ and $U \in \check{\Gamma}$. Then define $A(x) \leftrightarrow \neg U(x,x)$. Since Δ is closed under recursive substitutions, then we have $A \in \Gamma$. So there exists a $z \in \mathbb{R}$ such that $A = U_z$, but now we have $A(z) \leftrightarrow U(z,z) \leftrightarrow \neg A(z)$. Contradiction!

Therefore, by Wadge's lemma we must have that $\Gamma \subseteq \bigcup_\beta \Lambda$. Since Λ is a projective hierarchy then $\exists^{\mathbb{R}}\Gamma \subseteq \bigcup_\beta \Lambda$.

We say that a union $A = \bigcup_{\alpha<\delta} A_\alpha$ is $\utilde{\Sigma}^1_1$-bounded if

$$\text{for any } \utilde{\Sigma}^1_1 \text{ set } S \subseteq A, \text{ there exists a } \gamma < \delta \text{ such that } S \subseteq A_\gamma.$$

Let Γ_1 be the pointclass of Σ^1_1-bounded β length unions of Λ sets. Using the above set up, it is then shown in [27] and [17] that $\Gamma = \Gamma_1$. So the Steel pointclass corresponding to the projective algebra Λ can be characterized as all sets which are Σ^1_1-bounded β length unions of sets in Λ. We proceed to show that the Steel pointclass has the prewellordering property (see [27]). This will motivate a different characterization of the Steel pointclass which we will adopt in the rest of the section.

Theorem 2.3 (Steel). *Let Λ be a projective algebra with $\alpha = o(\Lambda)$ and assume that $\omega < cof(\alpha)$. let Γ be the Steel pointclass corresponding to Λ. Then $PWO(\Gamma)$.*

Proof. Let $\beta = cof(\alpha)$ and let $A \subseteq \mathbb{R}$ be a complete Γ set of reals.Let $A = \bigcup_{\gamma<\beta} A_\gamma$ be an increasing Σ^1_1 bounded β length union of sets such that for each $\gamma < \beta, A_\gamma \in \Lambda$. Let φ be the natural norm in A such that for $x \in A$, $\varphi(x) =$ least ξ such that $x \in A_\xi$. The norm $<^*_\varphi$ associated to φ can be written as $\bigcup_{\gamma<\beta} B_\gamma$ where $B_\gamma(x,y) \leftrightarrow x \in A_\gamma \wedge y \notin A_\gamma$. Then for each $\gamma < \beta$, $B_\gamma \in \Lambda$. It remains to show that $<^*_\varphi \in \Gamma$. We proceed to show that $<^*_\varphi$ is Σ^1_1 bounded. So let $S \subseteq \mathbb{R} \times \mathbb{R}$ be a $\utilde{\Sigma}^1_1$ and $S \subseteq <^*_\varphi$. Notice that if $S(x,y)$ holds then $x \in A$. Since by assumption $\bigcup_{\gamma<\beta} A_\gamma$ is a Σ^1_1 bounded union, there is a $\gamma_0 < \beta$ such that whenever $S(x,y)$ holds $x \in A_{\gamma_0}$.If $\varphi(x) < \varphi(y)$, then there is a $\gamma < \gamma_0$ such that $x \in A_\gamma$ ad $y \notin A_\gamma$ and $B_\gamma(x,y)$ holds. So $<^*_\varphi \in \Gamma$. A similar computation shows that $\leq^*_\varphi \in \Gamma$. So PWO($\Gamma$).

$\square$

Gathering all the facts above we characterize the Steel pointclass as follows:

DEFINITION 2.4 (Steel pointclass). If Δ is selfdual, closed under real quantifiers, $o(\Delta)$ has uncountable cofinality, Δ is not closed under well-ordered unions, then the Steel pointclass is the pointclass Γ such that $\Delta = \Gamma \cap \check{\Gamma}$, Γ is closed under $\forall^{\mathbb{R}}$ and PWO(Γ).

Since the Steel pointclass is nonselfdual and closed under $\forall^{\mathbb{R}}$ then it is closed under $\wedge$. A natural question which arises then is whether the Steel pointclass is closed under $\vee$. The following theorem below shows that what prevents closure of the Steel pointclass under $\vee$ is the singularity of $o(\Delta)$.

To introduce the following theorem, recall that if Γ is a nonselfdual pointclass closed under $\forall^{\mathbb{R}}$ and $\vee$, and if $\varphi : A \to \kappa$ is a regular Γ-norm on a Γ-complete set A, then for every $B \in \check{\Gamma}$ such that $B \subseteq A$, there is a $\eta < \kappa$ such that $\sup\{\varphi(x) : x \in B\} = \eta$ [1]. In this case we say that φ is $\check{\Gamma}$-bounded. Similarly say that a norm is κ-Suslin bounded if for every set $B \subseteq A$ which is κ-Suslin , $\sup\{\phi(x) : x \in B\} < \gamma$ for $\phi : A \to \gamma$.

THEOREM 2.5 (Steel, [27]). *Suppose Sep*$(\check{\Gamma})$ *and suppose* $\Delta = \Gamma \cap \check{\Gamma}$ *is closed under* $\exists^{\mathbb{R}}$*. Assume* $A \in \Delta$ *and that there is a norm* $\varphi : A \twoheadrightarrow \lambda$ *which is* $\underset{\sim}{\Sigma}^1_1$*-bounded, where* $\lambda = cof(o(\Delta))$*. Then there is a* $B \in \check{\Gamma}$ *such that* $A \cap B \notin \check{\Gamma}$*.*

A variation of the proof of the above theorem, shows the following limitation to the closure of the Steel pointclass under $\vee$.

THEOREM 2.6 (Steel). *Suppose Sep*$(\check{\Gamma})$ *and suppose* $\exists^{\mathbb{R}}\Delta \subseteq \Delta$ *and* $o(\Delta)$ *is singular. Then* $\check{\Gamma}$ *is not closed under intersections with* Δ *sets.*

PROOF. Let $\alpha = cof(o(\Delta)) < o(\Delta)$ and let $\{\kappa_\gamma : \gamma < \alpha\}$ be a cofinal sequence in $o(\Delta)$. Let U be a universal $\check{\Gamma}$ set. Let $A \in \Delta$ and let $\varphi : A \to \alpha$ be a Δ norm of length α. By the

[1] see [22], $4C.11$ for a proof of this fact

coding lemma there is a relation P such that

$$P(x,\varepsilon) \leftrightarrow \forall x \exists \varepsilon (x \in A \to U_{(\varepsilon)_0} = U^c_{(\varepsilon)_1} \wedge |U_{(\varepsilon)_0}|_W \geq \kappa_{\varphi(x)})$$

Notice that $P \in \Delta$. Now define the relation R as follows:

$$R(x,\varepsilon) \leftrightarrow x \in A \wedge (\varepsilon)_0 \notin U_{(\varepsilon)_1}$$

Then $R \in \Gamma$. But since the set $\{|R_x|_W : x \in A\}$ is cofinal in $o(\Delta)$, then $R \notin \Delta$ and so $R \notin \check{\Gamma}$. Also R can be written as:

$$R(x,\varepsilon) \leftrightarrow x \in A \wedge (\varepsilon)_0 \in U_{(\varepsilon)_0}$$

and so R is the intersection of a set in Δ and a set in $\check{\Gamma}$ which is not in $\check{\Gamma}$.

□

Steel conjectures whether the regularity of $o(\Delta)$ would imply closure of $\check{\Gamma}$ under intersections.

CONJECTURE 2.7 (Steel, [27]). If Γ is the Steel pointclass such that $o(\Delta)$ is regular and $\exists^{\mathbb{R}}\Delta \subseteq \Delta$ then Γ is closed under $\vee$.

Notice that the conjecture can be rephrased by asking that if Sep($\check{\Gamma}$), $\exists^{\mathbb{R}}\Delta \subseteq \Delta$ and $o(\Delta)$ is a regular cardinal, then $\bigcap_2 \check{\Gamma} \subseteq \check{\Gamma}$, and this is actually how the conjecture was originally stated.

The proof of the conjecture relies on a generalization of the boundedness property which we discussed briefly above. As in [27], let

$$C \doteq \{o(\Delta) : \exists^{\mathbb{R}}\Delta \subseteq \Delta \wedge \Delta \text{ is a selfdual pointclass}\}$$

Notice that there are cofinally many in Θ such ordinals $\kappa \in C$, since these are the places where we are at the base of a projective-like hierarchy of type **II**, **III** or **IV**. If $\kappa \in C$ and $cf(\kappa) > \omega$ then, as noted above, Steel shows in [27] that there is a Steel pointclass Γ such that $o(\Delta) = \kappa$.

The following is a weaker version of the main conjecture. Essentially it says that the Steel pointclass is closed under unions if Δ contains the κ-Suslin sets where $\kappa < cof(o(\Delta))$.

The proof uses the Martin-Monk method which exploits the fact that a certain strategy flips membership to construct two disjoint sets which are comeager.

Theorem 2.8 (Steel, [27]). *Let Γ be nonselfdual, closed under $\forall^{\mathbb{R}}$ and such that $PWO(\Gamma)$. Suppose that $\exists^{\mathbb{R}}\Delta \subseteq \Delta$. Then Γ is closed under union with κ-Suslin sets for $\kappa < cf(o(\Delta))$.*

This is turn gives the following boundedness principle:

Theorem 2.9 (Steel, see [6]). *Let $\gamma < \Theta$ be a limit ordinal. Then there is a set $A \subseteq \mathbb{R}$ and a norm $\varphi : A \to \gamma$ which is onto and κ-Suslin bounded for all $\kappa < cf(\gamma)$.*

Therefore Steel's conjecture is true in the least initial segment of the Wadge hierarchy containing the inductive sets, IND, since by a result of Kechris, every $A \subseteq \mathbb{R} \in$ HYP is κ-Suslin for $\kappa < \kappa^{\mathbb{R}}$ and scales can be localized to smaller pointclass within HYP. This implies the following corollary:

Corollary 2.10. *If Γ is the Steel pointclass and $IND \subseteq \Gamma$, then for $A \in IND, B \in \Gamma$, we have that $A \cup B \in \Gamma$.*

Our goal is to generalize the above boundedness principle to all sets in Δ associated to the Steel pointclass Γ.

Let Δ be a selfdual pointclass such that $\exists^{\mathbb{R}}\Delta \subseteq \Delta$. Let $\kappa = o(\Delta)$ be such that κ is regular. Let Γ be the Steel pointclass above Δ, so we have $\forall^{\mathbb{R}}\Gamma \subseteq \Gamma$ and PWO(Γ). We will show that Δ sets are bounded in the norm, which turns out to be the same as Γ being closed under $\vee$ by the lemma below.

First, we introduce the pointclass $\underset{\sim}{\Sigma}^1_1(A)$, for some $A \subseteq \mathbb{R}$. We will need this notion in the proof below.

Definition 2.11. Let $A \subseteq \mathbb{R}$. $\underset{\sim}{\Sigma}^1_1(A)$ is the pointclass of all sets B such that:

$$B(x) \leftrightarrow C(x) \vee \exists y(\forall n(y)_n \in A \wedge D(\langle x, y\rangle)),$$

where C and D are $\underset{\sim}{\Sigma}^1_1$ sets.

Notice that $\underset{\sim}{\Sigma}^1_1(A)$ is a pointclass which contains A, is closed under $\exists^{\mathbb{R}}, \vee, \wedge$. Let

$C = \emptyset$, then we have $A(x) \leftrightarrow \exists y(\forall n(y)_n \in A \wedge D(\langle x, y\rangle))$, where $D(z) \leftrightarrow \forall i, j(((z)_1)_i = (z)_1)_j \wedge x = ((z)_1)_0)$. D is a $\underset{\sim}{\Sigma}^1_1$ set and this shows that $A \in \underset{\sim}{\Sigma}^1_1(A)$. Also notice that $\underset{\sim}{\Sigma}^1_1(A)$ is indeed a pointclass since taking the preimage of a set in $\underset{\sim}{\Sigma}^1_1(A)$ yields another set with complexity $\underset{\sim}{\Sigma}^1_1(A)$. Next we show closure of $\underset{\sim}{\Sigma}^1_1(A)$ under $\vee$. Let $B, B' \in \underset{\sim}{\Sigma}^1_1(A)$ be written as $B(x) \leftrightarrow C(x) \vee \exists y(\forall n(y)_n \in A \wedge D(\langle x, y\rangle))$ and $B'(x) \leftrightarrow C'(x) \vee \exists z(\forall n(z)_n \in A \wedge D'(\langle x, z\rangle))$ where $C, C', D, D' \in \underset{\sim}{\Sigma}^1_1$. Then we have

$$[C(x) \vee \exists y(\forall n(y)_n \in A \wedge D(\langle x, y\rangle))] \vee [C'(x) \vee \exists z(\forall n(z)_n \in A \wedge D'(\langle x, z\rangle))] \leftrightarrow$$
$$F(x) \vee \exists w(\forall n(w)_n \in A \wedge (G(\langle x, y\rangle) \vee G(\langle x, z\rangle)))$$

where $F = C \cup C'$ is a $\underset{\sim}{\Sigma}^1_1$ set since $\underset{\sim}{\Sigma}^1_1$ is closed under arbitrary unions and $G = D' \cup D$ is a $\underset{\sim}{\Sigma}^1_1$ set since $\underset{\sim}{\Sigma}^1_1$ is closed under recursive substitutions. We next show that $\underset{\sim}{\Sigma}^1_1(A)$ is closed under $\exists^{\mathbb{R}}$. Let $B \in \underset{\sim}{\Sigma}^1_1(A)$ be given by $B(\langle x, z\rangle) \leftrightarrow C(\langle x, z\rangle) \vee \exists y(\forall n(y)_n \in A \wedge D(\langle\langle x, z\rangle, y\rangle))$ and let $U(x) \leftrightarrow \exists z B(\langle x, z\rangle)$ with $C, D \in \underset{\sim}{\Sigma}^1_1$. We show that $U \in \underset{\sim}{\Sigma}^1_1(A)$. But notice that

$$\exists z[C(\langle x, z\rangle) \vee \exists y(\forall n(y)_n \in A \wedge D(\langle\langle x, z\rangle, y\rangle))]$$

is logically equivalent to

$$\exists z C(\langle x, z\rangle) \vee \exists y(\forall n(y)_n \in A \wedge \exists z D(\langle\langle x, z\rangle, y\rangle)),$$

using that $\underset{\sim}{\Sigma}^1_1$ is closed under existential quantification. Finally $\underset{\sim}{\Sigma}^1_1(A)$ is closed under $\wedge$. To see this again let $B, B' \in \underset{\sim}{\Sigma}^1_1(A)$ be written as $B(x) \leftrightarrow C(x) \vee \exists y(\forall n(y)_n \in A \wedge D(\langle x, y\rangle))$ and $B'(x) \leftrightarrow C'(x) \vee \exists z(\forall n(z)_n \in A \wedge D'(\langle x, z\rangle))$ where $C, C', D, D' \in \underset{\sim}{\Sigma}^1_1$. We want to see that $B(x) \wedge B'(x) \in \underset{\sim}{\Sigma}^1_1(A)$. Then we consider

$$[C(x) \vee \exists y(\forall n(y)_n \in A \wedge D(\langle x, y\rangle))] \wedge [C'(x) \vee \exists z(\forall n(z)_n \in A \wedge D'(\langle x, z\rangle))].$$

To compute this just notice that when the whole expression is unfolded, the $\underset{\sim}{\Sigma}^1_1$ set C' can be pushed in the second disjunct defining the set B past the quantification over y so that we have

$$C'(x) \wedge \exists y(\forall n(y)_n \in A \wedge D(\langle x, y\rangle)) \leftrightarrow \exists y(\forall n(y)_n \in A \wedge D(\langle x, y\rangle) \wedge C')$$

and $D(\langle x,y\rangle)\wedge C'$ is now a $\underset{\sim}{\Sigma}^1_1$ set. Similarly for C and $\exists z(\forall n(z)_n \in A\wedge D'(\langle x,z\rangle))$. Also when the expression is unfolded one writes $\exists y(\forall n(y)_n \in A\wedge D(\langle x,y\rangle))\wedge\exists z(\forall n(z)_n \in A\wedge D'(\langle x,z\rangle))$ as

$$\exists w(\forall n(w)_n \in A \wedge \exists \varepsilon_0, \varepsilon_1(D(\langle x, \varepsilon_0\rangle) \wedge D'(\langle x, \varepsilon_1\rangle)) \wedge \forall j((\varepsilon_0)_j = (w)_{2j} \wedge (\varepsilon_1)_j = (w)_{2j+1}).$$

So w is now a single real witnessing the above conjunction in a "zig-zag" way. Notice that $\exists \varepsilon_0, \varepsilon_1(D(\langle x, \varepsilon_0\rangle) \wedge D'(\langle x, \varepsilon_1\rangle))$ is still a $\underset{\sim}{\Sigma}^1_1$ set and $\forall j((\varepsilon_0)_j = (w)_{2j} \wedge (\varepsilon_1)_j = (w)_{2j+1})$ is $\underset{\sim}{\Delta}^1_1$

These closure properties of $\underset{\sim}{\Sigma}^1_1(A)$ will be important below. The pointclass $\underset{\sim}{\Sigma}^1_1(A)$ also has a universal set which comes from the universal set for $\underset{\sim}{\Sigma}^1_1$ sets in a natural way. Let $U \subseteq \mathbb{R}^2$ be universal for $\underset{\sim}{\Sigma}^1_1$ sets of reals. Then define $V(\varepsilon, x) \leftrightarrow U(\varepsilon_0, x) \vee \exists y(\forall n(y)_n \in A \wedge U(\varepsilon_1, \langle x, y\rangle))$. Then $V \in \underset{\sim}{\Sigma}^1_1(A)$ and is universal for $\underset{\sim}{\Sigma}^1_1(A)$ sets of reals by letting $C(x) \leftrightarrow U_{\varepsilon_0}(x)$ and $D(\langle x,y\rangle) \leftrightarrow U_{\varepsilon_1}(\langle x,y\rangle)$ be the two $\underset{\sim}{\Sigma}^1_1$ sets coded by ε_0 and ε_1. Since we sometimes use the recursion theorem, we go ahead and recall the statements of the s-m-n and the recursion theorem:

Theorem 2.12 (s-m-n-theorem, recursion theorem, Kleene). *Let Γ be a pointclass with a universal set. Then there are universal sets $U_{\mathcal{X}} \subseteq \mathbb{R} \times \mathcal{X}$, for all perfect product spaces $\mathcal{X}$ with the following properties:*

(1) *(smn-theorem)*

For every $\mathcal{X} = X_1 \times ... \times X_n$ and $\mathcal{Y} = X_1 \times ... \times X_n \times ... \times X_m$, where $m > n$, there is a continuous function $s_{\mathcal{Y},\mathcal{X}} : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ such that

$$U_{\mathcal{Y}}(y, x_1, ..., x_n, .., x_m) \longleftrightarrow U_{\mathcal{X}'}(s_{\mathcal{Y},\mathcal{X}}(y, x_1, .., x_n), x_{n+1}, .., x_m),$$

where $\mathcal{X}' = X_{n+1} \times ... \times X_m$

(2) *(Recursion theorem)*

For every perfect product space $\mathcal{X} = X_1 \times ... \times X_n$ and Γ set $A \subseteq \mathbb{R} \times \mathcal{X}$, there is a $y^* \in \mathbb{R}$ such that for all $\vec{x} \in \mathcal{X}$,

$$U_{\mathcal{X}}(y^*, \vec{x}) \longleftrightarrow A(y^*, \vec{x})$$

We next show the following theorem, which reduces Steel's conjecture to the question of whether Δ sets are *bounded in the norm.* We say that Δ sets are bounded in the norm if there is a Δ-bounded norm, that is a norm $\varphi : P \to \kappa$ for some ordinal κ and a set $P \subseteq \mathbb{R}$ such that for every Δ set $S \subseteq P$, $\sup\{\varphi(x) : x \in S\} < \kappa$.

THEOREM 2.13. *Let Γ be the Steel pointclass and let $\Delta = \Gamma \cap \check{\Gamma}$ be such that $\exists^{\mathbb{R}}\Delta \subseteq \Delta$. Then the following are equivalent:*

(1) $\bigcup_2 \Gamma \subseteq \Gamma$,

(2) $\bigcup_\omega \Gamma \subseteq \Gamma$,

(3) Γ *is closed under union with Δ sets,*

(4) Δ *sets are bounded in the norm.*

PROOF. Let Γ be a nonselfdual pointclass such that $\exists^{\mathbb{R}}\Delta \subseteq \Delta$, PWO($\Gamma$) and Γ is closed under $\forall^{\mathbb{R}}$. (1) $\longrightarrow$ (2) holds because we have $\neg$Sep(Γ), this is theorem 2.2 in [27]. (2) $\longrightarrow$ (1) is immediate. That clause (2) implies clause (3) is also immediate. We next show that (3) implies (2). So let $A, B \in \Gamma$. We show that $A \cup B \in \Gamma$. Since Red(Γ) holds, we may assume that $A \cap B = \emptyset$. Let $A = \bigcup_{\beta<\alpha} A_\beta$ and $B = \bigcup_{\beta<\alpha} B_\beta$ where α is the ordinal such that $\bigcup_\alpha \Delta \nsubseteq \Delta$. Define

$$\Gamma^* = \{ \bigcup_{\alpha<o(\Delta)} A_\alpha : \forall\alpha(A_\alpha \in \Delta) \wedge \bigcup_{\alpha<o(\Delta)} A_\alpha \text{ is } \Delta \text{ bounded}\}$$

CLAIM 2.14. $\Gamma^* = \Gamma$

PROOF. We have $\Gamma^* \subseteq \Gamma$ since every set on Γ^* is a $\underset{\sim}{\Sigma}^1_1$-bounded union of set Δ sets. We next show that $\Gamma \subseteq \Gamma^*$. So let $A \in \Gamma \setminus \check{\Gamma}$. Let $A = \bigcup_{\beta<\alpha} A_\beta$ with $A_\beta \in \Delta$ for every $\beta < \alpha$ and α is least such that $\bigcup_\alpha \Delta \nsubseteq \Delta$. We may assume that the union is increasing. Let $\varphi : A \to \alpha$ be a $\underset{\sim}{\Sigma}^1_1$-bounded Γ-norm on A. Let $\{\kappa_\beta : \beta < \alpha\}$ be cofinal in $o(\Delta)$. Let U be a universal Γ set. Define the Solovay game as follows:

I	x
II	$\langle w, y, z\rangle$

The payoff condition is then defined by:

$$\text{Player II wins iff } x \in A \to (U_y = U_z^c = A_{\varphi(w)} \wedge |U_y|_W \geq \kappa_{\varphi(x)}).$$

Since φ is $\underset{\sim}{\Sigma}_1^1$-bounded then Player II has a wining strategy τ for this game. Then let

$$R(x, w, y) \leftrightarrow x \in A \wedge w = \tau(x)_0 \wedge U_{\tau(x)_1} = A_{\varphi(w)} \wedge y \notin U_{\tau(x)_2}.$$

Then we have that $\{|R_x|_W : x \in A\}$ is unbounded in $o(\Delta)$ and so $\{|A_\beta|_W : \beta < \alpha\}$ is unbounded in $o(\Delta)$.

Next for $\beta < \alpha$, let

$$C_\beta = \{(x, y) : y \in A_{\beta+1} \setminus A_\beta \wedge x \text{ codes a continuous function } f_x \text{ s.t } f_x^{-1}(A_\beta) \subseteq A\}.$$

Then for every $\beta < \alpha$, C_β is defined as $(\Delta \wedge \forall^{\mathbb{R}}(\Delta \vee \Gamma))$ and so because we are assuming that Γ is closed under unions with Δ sets, we have for every $\beta < \alpha$, $C_\beta \in \Gamma$. Let $C = \bigcup_{\beta<\alpha} C_\beta$. Then another Solovay game argument as above shows that $C \in \exists^{\mathbb{R}}\Gamma$. Actually one can show that $C \in \Gamma$. Notice that because $\exists^{\mathbb{R}}\Delta = \Delta$ and because $\Gamma = \bigcup_\alpha \Delta$, then $\exists^{\mathbb{R}}\Gamma \subseteq \bigcup_\alpha \Delta$. So let $D_\beta \in \Delta$ for every $\beta < \alpha$ such that $C = \bigcup_{\beta<\alpha} D_\beta$. We may assume that the union is increasing. Define the sets B_β by

$$B_\beta(z) \leftrightarrow \exists (x, y) \in D_\beta \exists \gamma \leq \beta (y \in A_{\gamma+1} \setminus A_\gamma \wedge f_x(z) \in A_\gamma).$$

Then $B_\beta \in \Delta$. Notice that $A = \bigcup_{\beta<\alpha} B_\beta$ and $\bigcup_{\beta<\alpha} B_\beta$ is Δ-bounded since every Δ set is coded as a set $f_x^{-1}(A_\beta)$ for some $\beta < \alpha$.

$\square$

Now recall that $A = \bigcup_{\beta<\alpha} A_\beta$ and $B = \bigcup_{\beta<\alpha} B_\beta$. These unions are Δ-bounded and increasing with each A_β and B_β in Δ. We show that $\bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)$ is Δ bounded. Then let $C \subseteq \bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)$ with $C \in \Delta$. Then $C \cap A \in \Gamma$ as Γ is closed under intersections. Also $C \cap A = C \cap B^c$ and $C \cap B^c \in \check{\Gamma}$, since by assumptions $\check{\Gamma}$ is closed under intersections with Δ sets. So $C \cap A \in \Delta$ and $\exists \gamma_1 < \alpha$ such that $C \cap A \subseteq A_{\gamma_1}$. Similarly, there exists a $\gamma_2 < \alpha$ such that $C \cap B \subseteq B_{\gamma_2}$. Let $\gamma = \max(\gamma_1, \gamma_2)$. Then $C \subseteq A_\gamma \cup B_\gamma$. So $A \cup B \in \Gamma$ and $\bigcup_2 \Gamma \subseteq \Gamma$.

Finally it just remains to show that Δ sets are bounded in the norm if and only if Γ is closed under unions with Δ sets. Recall that $o(\Delta) = \kappa$ is regular. We'll make use of this in the proof. Suppose first that Δ sets are bounded in the norm. We need to see that Γ is closed under unions with Δ sets. So let $A \in \Gamma$ such that $A = \bigcup_{\beta<\kappa} A_\beta$ with $A_\beta \in \Delta$ for every $\beta < \kappa$ and let $B \in \Delta$ such that $B = \bigcup_{\beta<\alpha} B_\beta$ for some $\alpha < \kappa$ with $B_\beta \in \Delta$ for every $\beta < \alpha$. It suffices to show that $A \cup B$ is Δ-bounded. We may assume that the unions are increasing and continuous, that is at all limit ordinal $\gamma < \kappa$ we have $A_\gamma = \bigcup_{\beta<\gamma} A_\beta$. So let $C \subseteq A \cup B$ such that $C \in \Delta$. We also have that

$$A \cup B = \bigcup_{\beta<\kappa} A_\beta \cup \bigcup_{\beta<\alpha} B_\beta = \bigcup_{\beta<\alpha} (A_\beta \cup B_\beta) \cup \bigcup_{\alpha<\xi<\kappa} A_\xi.$$

But notice that we must have $\bigcup_{\beta<\alpha}(A_\beta \cup B_\beta) \in \Delta$ since κ is a regular cardinal, $\alpha < \kappa$ and since $cof(\kappa) = \kappa$ is least such that $\bigcup_{cof(\kappa)} \Delta \nsubseteq \Delta$. So let $D = \bigcup_{\beta<\alpha}(A_\beta \cup B_\beta)$. Then $C \cup D \in \Delta$. So we have $C \cup D \subseteq \bigcup_{\alpha<\xi<\kappa} A_\xi =_{def} A' = A$, since the union is continuous. Let $\varphi : A' \to \kappa$ be the natural norm defined by $\varphi(x) =$ the least $\xi < \kappa$ such that $x \in A_\xi$. Since Δ sets are bounded in the norm and since κ is regular, there exists a $\xi_1 < \kappa$ be such that $C \cup D \subseteq A_{\xi_1}$. So the union $A \cup B$ is Δ bounded. Next we must show that a union is Δ-bounded union of Δ sets if and only if it is a Γ-complete set. This will ensure that $A \cup B$ is in $\Gamma \setminus \check{\Gamma}$. So let $A = \bigcup_{\alpha<\kappa} A_\alpha$ be a Δ-bounded union of Δ sets. We need to see that A is $\Gamma \setminus \check{\Gamma}$. We start first by showing that our assumption implies that if $A \in \Gamma \setminus \check{\Gamma}$ then A is a Δ bounded union of Δ sets. By PWO(Γ), let $\varphi : A \to \kappa$ be a Γ norm. Since Δ sets are bounded in the norm then for any Δ subset of $A_\alpha \subseteq A$, there exists an $\beta < \kappa$ such that elements of A_α are sent before β. In addition every initial segment of the norm φ is a Δ set. So A is a union of Δ sets which are Δ bounded. Now we justify why any Δ-bounded union of Δ sets is in $\Gamma \setminus \check{\Gamma}$. So let $A = \bigcup_{\alpha<\kappa} A_\alpha$ be a Δ bounded union of Δ sets. We may assume that the union is increasing and continuous. Consider the following game:

I $\quad x$

$$\text{II} \quad \langle w, y, z \rangle$$

The pay off condition is determined by player II wins the run of the game if and only if

$$x \in A \to \exists \alpha (U_w = U_y^c = A_\alpha \wedge x \in U_w \wedge z \in U_w)$$

Then player II has a winning strategy τ. Next notice that $x \in \bigcup A_\alpha \leftrightarrow x \in U_{\tau(x)_0} \wedge U_{\tau(x)_0} = U^c_{\tau(x)_1} \wedge \tau(x)_2 \in U_{\tau(x)_0}$. Then $\bigcup_{\alpha<\kappa} A_\alpha$ is in $\Gamma \setminus \Delta$. Thus $\bigcup_{\alpha<\kappa} A_\alpha \in \Gamma \setminus \check{\Gamma}$.

Finally notice that if Γ is closed under unions with Δ sets, then Γ is closed under finite unions by the above and thus Moschovakis argument (see $4.C.11$ in [22]) applies and this implies that Δ sets are bounded in the norm. This finishes the proof.

□

The following now shows Steel's conjecture. From it we obtain the above Δ-boundedness principle.

Theorem 2.15 (A, Jackson). *Assume ZF+DC+AD. Let κ be a cardinal such that $o(\Delta_\kappa) = \kappa$ where $\Delta_\kappa = \Gamma_\kappa \cap \check{\Gamma}_\kappa$ and Δ_κ is closed under $\exists^{\mathbb{R}}$, $\wedge$ and $\vee$. Assume Sep($\check{\Gamma}_\kappa$). Let $\lambda < cof(\kappa)$ be a cardinal such that $o(\Delta_\lambda) = \lambda$ and Δ_λ is closed under $\exists^{\mathbb{R}}$, $\wedge$ and $\vee$, where $\Delta_\lambda = \Gamma_\lambda \cap \check{\Gamma}_\lambda$. Assume Sep($\check{\Gamma}_\lambda$). Suppose that $\check{\Gamma}_\kappa \cap \Delta_\lambda \subseteq \check{\Gamma}_\kappa$. Then*

1. *$\check{\Gamma}_\kappa \cap \Gamma_\lambda \subseteq \check{\Gamma}_\kappa$ and more generally if Σ is the pointclass of λ length unions of Δ_λ sets, then $\check{\Gamma}_\kappa \cap \Sigma \subseteq \check{\Gamma}_\kappa$.*
2. *Γ_λ is not closed under real quantifiers then $\check{\Gamma}_\kappa \cap \check{\Gamma}_\lambda \subseteq \check{\Gamma}_\kappa$.*
3. *Suppose $cof(\lambda) = \omega$ and let Λ be the pointclass of all countable intersections of Δ_λ sets, i.e $\Lambda = \bigcap_\omega \Delta_\lambda$ then $\check{\Gamma}_\kappa \cap \Lambda \subseteq \check{\Gamma}_\kappa$.*
4. *Suppose $cof(\lambda) = \omega_1$ and let Λ be the pointclass of all length ω_1 intersections Δ_λ sets, i.e $\Lambda = \bigcap_{\alpha<\omega_1} \Delta_\lambda$ then $\check{\Gamma}_\kappa \cap \Lambda \subseteq \check{\Gamma}_\kappa$. Moreover if $\lambda < \kappa$ is a regular cardinal, then $\check{\Gamma}_\kappa \cap \Lambda \subseteq \check{\Gamma}_\kappa$ where Λ is the pointclass of all intersections of Δ_λ sets of length λ.*

PROOF. We begin by showing $\check{\Gamma}_\kappa \cap \Gamma_\lambda \subseteq \check{\Gamma}_\kappa$. Let then $A \in \Gamma_\lambda$ and $B \in \check{\Gamma}_\kappa$. Let $A = \bigcup_{\alpha<\lambda} A_\alpha$ where for every $\alpha < \lambda$, $A_\alpha \in \Delta_\lambda$.

Let σ be a winning strategy for player I in the Wadge game $G_{A\cap B,B}$, that is:

$$x \notin B \to \sigma(x) \in A \cap B$$

$$x \in B \to \sigma(x) \notin A \cap B$$

As in Steel [27], we define a sequence of winning strategies $\langle \sigma_n : n \in \omega \rangle$ for I in the game $G_{A\cap B,B}$. Suppose σ_k is defined for all $k < n$. We also let τ be the copying strategy for II. For any $x \in \mathbb{R}$ we let

$$\tau_n = \begin{cases} \sigma_n & \text{if } x(n) = 0 \\ \tau & \text{if } x(n) = 1 \end{cases}$$

....	τ_3	τ_2	τ_1	τ_0
....	$x_3(0)$	$x_2(0)$	$x_1(0)$	$x_0(0)$
....	....	$x_2(1)$	$x_1(1)$	$x_0(1)$
....	....	....	$x_1(2)$	$x_0(2)$
....	....	....	....	$x_0(3)$
....	....	....	....	....
....	x_3	x_2	x_1	x_0

TABLE 2.1. Diagram of Martin-Monk games

At stage n we have a pair of Δ_κ inseparable sets C and D such that $D \in \check{\Gamma}_\kappa$. That is we have $C \subseteq B^c$ and $D \subseteq B$ with $D \in \check{\Gamma}_\kappa$ and B as above. Let $E_\alpha = \{x : \sigma(x) \in A_\alpha\}$. Then we have $E_\alpha \in \Delta_\lambda$. Now by assumption we have that $D \cap E_\alpha = D_\alpha \in \check{\Gamma}_\kappa$. We show the following claim:

CLAIM 2.16. *For some $\alpha < \lambda$, $C \cap E_\alpha$ is Δ_κ-inseparable from $D \cap E_\alpha$.*

Proof. Notice that since $\lambda < cof(\kappa)$ and by the Coding lemma applied to $\check{\Gamma}_\kappa$, for some $\alpha < \lambda$, $C_\alpha = C \cap E_\alpha$ must be Δ_κ-inseparable from D (otherwise C and D would not be Δ_κ inseparable, since $C = \bigcup_{\alpha<\lambda} C_\alpha$. This then implies that C_α is Δ_κ inseparable from $D_\alpha = D \cap E_\alpha$ since if not then let $F \in \Delta_\kappa$ separate C_α from D_α, that is we have $C_\alpha \subseteq F$ and $F \cap D_\alpha = \emptyset$. This would then imply that $F \cap E_\alpha$ separates C_α from D.

□

Next, consider the game in which player I plays x and player II plays y and player I wins iff

$$x \notin B \to y \in C_\alpha$$

$$x \in B \to y \in D_\alpha$$

Notice that player II cannot have a winning strategy τ in this game since if τ is a winning strategy then we have $y \in C_\alpha \to \tau(y) \in B$ and $y \in D_\alpha \to \tau(y) \notin B$. But this then implies that $C_\alpha \subseteq \tau^{-1}(B)$ and $\tau^{-1}(B) \cap D_\alpha = \emptyset$. But $\tau^{-1}(B), D_\alpha \in \check{\Gamma}_\kappa$ so by $\text{Sep}(\check{\Gamma}_\kappa)$, there is a Δ_κ set which separates C_α from D_α, contradiction!

So fix a winning strategy ρ for player I in the separation game and let $\sigma_n = \sigma \circ \rho$. Notice then that $x \notin B \to \rho(x) \in C_\alpha \subseteq E_\alpha$, so we have that $\sigma \circ \rho(x) \subseteq A_\alpha \subseteq A$. Also $x \in B \to \rho(x) \in D_\alpha \subseteq E_\alpha$ so we have that $\sigma \circ \rho(x) \in A_\alpha \subseteq A$. Therefore the strategies σ_n always give a play which is in A. We also need to see that σ_n flips membership in B for every $n \in \omega$. Notice that $x \notin B \to \rho(x) \in C_\alpha \subseteq B^c$ so $\sigma \circ \rho(x) \in B$. Also $x \in B \to \rho(x) \in D_\alpha$ and $\sigma \circ \rho(x) \in A$. Therefore $\sigma \circ \rho(x) \notin B$. So we have $x \notin B \to \sigma \circ \rho(x) \in A \cap B$ and $x \in B \to \sigma \circ \rho(x) \in A \cap B^c$. This now allows us to derive a contradiction as in Martin-Monk proof that $\leq_W$ is a prewellorder. Namely, let $I = \{x \in \mathbb{R} : \forall^\infty n x(n) = 0\}$ and let $M = \{x \in I : x_0 \in B\}$. M has the Baire property so there is a cone N_s determined by some $s \in \omega^{<\omega}$ on which M is meager or comeager. Let $i \notin dom(s)$ and let

$$T(x)(k) = \begin{cases} x(k) & \text{if } i \neq k \\ 1 - x(k) & \text{if } i = k \end{cases}$$

T is a homeomorphism and we have $T"N_s = N_s$. Recall that x_k is the real obtained after filling the diagram of Martin-Monk game. Then if $x \in I$ then $T(x)_k = x_k$ for $i < k$ and $T(x)_k \in B$ if and only if $x_k \notin B$ if $k \leq i$. So we have $T"(M \cap I \cap N_s) = M^c \cap I \cap N_s$. But since I was comeager, this is a contradiction. This finishes the proof in the case where $\check{\Gamma}_\kappa \cap \Gamma_\lambda \subseteq \check{\Gamma}_\kappa$. Next we show the theorem in the case where $\check{\Gamma}_\kappa \cap \check{\Gamma}_\lambda \subseteq \check{\Gamma}_\kappa$.

Next let $A = \bigcap_{\alpha<\lambda} A_\alpha$ with $A_\alpha \in \Delta_\lambda$, so that $A \in \check{\Gamma}_\lambda$. That is A is a $\underset{\sim}{\Sigma}^1_1$ bounded intersection of sets in Δ_λ, that is the collection $\{A^c_\alpha\}_{\alpha<\lambda}$ is a $\underset{\sim}{\Sigma}^1_1$ bounded union of sets in Δ_λ. Let $B \in \check{\Gamma}_\kappa$. Next let φ be a prewellordering on a set $F \subseteq \mathbb{R}$ of length λ such that

(1) All initial segments of φ are in Δ_λ.

(2) $F \in \Gamma_\lambda$,that is F is a $\underset{\sim}{\Sigma}^1_1$ bounded union of Δ_λ sets.

This is always possible since if Γ_λ is a Steel pointclass closed under $\forall^\mathbb{R}$ we can define $\varphi \in \Gamma_\lambda$. We will denote F by F_φ. F_φ is of course in Γ_λ. We will also let $\{F_\alpha\}_{\alpha<\lambda}$ be a λ sequence of Δ_λ sets such that $F_\varphi = \bigcup_{\alpha<\lambda} F_\alpha$ is a $\underset{\sim}{\Sigma}^1_1$-bounded union of Δ_λ sets. For every $\alpha < \lambda$, we then consider the game where player I plays a real x and player II plays a real y and player II wins the run of the game iff $x \notin A \to \exists\alpha\exists\beta(y \in F_\alpha \wedge \varphi(y) = \beta \wedge x \notin A_{\varphi(y)})$. Then II has a winning strategy ρ for this game by $\underset{\sim}{\Sigma}^1_1$-boundedness [2]. Let σ be as in the previous case. We want to define a sequence of strategies $\langle\sigma_n : n \in \omega\rangle$. At stage n we have σ_n and a pair of Δ_κ-inseparable sets C_n and D_n, where $C_n \subseteq B^c$ and $D_n \subseteq B$. For $\alpha < \lambda$, let $E_\alpha = \{x : \rho \circ \sigma(x) \notin F_\alpha \vee (|\rho \circ \sigma(x)| = \alpha \wedge \sigma(x) \in A_\alpha)\}$. Notice that we have $F_\alpha \in \Delta_\kappa$. We also let as above $C_\alpha = C \cap E_\alpha$ and $D_\alpha = D \cap E_\alpha$. Then again by the coding lemma (and since $\lambda < cof(\kappa)$). we must have that for some $\alpha < \lambda$, C_α must be Δ_κ-inseparable from D since if not D and C would not be Δ_κ-inseparable. We must then have that C_α must be Δ_κ-inseparable from D_α. Notice also that

$$D_\alpha = D \cap \{x : \rho \circ \sigma(x) \notin F_\alpha\} \cup (D \cap \{x : |\rho \circ \sigma(x)| = \alpha \wedge \sigma(x) \in A_\alpha\}).$$

[2]recall that A^c is a $\underset{\sim}{\Sigma}^1_1$ bounded union of Δ_λ sets

Then since the set $\{x : |\rho \circ \sigma(x)| = \alpha \wedge \sigma(x) \in A_\alpha\}$ and the set F_α are both in Δ_λ and since $D \in \check{\Gamma}_\kappa$ then D_α must be in $\check{\Gamma}_\kappa$. Now as above we consider the separation game in which player I plays a real x and player II plays a real y and player I wins iff

$$x \notin B \rightarrow y \in C_\alpha$$

$$x \in B \rightarrow y \in D_\alpha$$

Player II cannot have a winning strategy τ in this game since then if τ is winning for II then we have $y \in C_\alpha \rightarrow \tau(y) \in B$ and $y \in D_\alpha \rightarrow \tau(y) \notin B$. This would then imply that $C_\alpha \subseteq \tau^{-1}(B)$ and $\tau^{-1}(B) \cap D_\alpha = \emptyset$. But since both $\tau^{-1}(B)$ and D_α are in $\check{\Gamma}_\kappa$, then by $\mathrm{Sep}(\check{\Gamma}_\kappa)$, there is a Δ_κ set which separates C_α from D_α, contradiction!

So we fix a winning strategy ε for player I and we let $\sigma_n = \sigma \circ \varepsilon$. Notice that ε is winning for I for every $\alpha < \lambda$. Suppose first that $\rho \circ \sigma \circ \varepsilon(x) \in F_\alpha$. Then we have $x \notin B \rightarrow \varepsilon(x) \in C_\alpha \subseteq E_\alpha$ and so we have $\sigma \circ \varepsilon(x) \in A_\alpha$ for every $\alpha < \lambda$. Also $x \in B \rightarrow \varepsilon(x) \in D_\alpha \subseteq E_\alpha$ so we have $\sigma \circ \varepsilon(x) \in A_\alpha$ for every $\alpha < \lambda$. In both cases, if $\rho \circ \sigma \circ \varepsilon(x) \notin F_\alpha$, for every $\alpha < \lambda$, then $\sigma \circ \varepsilon(x) \in A$, since ρ is winning for player II in the above game involving F_α.

Now as above this gives a contradiction by the Martin-Monk argument.

Finally, we show that if $cof(\lambda) = \omega$ and let Λ be the pointclass of all countable intersections of Δ_λ sets, i.e $\Lambda = \bigcap_\omega \Delta_\lambda$ then $\check{\Gamma}_\kappa \cap \Lambda \subseteq \check{\Gamma}_\kappa$. Notice that $\check{\Gamma}_\lambda \subseteq \bigcap_\omega \Delta_\lambda$. We let $A \in \bigcap_\omega \Delta_\lambda$ and $B \in \check{\Gamma}_\kappa$. We need to see that $A \cap B \in \check{\Gamma}_\kappa$. Let $A = \bigcap_{n<\omega} A_n$, where for every $n < \omega$, $A_n \in \Delta_\lambda$. As above suppose not. Then this means that player I wins the following Wadge game:

$$x \notin B \rightarrow \sigma(x) \in A \cap B$$

$$x \in B \rightarrow \sigma(x) \notin A \cap B$$

σ is a winning strategy for player I in the Wadge game $G_{A\cap B,B}$. We wish to define strategies σ_n as above such that we can fill the diagram of Martin-Monk games and derive a contradiction using the usual Martin-Monk argument. We then define the strategies σ_n inductively. Suppose σ_n has been defined at stage n. We show how to define σ_{n+1} at stage $n+1$. Define the set X_i as follows: $X_i = \{x : \sigma(x) \in A \wedge \exists i(\sigma \circ \sigma_n \circ ... \circ \sigma_i(x) \notin A_i)\}$. Notice that $X_i \in \Delta_\kappa$. Then there is an i such that $B^c \cap X_i$ is Δ_κ inseparable from $B \cap A_i$, since $B^c \cap X_i$ is Δ_κ inseparable from B. In addition we have $\bigcap_{i<\omega} B \cap A_i = B \cap A$. This means that we can run the separation game argument: player I wins the following game

$$x \notin B \rightarrow y \in B^c \cap X_i$$

$$x \in B \rightarrow y \in B \cap A_i$$

The Martin-Monk contradiction can be carried out as above now.

□

Next we show that if λ has cofinality ω_1, then $\check{\Gamma}_\kappa$ is closed under intersections with the pointclass Λ of ω_1 length intersections of Δ_λ sets. So let $A \in \Lambda$ be such that $A = \bigcap_{\alpha<\omega_1} A_\alpha$ where $A_\alpha \in \Delta_\lambda$ for every $\alpha < \omega_1$. Let $B \in \check{\Gamma}_\kappa$. Suppose again that $A \cap B \notin \check{\Gamma}_\kappa$. Therefore we can fix a winning strategy σ for player I in the Wadge game $G_{A,A\cap B}$. Again our goal will be to define a sequence of winning strategies $\langle \sigma_n : n < \infty \rangle$ for which we can carry out the Martin-Monk contradiction. Recall that the Wadge game $G_{A,A\cap B}$ is given by:

$$x \notin B \rightarrow \sigma(x) \in A \cap B$$

$$x \in B \rightarrow \sigma(x) \notin A \cap B$$

Notice that σ flips membership in B if $\sigma(x) \in A$. For every $\alpha < \omega_1$ there are strategies for player I, $\sigma^0_\alpha, \sigma^1_\alpha, \sigma^2_\alpha, ...$ such that the following Martin-Monk diagram of games is filled up, that is for any $z \in 2^\omega$ the strategies σ^n_α are picked. Notice that we cannot pick the strategies σ^n_α in function of α.

....	τ	τ	τ	τ
....	σ_α^3	σ_α^2	σ_α^1	σ_α^0
....	$x_3(0)$	$x_2(0)$	$x_1(0)$	$x_0(0)$
....	....	$x_2(1)$	$x_1(1)$	$x_0(1)$
....	....	....	$x_1(2)$	$x_0(2)$
....	....	....	....	....
....	....	....	....	....
....	x_3	x_2	x_1	x_0

TABLE 2.2. Diagram of Martin-Monk games in the $cf(\omega_1)$ case

and such that for $z \in 2^\omega$, the digits of z chose either the copying strategy τ or σ_α^n for a given n.The strategies σ_α^n have the following property. For every n,

(1) If $x_{n+1} \notin B$, then $\sigma_{j,n}(x_{n+1}) = \sigma_j \circ ... \circ \sigma_n(x_{n+1}) \in A, \forall j \leq n$,
(2) If $x_{n+1} \in B$ then $\sigma_{j,n}(x_{n+1}) \in A_\alpha, \forall j \leq n$ and
(3) If $x_{n+1} \notin B$ then $\sigma_\alpha^n(x_{n+1}) \in B$ and if $x_{n+1} \in B$ and $\sigma_\alpha^n(x_{n+1}) \in A$ then we have $\sigma_\alpha^n(x_{n+1}) \notin B$.

We now show the following claim:

CLAIM 2.17. *The strategies σ_α^n exist, for any $\alpha < \omega_1$.*

PROOF. We start with the case $n = 0$. First notice that if $x \notin B$ then $\sigma(x) \in B \cap A \subseteq B \cap A_\alpha$ and $B \cap A_\alpha \in \check{\Gamma}_\kappa$. Now B and B^c cannot be separated by a Δ_κ set, therefore B^c cannot be separated by a Δ_κ set from $B \cap \{x : \sigma(x) \in A_\alpha\}$, which is in $\check{\Gamma}_\kappa$ so there is a strategy ρ for player I in the separation game such that if $x \notin B$ then $\rho(x) \notin B$ and if $x \in B$ then $\rho(x) \in B \cap \sigma^{-1}" A_\alpha$. Then let $\sigma_\alpha^0 = \sigma \circ \rho$. σ_α^0 has the above properties and flips membership. We now show the general case. Assume that $\sigma_\alpha^0, ..., \sigma_\alpha^{n-1}$ are defined. We show how to define σ_α^n. As in Steel [27], this is done in 2^n steps, depending on whether $z \in 2^\omega$ chooses τ or σ_α^n. Let

$$X_{n+1} = \{x_{n+1} : \sigma(x_{n+1}) \in A \wedge \exists i \leq n(x_i \notin A)\}.$$

Notice that $B \cap X_{n+1} = \emptyset$. Then $B^c \setminus X_{n+1}$ and B are Δ_κ inseparable. This then implies that $B^c \setminus X_{n+1}$ and

$$B \cap \{x_{n+1} : \forall i \leq n \sigma_\alpha^i \circ \sigma_\alpha^{i+1} \circ ... \circ \sigma_\alpha^{n-1} \circ \sigma(x_{n+1}) \in A_\alpha\}$$

are Δ_κ inseparable. Then by the separation game we have a wining strategy ρ for player I such that if $x_{n+1} \notin B$ then $\rho(x_{n+1}) \in B^c \setminus X_{n+1}$ and if $x \in B$ then we have

$$\rho(x_{n+1}) \in B \cap \{x_{n+1} : \sigma_\alpha^i \circ ... \circ \sigma_\alpha^{n-1} \sigma(x_{n+1}) \in A_\alpha \text{ for all } i \leq n\}.$$

Then let $\sigma_\alpha^n = \sigma \circ \rho$.

□

Next by the Coding lemma and by uniformization we have function $f : x \to \sigma_x^n$ on the set WO such that for $x \in$ WO, the strategies $\{\sigma_x^n\}$ are as in $\{\sigma_\alpha^n\}$ for $\alpha = |x|$. We will use the theory of generic codes of Kechris and Woodin. Fix then a generic coding function $f : \omega_1^\omega \to \mathbb{R}$. Recall that α^ω equipped with the product of the discrete topology carries all notions of category. The function $f : \alpha^\omega \to \mathbb{R}$ is such that $\forall \alpha < \omega_1, \forall \vec{\alpha} \in \alpha^\omega, f(\alpha^\frown \vec{\alpha}) \in$ WO and $\forall^* \vec{\alpha} \in \alpha^\omega |f(\alpha^\frown \vec{\alpha})| = x$, where $|x| = \alpha$. We now define a branch $b \in \omega_1^\omega$ which will be used to witness that we have strategies for player I $\tilde{\sigma}_0, \tilde{\sigma}_1, ..., \tilde{\sigma}_n, ...$, from which we obtain the usual Martin-Monk contradiction. We define $b = lim_n b_n$, and $b_n \in \omega_1^{<\omega}$. Suppose then that b_{n-1} is defined. We show how to define b_n. In addition, we define a sequence of ordinals θ_n as we define the b_n for all n. We also let $b_0 \subseteq b_1 \subseteq ... \subseteq b_n \subseteq ...$ and $b = \bigcup_n b_n$. First extend b_{n-1} to b_n' such that there is a sequence $t_n \subseteq s_n$, where $s_n \in 2^{<\omega}$ and t_n is the n^{th}-sequence in an enumeration of sequence in $2^{<\omega}$, such that

$$\forall^*_{W_1^1} \alpha < \omega_1 \forall^*_{b_n'} \vec{\alpha} \in \alpha^\omega, \sigma^0_{f(\alpha^\frown \vec{\alpha})} \restriction n, ..., \sigma^n_{f(\alpha^\frown \vec{\alpha})} \restriction n$$

are fixed. Here $\sigma^i_{f(\alpha^\frown \vec{\alpha})} \restriction n$ means we use $z \in 2^\omega$ to decide whether we use τ or $\sigma^i_{f(\alpha^\frown \vec{\alpha})}$ to fill the Martin-Monk diagram. This fixes the values of $\tilde{\sigma}_0 \restriction n, ..., \tilde{\sigma}_n \restriction n$.

Next fix a relation $R(x, y) \leftrightarrow x \in$ WO $\wedge\, y \in A_{|x|}$. Let $\vec{\psi}_n$ be a scale on R. We now define $\theta_n(z)$ for all $z \in N_{t_n}$. By additivity of category, we will get $t_n \subseteq s_n$ such that $\forall^*_{s_n} z b_n(z) = b_n \wedge \theta_n(z) = \theta_n$. So fix $z \in N_{s_n}$ and define $\theta_n(z)$ as follows. Consider the game

$G^z_{\alpha,\vec{\alpha}}$ as in Becker and Kechris: player I plays a real $x_1 \in 2^\omega$ and a sequence of ordinals below α, $\vec{\alpha}_n \in \alpha^\omega$. Player II answers by playing a real $x_2 \in 2^\omega$, a sequence of ordinals $\vec{\beta}_n \in \alpha^\omega$, finitely many reals $y_0, ..., y_n$, finitely many sequences of ordinals $\vec{\xi}^0, ..., \vec{\xi}^n < \sup\{\vec{\psi}_n\}$, finitely many reals $w_0, ..., w_n$, finitely many sequences of ordinals $\vec{\gamma}_0, ..., \vec{\gamma}_n$ each ordinals of which is below ω_1 and finitely many sequences of integers $\vec{\eta}_0, ..., \vec{\eta}_n$. In addition player II must play so that $y_i \restriction n = \sigma^i_{f(\alpha^\frown\vec{\alpha})}(z) \restriction n$. The payoff is defined as follows: player II wins provided

$$(x_1 \restriction n, \alpha^\frown\vec{\alpha} \restriction n) \in T_{\text{wo}} \to ((x_2 \restriction n, \alpha^\frown\vec{\beta} \restriction n) \in T_{\text{wo}} \wedge (x_1 \restriction n, x_2 \restriction n, w_1 \restriction n, ..., w_n \restriction n, \eta_0 \restriction n, ..., \eta_n \restriction n) \in S),$$

where S is a tree on ω^6 witnessing that

$$(x_1 \restriction n, |w_1|, ..., |w_n|) \in T_{\text{wo}} \restriction |x_2| \text{ and } (x_2, y_i, \xi^i) \in T_{\vec{\psi}},$$

where T_{ψ_i} is the tree from the scale $\vec{\psi}$. The relation $(x_1 \restriction n, |w_1|, ..., |w_n|) \in T_{\text{wo}} \restriction |x_2|$ is Σ^1_1 in the codes for $w_1, ..., w_n, x_1, x_2$. This is closed game for II for if the run of the game in infinite then II wins. For each $z \in N_{s_n}$ and for each $\alpha < \omega_1$ and each $\vec{\alpha} \in \alpha^\omega$, II has a canonical winning strategy in $G^z_{\alpha,\vec{\alpha}}$. We call this canonical wining strategy $\tau^z_{\alpha,\vec{\alpha}}$. We define $\theta_n(z) = \langle \theta^{\pi_0}_n(z), ..., \theta^{\pi_k}_n(z)\rangle$ and b_n extending b'_n to satisfy the following. We first extend successively b'_n to $b^{\pi_0}_n, b^{\pi_1}_n, ..., b^{\pi_n}_n$ to obtain $b^{\pi_0}_n \subseteq b^{\pi_1}_n \subseteq ... \subseteq b^{\pi_n}_n$. We will then let $b_n = b^{\pi_n}_n$, so that b_n does not depend on which permutation we consider. Let $\pi = \pi_i$ be possible permutations of $n-1$. Let

$$b^\pi_n = [(\alpha_0, ..., \alpha_{n-1}) \to b^\pi_n(\alpha_0, ..., \alpha_{n-1})]_{W^{n-1}_1}.$$

This defines b_n if we define $b^\pi_n(\alpha_0, ..., \alpha_{n-1})$. Now we define $\theta^\pi_n(z)(\alpha_0, ..., \alpha_{n-1}, \alpha)$ and $b^\pi_n(\alpha_0, ..., \alpha_{n-1})$ by the following equation:

$$\forall^*_{W^n_1}(\alpha_0, ..., \alpha_{n-1}, \alpha)\forall^*_{b_n(\alpha_0,...,\alpha_{n-1})}\vec{\alpha} \in \alpha^\omega \tau^{z,\pi}_{\alpha,\vec{\alpha}} = \tau^z_{\alpha,\vec{\alpha}}(\alpha_0, ..., \alpha_{n-1}, \alpha) = \theta^\pi_n(z)(\alpha_0, ..., \alpha_{n-1}, \alpha),$$

where $x_1 \restriction n \cong \pi$ and $\tau^{z,\pi}_{\alpha,\vec{\alpha}}$ is restricted to sequences $\vec{\alpha}$ order-isomorphic to the permutation π. Notice that on a measure one set the strategies $\tilde{\sigma}_0, ..., \tilde{\sigma}_n$ are defined.

We then have a comeager set $G \subseteq 2^\omega$, which is the intersection of the comeager sets N_{s_n} defined above, where the s_n are dense in $2^{<\omega}$. By countable additivity of the measures W_1^n we can fix the s_n and by additivity of category, a comeager set for each s_n.

We now show this next claim:

Claim 2.18. *For any $z \in G$ if we fill the diagram using the strategies $\tilde{\sigma}_n$ if $z(n) = 1$ and τ if $z(n) = 0$ then the resulting $x_0, x_1, ..., x_n, ...$ are in A.*

Proof. We show that $x_i \in A_{\alpha_0}$ for all α_0 and for all i. Fix a measure one sets A_n with respect to W_1^n, so that we have

$$\forall^*_{W_1^n}(\alpha_0, ..., \alpha_{n-1}, \alpha)\forall^*_{b_n(\alpha_0,...,\alpha_{n-1})}\vec{\alpha} \in \alpha^\omega \tau^{z,\pi}_{\alpha,\vec{\alpha}} = \tau^z_{\alpha,\vec{\alpha}}(\alpha_0, ..., \alpha_{n-1}, \alpha) = \theta^\pi_n(z)(\alpha_0, ..., \alpha_{n-1}, \alpha),$$

for all $(\alpha_0, ..., \alpha_{n-1}, \alpha) \in A_n$. Let $C_n \subseteq \omega_1$ be c.u.b sets generating the A_n and let $C = \bigcap_n C_n$. Let $\alpha > \alpha_0$ be a closure point of C. Let $x_1 \in \mathrm{WO}$ such that $|x_1| = \alpha$. Let $(\alpha_0, \alpha_1, ...) \in C^\omega$ be such that $(x_1, \alpha, \alpha_0, \alpha_1, ...) \in T_{\mathrm{WO}}$ by homogeneity of T_{WO}. This then defines the sequence $b_0 = b(\alpha_0), b_1 = b(\alpha_0, \alpha_1)$. Let $\pi_n \cong x_n \restriction n$. From the equation we have, we can fix $\pi_0, \pi_1, ...$ such that

$$\forall^*_{b_n(\alpha_0,...,\alpha_{n-1})}\vec{\alpha} \in \alpha^\omega \theta^{\pi_n}_n(z)(\alpha_0, ..., \alpha_n, \alpha) = \tau^z_{\alpha,\vec{\alpha}}(\alpha_0, ..., \alpha_{n-1}, \alpha)$$

a run of $G^z_{\alpha,\vec{\alpha}}$ in which II has not yet lost. This then shows that II wins $G^z_{\alpha,\vec{\alpha}}$ where player I plays x_1 and $(\alpha_1, \alpha_1, ...)$ as above. In this run of $G^z_{\alpha,\vec{\alpha}}$ the reals $y_0, y_1, ...$ produced are equal to $\tilde{\sigma}_0(z), \tilde{\sigma}_1(z), ...$ So we have $\tilde{\sigma}_n(z) \in A_\alpha$ for all n. Contradiction!

$\square$

Finally the following claim concludes the proof.

Claim 2.19. *$\forall n, \tilde{\sigma}_n$ flips membership in B in that if $x \notin B$ then $\tilde{\sigma}_n(x) \in B$ and if $x \in B$ and $\tilde{\sigma}_n(x) \in A$ then $\tilde{\sigma}_n(x) \notin B$.*

Proof. We just have to modify the above game so that player I has to produce ordinals $\delta_0, \delta_1, ...$ which witness $(\tilde{\sigma}_n \restriction n, \vec{\delta} \restriction n)$ are in the tree witnessing the above two properties of σ. Therefore the $\tilde{\sigma}_n$ have the above two properties. So for $z \in G$, the $\tilde{\sigma}_n$ then give a contradiction in the Martin-Monk argument.

$\square$

We next outline how to extend to the previous argument to work for any $\lambda < \kappa$ with λ a regular cardinal. The set up is basically the same except we need to modify the definition of the generic coding function f. We then start out by fixing a regular cardinal λ and assume that we are within scales. We fix a scale $\vec{\varphi}$ on a universal Γ_λ set W. Again for every $\alpha < \lambda$, one can show that the strategies σ_α^n exists. We may pick a $\lambda' > \lambda$ with $\lambda' < \kappa$ such that the scale $\vec{\varphi}$ appears. Notice that the scale $\vec{\varphi}$ may be a lot more complicated than $\Gamma_{\lambda'}$. We also let T_W be the tree from the scale and assume for notational simplicity that it is a tree on $2 \times \lambda'$.

Once the strategies σ_α^n are shown to exist for every $\alpha < \lambda$ then by the Coding lemma and by uniformization we have a function $f : W \to \{\sigma_{|x|}^n\}$ such that the strategies $\{\sigma_{|x|}^n\}$ are as expected. Next we then define the generic coding function $f : (\lambda')^\omega \to \mathbb{R}$. The only difference is that now we need to take the supercompactness measures on ω_1 into account since these appear in the general definition of the generic coding function. Notice that f has the following two properties:

(1) $\forall \alpha < \lambda \forall \vec{\alpha} \in \alpha^\omega f(\alpha, \vec{\alpha}) \in W$

(2) $\forall \alpha < \lambda \forall_\nu^* S \in \mathcal{P}_{\omega_1}(\lambda') \forall \vec{\alpha} \in S^\omega |f(\alpha, \vec{\alpha})| = \alpha$, where $f(\alpha, \vec{\alpha}) = x$ and $|x| = \alpha$.

The main points are the following. First we fix homogeneity measure $\langle \mu_u : u \in 2^{<\omega}$ for the tree T_W. As above we must define a branch b_n and the ordinals $\theta_n^u(\alpha_0, ..., \alpha_n)$ which correspond to canonical strategies in the Becker-Kechris game. We then fix a neighborhood determined by t_n (recall these correspond to $z \in 2^\omega$ which determines which strategies to chose to fill up the Martin-Monk diagram) We then define for sequences $u \in 2^{<\omega}$ such that $lh(u) = n$ the product measure $\mu_n = \prod_{\{u : lh(u) = n\}} \mu_u$. We do this in order to handle all possible sequences u of a speficic length in our quantifiers computations. Notice that if $u_0 \subseteq u_1$ then by homogeneity the measure μ_{u_1} naturally projects to μ_{u_0}. However if we have two sequence u_0 and u_1 such that $u_0 \not\subseteq u_1$ and $u_1 \not\subseteq u_0$ then we must go to a more general measure which projects to both μ_{u_0} and μ_{u_1} in order to define the ordinal, θ^u. Notice that the product measure μ_n projects to each μ_{u_i} for $i \leq k$, some $k < \omega$ and need not be normal.

We define θ_n^u as follows:

$$\forall^*_{t_n} z \forall u \in 2^n \forall^*_{\mu_n}(\alpha_0, ..., \alpha_{n-1}) \forall^*_\nu S \in \mathcal{P}_{\omega_1}(\lambda') \forall^*_{b_n(\alpha_0,...,\alpha_{n-1})} \vec{\alpha} \in S^\omega [\theta_n^u(\pi_u(\alpha_0, ..., \alpha_{n-1}) = \tau^{\alpha,\vec{\alpha}}(\pi_u(\alpha_0, ..., \alpha_{n-1})]$$

and similarly for the definition of b_n^u, where π_u is the projection map from the product measure μ_n to the homogeneity measure μ_u. The main important points is that when extending b_{n-1} to b_n we must use normality of the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\lambda')$ to stabilize the extension of b_{n-1}. The rest of the proof involving the Becker-Kechris game with the appropriate modifications is now as above.

Finally we show the following lemma of independent interest:

Lemma 2.20. *Let κ be a regular cardinal, then Γ_κ is closed under $< \kappa$ intersections.*

Proof. Suppose not. Then we have that $\check{\Gamma}_\kappa$ is not closed under $< \kappa$ unions. So let let $\delta < \kappa$ be such that $\{A_\alpha\}_{\alpha<\delta}$ be in $\check{\Gamma}_\kappa$ and $A = \bigcup_{\alpha<\delta} A_\alpha \notin \check{\Gamma}_\kappa$. Then by Wadge's lemma we have that $A = \bigcup_{\alpha<\delta} A_\alpha \in \Gamma_\kappa$. By Sep($\check{\Gamma}_\kappa$), for every $\alpha < \delta$, there is a Δ_κ set which separates A_α from A^c. Since κ is a regular cardinal and since $\delta < \kappa$ then there is a $\theta < \kappa$ such that for each Δ_κ sets separating A_α from A (call them C_α), we have that $|C_\alpha|_W \leq \theta$. Next let Γ_0 be a pointclass such that $\theta < o(\Gamma)$ and $\exists^{\mathbb{R}}\Gamma_0 \subseteq \Gamma_0$. Then by the coding lemma we have a Γ_0 relation R such that R is the set of codes of Γ_0 sets which separate A_α from A^c. But then $A \in \Gamma_0$. Contradiction!

□

In the next section we analyze projective-like hierarchies by means of the ordinal associated to the base of the projective-like hierarchy, $o(\Delta)$.

2.2. Characterization of Projective-Like Hierarchies by the Associated Ordinals

Before we move on, we discuss the situation on the projective-like hierarchies of type **II** and **III** which arises from the above theorem. We will then introduce a conjecture pertaining to the characterization of type **IV** projective-like hierarchies in terms of the associated ordinal and we will give a proof to the conjecture.

First we briefly recall the situation at the level of type **I** projective-like hierarchies. Let Λ be a projective algebra. Let $\Gamma_1, \Gamma_2, \Gamma_3...$ be the projective like hierarchy generated by Λ. Let α be the ordinal associated with Λ, that is $\alpha = o(\Lambda) = \sup\{|A|_W : A \in \Lambda\}$. Kechris, Solovay and Steel conjectured in [17] that α alone determines which projective-like hierarchy arises. If $cof(\alpha) = \omega$ then we are in the situation of a projective-like hierarchy of type **I**. We briefly recall the set up. Let $\{A_n\}$ be sets such that for every $n < \omega$, we have $|A_n|_W = \alpha_n < \alpha$. Assume that $|A_n|_W < |A_{n+1}|_W$. We then let $A = \oplus A_n$ be the join of the sets A_n. Then at A we have a selfdual degree, that is $A \equiv_W A^c$. Let $\Sigma_0 = \bigcup_\omega \Lambda$ be the pointclass of sets which are countable unions of sets in Λ. Then $A \in \Sigma_0$ and Σ_0 is closed under countable unions by definition. Σ_0 is closed under $\exists^{\mathbb{R}}$, since if $A(x) \leftrightarrow \exists y B(x,y)$ with $B \in \Sigma_0$ and $B = \bigcup_\omega B_n$ with $B_n \in \Lambda$, then we have $A(x) \leftrightarrow \exists y B(x,y) \leftrightarrow \exists y \exists n B_n(x,y) \leftrightarrow \exists n \exists y B_n(x,y)$, and this last set is in Σ_0 by definition. In addition Σ_0 is nonselfdual pointclass. To see this, assuming all A_n as above are nonselfdual degrees, define universal sets U_n for the intermediate pointclasses $\{B : B \leq_W A_n\}$. If we let $U(x,y) \leftrightarrow \exists n U_n((x)_n, y)$ then U is universal for Σ_0. Also Σ_0 cannot be closed under countable intersections since if it were then it would contain $\Pi_0 = \check{\Sigma}_0$ and therefore would not be nonselfdual. Then a type **I** projective-like hierarchy is generated in the usual way starting from Σ_0. Notice that we have PWO(Σ_0) since we can define the natural norm φ on $A = \bigcup_n A_n$, for $A_n \in \Lambda$ by $\varphi(x) =$ the least n such that $x \in A_n$. Then $\leq_\varphi$ and $<_\varphi$ are both countable unions of sets in Λ.

Next if $\omega < cof(\alpha)$ and α is singular then $\Gamma_1, \Gamma_2, \Gamma_3, ...$ is a type **II** projective-like hierarchy. If not then $\Lambda = \Gamma_1 \cap \check{\Gamma}_1$ and we are in a type **III** projective-like hierarchy, so by results of [17], we have PWO(Γ_1). Since Γ_1 is closed under $\forall^{\mathbb{R}}$, letting

$$\alpha = \sup\{\xi : \xi \text{ is the length of a } \Delta_1 \text{ prewellordering}\}$$

and since Γ_1 is closed under $\wedge, \vee$, in this case by [22] we must have α is regular, contradiction. Notice that this can be seen directly using the above theorem of Steel which shows that the singularity of α implies the non-closure of Γ under $\vee$. Then by the above theorem which give a solution to Steel's conjecture, it is true that whenever α is regular, Λ generates a projective-like hierarchy of type **III** or **IV**. So there are no projective-like hierarchies of type **II** for which α is regular: if $\beta = cof(\alpha) < \alpha$, then the Steel pointclass in within a type **II** projective-like hierarchy and if α is regular then the Steel pointclass is at least within a type **III** projective-like hierarchy. In the type **IV** case we speak of an *inductive-like* hierarchy instead of a projective-like hierarchy. To introduce the conjecture below which pertains to a characterization of type **IV** projective-like hierarchies in terms of the associated ordinal, we recall some definitions from [12]. For any ordinal α, let $B_\alpha = \{x : \exists\gamma < \alpha, x \subseteq L_\gamma\}$. Notice that $L_\alpha \subseteq B_\alpha$ and B_α is a transitive set. The set of Δ_0 formulas is the closure under boolean combinations and bounded quantification of atomic formulas. A formula in the language of set theory is Π_2 if it is of the form $\forall y \exists x \varphi$ where $\varphi \in \Delta_0$.

DEFINITION 2.21. A cardinal α is ${}^b\Pi^1_2$-indescribable if for every $X \subseteq L_\alpha$ and for every Π_2 formula φ of the language of set theory with parameters from B_α we have

$$(B_\alpha, \in, X) \vDash \varphi \to \exists\beta < \alpha \text{ s.t } (B_\beta, \in, X \cap L_\beta) \vDash \varphi$$

Given the above picture of the Wadge hierarchy, we then have the following conjecture as in [17]:

CONJECTURE 2.22. Let Γ be any pointclass closed under $\forall^{\mathbb{R}}$ and suppose PWO(Γ). Suppose $\exists^{\mathbb{R}}\Delta \subseteq \Delta$ and $o(\Delta) = \kappa$ is ${}^b\Pi^1_2$-indescribable and Mahlo. Then Γ is closed under $\exists^{\mathbb{R}}$.

Using the above notion of ${}^b\Pi^1_2$-indescribability, Kechris has shown that if κ is a Suslin cardinal such that $\omega < cof(\kappa)$, then $S(\kappa)$ is closed under $\forall^{\mathbb{R}}$ if and only if κ is ${}^b\Pi^1_2$-indescribable, where $S(\kappa)$ is the pointclass of all κ-Suslin sets. It is standard that $S(\kappa)$ is closed under $\exists^{\mathbb{R}}$ (see [22]). Therefore the conjecture is true if we assume that $\Lambda \subseteq$ **IND**, where **IND** is the boldface pointclass of the inductive sets and where Λ generates Γ, since by a result of Kechris every set in **IND** is κ-Suslin for some $\kappa < \kappa^{\mathbb{R}}$. Recall that an interval of ordinals $[\alpha, \beta]$ is a Σ_1-gap if and only if

(1) $L_\alpha(\mathbb{R}) \prec^{\mathbb{R}}_1 L_\beta(\mathbb{R})$
(2) $\forall\xi < \alpha(L_\xi(\mathbb{R}) \not\prec^{\mathbb{R}}_1 L_\alpha(\mathbb{R}))$

(3) $\forall\gamma > \beta(L_\beta(\mathbb{R}) \not\prec_1^{\mathbb{R}} L_\gamma(\mathbb{R}))$

The scale property is depends on whether we are in a Σ_1-gap. Basically, new scales appear when new Σ_1 facts about the reals are verified in $L(\mathbb{R})$. Kechris has shown that once one is past the pointclass of inductive sets **IND** then the scale property no longer holds in a projective-like hierarchy of type **IV**. For example, consider $\Pi_1 = \forall^{\mathbb{R}}(\mathbf{IND} \vee \check{\mathbf{IND}})$. Then Π_1 does not have the scale property and no Π_n or Σ_n can have the scale property. This is a gap of length ω. Past this gap the scale property resumes, since Moschovakis has shown that the pointclass Σ_ω, the least pointclass closed under $\exists^{\mathbb{R}}$ and containing $\bigcup_n \Sigma_n$, has the scale property. But then, later on, longer and longer gaps occur. We feel that there are characterizations of the lengths of the Σ_1 gaps in terms of the associated ordinal of the pointclass which closes a gap, but we do not know how to precisely show this.

The above conjecture is true below the first nontrivial gap in scales. Past the first Σ_1 gap in scales, the conjecture remained unsolved. We show the conjecture below. In the proof we use the notion of ∞-Borel set which we first define:

Definition 2.23 (∞-Borel set). Let $A \subseteq \mathbb{R}$. Then A is ∞-Borel if and only if there is a set $S \subseteq \gamma$, for some $\gamma \in \mathrm{ORD}$ and a formula φ in the language of set theory such that

$$x \in A \leftrightarrow L[S,x] \models \varphi[S,x]$$

$(\varphi, S) \subseteq \mathrm{ORD}$ is the code of the ∞-Borel set A and we let $A = A_{\varphi,S}$.

Also, we use a theorem of Woodin which gives a bound on where the code of an ∞-Borel set appears.

Theorem 2.24 (Woodin). *Let $A \subseteq \mathbb{R}$ be an ∞-Borel set. Then there is a $\gamma < \Theta$ and a prewellorder $\preceq \in \underset{\sim}{\Pi}^1_2(A)$ of length γ such that $S \subseteq \gamma$ and S is the Borel code of A.*

We now show the above conjecture pertaining to inductive-like hierarchies.

Theorem 2.25 ($\mathrm{AD} + V = L(\mathbb{R})$). *Let Γ be a Steel pointclass, that is Γ is closed under $\forall^{\mathbb{R}}$, $PWO(\Gamma)$ and suppose that $\exists^{\mathbb{R}}\Delta \subseteq \Delta$. Suppose that $o(\Delta) = \kappa$. Then the following are equivalent:*

(1) *κ is ${}^b\Pi^1_2$-indescribable and Mahlo.*

(2) *Γ is closed under $\exists^{\mathbb{R}}$.*

Proof. Recall that we are in the situation where we have $\mathrm{Sep}(\check{\Gamma})$. Assume first that Γ is closed under $\exists^{\mathbb{R}}$. We need to see that κ is ${}^b\Pi^1_2$-indescribable. By theorem 3.1 of [14], we must have that for every inductive-like pointclass Γ, that κ is Mahlo. Let

$$\underset{\sim}{\delta} =_{def} \sup\{\xi : \xi \text{ is the length of a } \Delta \text{ prewellordering of } \mathbb{R}\}.$$

Then by the companion theorem of Moschovakis (see theorem 9E.1 in [21]), $\underset{\sim}{\delta}$ is the ordinal of its admissible companion $\mathcal{M}$ above $\mathbb{R}$. So $o(\mathcal{M}) = \underset{\sim}{\delta}$. Since every admissible ordinal is Π_2-reflecting and every set $A \subseteq L_{\underset{\sim}{\delta}+1}$ is Δ_1 over $\mathcal{M}$ by the coding lemma, and $|L_{\underset{\sim}{\delta}+1}| = \underset{\sim}{\delta}$, we have that $\underset{\sim}{\delta}$ is ${}^b\Pi^1_2$-indescribable.

We must now show that $\underset{\sim}{\delta} = \kappa$. The result is true for any projective algebra.

Claim 2.26. *Let $\Delta = \Gamma \cap \check{\Gamma}$ be a projective algebra. Then the following ordinals are equal:*

(1) $\underset{\sim}{\delta} = \sup\{\xi : \xi$ *is the length of a* Δ *prewellordering of* $\mathbb{R}\}$

(2) $o(\Delta) = \kappa = \sup\{|A|_W : A \in \Delta\}$

Proof. The following argument is due to Jackson. First let $\alpha < o(\Delta)$ such that for some $A \in \Delta$ we have $|A|_W = \alpha$. Then this initial segment determined by A in the Wadge hierarchy defines a prewellordering $\preceq$ in Δ of length α, since Δ is closed under quantifiers, $\vee$ and $\wedge$. We define $\preceq$ by $x \preceq y \leftrightarrow f_x^{-1}(A) \leq_w f_y^{-1}(A)$, where f_x, f_y are the Lipschitz continuous functions coded by x and y. Notice that for some $n \in \omega$, $\preceq \in \underset{\sim}{\Sigma}^1_n(A)$ and since Δ is closed under quantifiers, $\vee$ and $\wedge$ we have $\underset{\sim}{\Sigma}^1_n(\preceq) \in \Delta$. So $\alpha < \delta(\Delta)$, hence $o(\Delta) \leq \delta(\Delta)$.

Next let $\alpha < \delta(\Delta)$. We need to see that $\alpha < o(\Delta)$. We will use the jump function. Let $\preceq$ be a prewellordering in Δ of length α. We then construct an increasing sequence of Wadge degrees of length α. There is a function $F : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ such that

$$\text{for all } A \subseteq \mathbb{R}, A <_W F(A).$$

The function F is the *jump of* A, where we let $F(A) = A'$ be defined by

$$A'(x) \leftrightarrow (x(0) = 0 \wedge \tau_{x'}(x) \notin A) \vee (x(0) = 1 \wedge \tau_{x'}(x) \in A),$$

where x' is the shift of x, i.e $x'(n) = x(n+1)$ and $\tau_{x'}$ is the continuous function coded by x'. Notice that $F(A)$ is not Wadge reducible to either A or A^c and it has Wadge degree strictly higher to either A or A^c. For if $\tau_{x'}$ reduced A' to A then we would get $0^\frown x' \in A'$ iff $\tau_{x'}(0^\frown x') \in A$ but since

$$0^\frown x' \in A' \longleftrightarrow \tau_{x'}(0^\frown x') \notin A,$$

by definition, contradiction!

Next we define by induction on $\alpha < |\preceq|$ a Δ set A_α. Let $A_0 = \emptyset$ and let $A_{\alpha+1} = A'_\alpha$. If α is a limit ordinal then let $A_\alpha(x) \leftrightarrow (|x_0|_\preceq < \alpha \wedge x_1 \in A_{|x_0|_\preceq})$. Then by definition of the jump function and by induction the A_α are strictly increasing in Wadge degrees. Now we check that each $\mathrm{AD}_\alpha \in \Delta$. Let $R(x,y) \leftrightarrow x \in dom(\preceq) \wedge y \in A_{|x|_\preceq}$. We show that $R \in \Delta$. We define a relation W, for $i = 0, 1$ such that if $W(x,y,i,z,w,j)$ holds means that $i = 1$ and (z,w,j) witnesses that $R(x,y)$ holds and $i = 0$ and (z,w,j) witnesses that $\neg R(x,y)$ holds. Then define $W(x,y,i,z,w,j)$ as follows:

(1) $i = 1$ and x is an immediate successor of z in $\preceq$ and either $0 < y(0)$, $w = \tau_{y'}(y)$and $j = 0$ or $y(0) = 0$ and $w = \tau_{y'}(y)$ and $j = 1$,

(2) $i = 1$ and x has limit rank in $\preceq$, $y_0 \preceq x, y_0 = z, w = y_1$ and $j = 1$,

(3) $i = 0$ and either $x \notin dom(\preceq)$ or x is an immediate successor of z in $\preceq$ and either $0 < y(0)$, $w = \tau_{y'}(y)$ and $j = 1$ or $y(0) = 0$, $w = \tau_{y'}(y)$ and $j = 0$,

(4) $i = 0$ and either $x \notin dom(\preceq)$ or x has limit rank in $\preceq$ and the following hold: $\neg y_0 \prec x \vee (z = y_0 \wedge w = y_1 \wedge j = 0$,

(5) $i = 0$ and either $x \notin dom(\preceq)$ or $|x|_\preceq = 0$.

Then W is in Δ as $\preceq \in \Delta$. We then have:

$$R(x,y) \leftrightarrow \exists z, w, \varepsilon(z_0 = x \wedge w_0 = y \wedge \varepsilon(0) = 1 \wedge \forall i W(z_i, w_i, \varepsilon(i), z_{i+1}, w_{i+1}, \varepsilon(i+1)).$$

So $R \in \Delta$, and for every $\alpha < |\preceq|$, $A_\alpha \in \Delta$.

$\square$

This now finishes the proof of (2) $\to$ (1). Next we must show that whenever κ is ${}^b\Pi^1_2$-indescribable and Mahlo then Γ is closed under $\exists^{\mathbb{R}}$. Assume that κ is ${}^b\Pi^1_2$-indescribable. We must show that Γ is closed under $\exists^{\mathbb{R}}$. Specifically we show the following:

Claim 2.27. *Let Γ be a Steel pointclass such that $\exists^{\mathbb{R}}\Delta \subseteq \Delta$ and $\kappa = o(\Delta)$ is ${}^b\Pi^1_2$-indescribable. Then Γ is closed under $\exists^{\mathbb{R}}$.*

Proof. We make the general assumption that we are in the context where we do not have the scale property, since by the above remark if $\Gamma \subseteq \mathbf{IND}$ or Γ is not located in a Σ_1-gap, then we can localize scales to Γ or Γ sets are κ Suslin for some κ, and then by the result mentioned above of Kechris, see [12], the conjecture is true. We also work by contradiction below. Assume Γ is either located in a Σ_1-gap below the last Σ_1-gap $[\underset{\sim}{\delta}^2_1, \Theta]$, or that Γ is located in the last Σ_1 gap $[\underset{\sim}{\delta}^2_1, \Theta]$. Suppose that $o(\Delta)$ is ${}^b\Pi^1_2$-indescribable. We must see that Γ is closed under $\exists^{\mathbb{R}}$. So let $B \in \Gamma \setminus \check{\Gamma}$ and let $A(x) \leftrightarrow \exists y B(x,y)$. Under $\mathrm{AD} + V = L(\mathbb{R})$, every set of reals is ∞-Borel, so the set B is ∞-Borel, and thus there is a formula φ and a set of ordinals $S \subseteq \gamma$ for some γ such that

$$B(x,y) \leftrightarrow L[S,x,y] \vDash \varphi(x,y),$$

see [17]. By Woodin's theorem, the Borel code S can be taken to be subset of γ, where γ is the length of a $\underset{\sim}{\Pi}^1_2(B)$ prewellordering. So we have that $\gamma < \underset{\sim}{\delta}^1_2(B)$, where

$$\underset{\sim}{\delta}^1_2(B) = \sup\{\xi : \xi \text{ is the length of a } \Delta^1_2(B) \text{ p.w.o of } \mathbb{R}\}.$$

Since $\underset{\sim}{\Pi}^1_1(B) \subseteq \Gamma$, because Γ is closed under $\forall^{\mathbb{R}}$ and by the proof of Steel's conjecture, Γ is also closed under $\vee$ as κ is regular, and since there must be a Γ prewellordering of length $\underset{\sim}{\delta}^1_1(B) = o(\underset{\sim}{\Delta}^1_1(B))$ and $\underset{\sim}{\delta}^1_2(B) = (\underset{\sim}{\delta}^1_1(B))^+$, we may then assume that $S \subseteq \kappa$ and $\gamma \leq \kappa$, because $o(\Gamma) = \kappa + 1$ and since one can define a $\underset{\sim}{\Pi}^1_1(B)$ prewellordering of length $|B|_W$. We then have

$$A(x) \leftrightarrow \exists y L[S,x,y] \vDash \varphi(x,y).$$

Let (φ, S) be the Borel code of the set B. Thus

$$A(x) \leftrightarrow (B_{\kappa+1}, \in, x, (\varphi, S)) \vDash \text{“}\exists y L[S, x, y] \vDash \varphi(x, y)\text{”}.$$

This implies then that there is a $\kappa' < \kappa$ such that

$$(B_{\kappa'+1}, \in, x, (\varphi, S \restriction \kappa' + 1)) \vDash \text{“}\exists y L[S \restriction \kappa' + 1, x, y] \vDash \varphi(x, y)\text{”},$$

since "$\exists y L[S \restriction \kappa' + 1, x, y] \vDash \varphi(x, y)$" is a Π_2 formula, as the satisfaction relation is Δ_1. Hence we have $A(x) \leftrightarrow \exists y L[S_1, x, y] \vDash \varphi(x, y)$ for some $S_1 \subseteq \kappa' + 1 \leq \gamma$. Let then

$$\tilde{\Gamma} = \{A : A \text{ is an effective } \kappa \text{ union of } < \kappa\text{-Borel codes}\}$$

Notice then that we have $\Delta \subsetneq \tilde{\Gamma} \subsetneq \bigcup_\kappa \Delta \subsetneq \exists^{\mathbb{R}}\Gamma$. We first show that $\tilde{\Gamma}$ is closed under the $\forall^{\mathbb{R}}$ quantifier. Let then $B \in \tilde{\Gamma}$ and consider

$$A(x) \leftrightarrow \forall y B(x, y)$$

Now applying ${}^b\Pi^1_2$-indescribability again we have that $A(x) \leftrightarrow \exists \gamma < \kappa(\forall y L[T, x] \vDash \varphi(x, y))$, where T is a Borel code of size $\leq \gamma$. This shows that $A \in \tilde{\Gamma}$. So A is also in $\exists^{\mathbb{R}}\Gamma$. Notice that we must then have by Wadge $\tilde{\Gamma} = \Gamma$. It is then sufficient to notice that $\tilde{\Gamma}$ is closed under $\exists^{\mathbb{R}}$ to obtain the desired contradiction. This follows by a general argument using the Vopenka algebra to make any real of $L(\mathbb{R})$ generic over the image of $L[S, x]$ in an ultrapower by supercompactness measures (This is an argument of Caicedo and Ketchersid). This shows the theorem. However we explain briefly that the result follows directly from $\mathrm{AD}^{L(\mathbb{R})}$ using Turing-determinacy (which itself is equivalent to AD in the context of $L(\mathbb{R})$, by a result of Woodin), without having to refer to the Vopenka algebra. Let then $B \in \tilde{\Gamma}$, we wish to see that $A(x) \leftrightarrow \exists y B(x, y)$ is still in $\tilde{\Gamma}$. Let $\mathbf{d}$ denote a Turing degree. By $\forall^*\mathbf{d}A(\mathbf{d})$ we means that $\exists \mathbf{e}_0 \forall \mathbf{e} \geq \mathbf{e}_0 A(\mathbf{e})$, where $\leq$ is the Turing degree partial order: $x \leq \mathbf{d}$ means that $x \leq_T y$ for any y of Turing degree $\mathbf{d}$. The main point is that if we have a set $D \in \tilde{\Gamma}$, then we may replace all occurrences of $\forall^*\mathbf{d}\exists x D(x)$ by $\exists x \forall^*\mathbf{d} D(x)$ by Turing determinacy. $\square$

We next include facts about type **IV** projective-like hierarchies. Suppose that κ is ${}^b\Pi^1_2$-indescribable. Then Γ is closed under $\exists^{\mathbb{R}}$. Thus Γ is closed under both $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$, hence also under countable unions and intersections. Define the pointclass $\Pi_1 = \Gamma \wedge \check{\Gamma}$ and let $\Sigma_1 = \check{\Pi}_1$. A typical example of this type of hierarchy is letting $\Gamma = \mathrm{IND}$, the pointclass of inductive sets. In this case, since IND is closed under continuous substitutions,$\wedge, \vee$, we define $\Sigma_1^*(\Gamma) = \{A \subseteq \mathbb{R} : \exists B \in \Gamma, C \in \check{\Gamma}$ such that $x \in A \leftrightarrow \exists y(B(x,y) \wedge C(x,y))\}$. Then we let $\Pi_n^*(\Gamma) = \{A^c : A \in \Sigma_n^*(\Gamma)\}$ and $\Sigma_{n+1}^* = \{\exists y A(x,y) : A \in \Pi_n^*(\Gamma)\}$. Notice that Π_1 is closed under $\forall^{\mathbb{R}}$ since both Γ and $\check{\Gamma}$ are closed under $\forall^{\mathbb{R}}$ and $\exists^{\mathbb{R}}$. Assume that Π_1 can be characterized as the pointclass of all $\underset{\sim}{\Sigma}_1^1$ bounded unions of $\check{\Gamma}$ sets of length κ, that is

$$\Pi_1 = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall \alpha < \kappa (A_\alpha \in \check{\Gamma}) \wedge \bigcup_{\alpha<\kappa} A_\alpha \text{ is } \underset{\sim}{\Sigma}_1^1 \text{ bounded}\}.$$

Let $\Pi_1' = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall \alpha < \kappa (A_\alpha \in \check{\Gamma}) \wedge \bigcup_{\alpha<\kappa} A_\alpha$ is $\check{\Gamma}$ bounded$\}$. Our goal is to show that $\Pi_1 = \Pi_1'$ first and then later we verify that Π_1 can indeed be characterized as the pointclass of all sets which can be written as $\underset{\sim}{\Sigma}_1^1$-bounded unions of $\check{\Gamma}$ sets.

SUBCLAIM 2.28. $\Pi_1 = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall \alpha < \kappa (A_\alpha \in \check{\Gamma}) \wedge \bigcup_{\alpha<\kappa} A_\alpha$ is $\check{\Gamma}$ bounded$\} = \Pi_1'$.

PROOF. Every $\check{\Gamma}$-bounded union is $\underset{\sim}{\Sigma}_1^1$-bounded. Let $A \in \Pi_1 \setminus \Sigma_1$. and let $A = \bigcup_{\alpha<\kappa} A_\alpha$ where each $A_\alpha \in \check{\Gamma}$, the union is $\underset{\sim}{\Sigma}_1^1$-bounded and $\kappa = o(\Delta)$. We may assume that the A_α's are increasing and that the union is continuous. Then $\langle |A_\alpha|_W : \alpha < o(\Delta)\rangle$ is cofinal in $o(\Delta)$. Now for $\alpha < \kappa$ define the sets C_α by

$$C_\alpha =_{def} \{(x,y) : y \in A_{\alpha+1} \setminus A_\alpha \wedge x \text{ codes a continuous function } f_x \text{ s.t } f_x^{-1}(A_\alpha) \subseteq A\}.$$

Then notice that for each $\alpha < \kappa$, C_α is defined as $\check{\Gamma} \wedge \forall^{\mathbb{R}}(\Gamma \vee \Gamma) = \check{\Gamma} \wedge \Gamma$. Then by definition, $C_\alpha \in \Pi_1$. We have that if $C = \bigcup_{\alpha<\kappa} C_\alpha$, then the proof of subclaim 2.28 also shows that $C \in \exists^{\mathbb{R}}\Pi_1 = \Sigma_2$, since κ is regular. So let $C = \bigcup_{\alpha<\kappa} D_\alpha$ where each $D_\alpha \in \check{\Gamma}$ and the union is increasing. Define the sets B_α as follows

$$z \in B_\alpha \leftrightarrow \exists (x,y) \in D_\alpha \exists \beta \leq \alpha (y \in A_{\beta+1} \setminus A_\beta \wedge f_x(z) \in A_\beta)$$

Then for every $\alpha < \kappa$, we have that $B_\alpha \in \check{\Gamma}$, since $\check{\Gamma}$ is closed under $\exists^{\mathbb{R}}$, $\wedge$ and $\vee$, by the proof of Steel's conjecture. Then we have that $\bigcup_{\alpha<\kappa} B_\alpha = A$. In addition $\bigcup_{\alpha<\kappa} B_\alpha$ is a $\check{\Gamma}$-bounded union since any $\check{\Gamma}$ is of the form $f_x^{-1}(A_\beta)$ for some $\beta < \kappa$ and some $x \in \mathbb{R}$. So $A \in \Pi_1'$.

□

Finally we show that the pointclass $\Pi_1 = \Gamma \wedge \check{\Gamma}$ is the pointclass of all sets which can be written as $\underset{\sim}{\Sigma}^1_1$-bounded unions of $\check{\Gamma}$ sets.

SUBCLAIM 2.29. $\Pi_1 = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall\alpha < \kappa(A_\alpha \in \check{\Gamma}) \wedge \bigcup_{\alpha<\kappa} A_\alpha \text{ is } \underset{\sim}{\Sigma}^1_1 \text{ bounded}\}$.

PROOF. Let $\Omega = \{\bigcup_{\alpha<\kappa} A_\alpha : \forall\alpha < \kappa(A_\alpha \in \check{\Gamma}) \wedge \bigcup_{\alpha<\kappa} A_\alpha \text{ is } \underset{\sim}{\Sigma}^1_1 \text{ bounded}\}$. We must show that $\Pi_1 = \Omega$. Suppose that $A \in \Pi_1$. So let $B \in \Gamma$ and $C \in \check{\Gamma}$ such that $A = B \cap C$. Then since Γ is a Steel pointclass, let $B = \bigcup_{\alpha<\kappa} B_\alpha$ and the union is increasing and $\underset{\sim}{\Sigma}^1_1$-bounded and each $B_\alpha \in \Delta$. Then we have that $A = \bigcup_{\alpha<\kappa} B_\alpha \cap C$. This union is a $\underset{\sim}{\Sigma}^1_1$-bounded union of $\check{\Gamma}$ sets since $\check{\Gamma}$ is closed under $\wedge$ so in particular $\check{\Gamma}$ is closed under intersections with Δ sets. So we have $\Pi_1 \subseteq \Omega$.

Next notice that since $\check{\Gamma}$ is closed under $\forall^{\mathbb{R}}$ then Ω is also closed under $\forall^{\mathbb{R}}$ by Addison's argument. Let $\preceq$ be a Γ prewellordering of length κ, let φ be the Γ norm associated to $\preceq$ and let U be a universal $\check{\Gamma}$ set of reals. Apply the coding lemma to obtain a relation $R(w, \varepsilon) \in \Gamma$ such that

(1) $\varphi(w) = \varphi(\varepsilon) \to (R(w, \varepsilon) \leftrightarrow R(z, \varepsilon))$
(2) $R(w, \varepsilon) \to \varepsilon \in C$, where C is the set of codes of the sets in some sequence of $\check{\Gamma}$ sets $\{A_\alpha\}_{\alpha<\kappa}$.
(3) $\forall w \exists \varepsilon (R(w, \varepsilon) \wedge U_\varepsilon = A_{\varphi(w)})$.

Then we compute that $x \in \bigcup_{\alpha<\kappa} A_\alpha \to \exists w \exists \varepsilon (R(w, \varepsilon) \wedge x \in U_\varepsilon)$. Thefore we have $\bigcup_\kappa \check{\Gamma} \subseteq \exists^{\mathbb{R}}(\Gamma \wedge \check{\Gamma})$. Now since $\Pi_1 \subseteq \Omega \subseteq \Sigma_2$ and since Ω is closed under $\forall^{\mathbb{R}}$ then we must have that $\Pi_1 = \Omega$, since if not then by Wadge's lemma we have $\Omega \subseteq \Sigma_1$ and thus $\Pi_1 \subseteq \Sigma_1$, contradiction!

□

Now from the above we can show that PWO(Π_1). The following argument is due to Jackson.

SUBCLAIM 2.30. PWO(Π_1)

Proof. Let $A \in \Pi_1$ be such that $A = B \cap C$ for $B \in \Gamma$ where $B = \bigcup_{\alpha<\kappa} B_\alpha$ a $\underset{\sim}{\Sigma}^1_1$-bounded union of Δ sets and $C \in \check{\Gamma}$. Then we have $A = \bigcup_{\alpha<\kappa} B_\alpha \cap C$. Let $A_\alpha = B_\alpha \cap C$, so that for every $\alpha < \kappa$, $A_\alpha \in \check{\Gamma}$ and $A = \bigcup_{\alpha<\kappa}$ is a $\underset{\sim}{\Sigma}^1_1$ bounded union of $\check{\Gamma}$ sets. Let φ be the natural norm on A coming from the union, i.e $\varphi(x) =$ the least γ such that $x \in A_\gamma$. We must see that φ is a Π_1 norm. Since $C \in \check{\Gamma}$ then let $\mathbb{R} \setminus C = \bigcup_{\xi<\kappa} C_\xi$ where for every $\xi < \kappa$, C_ξ are Δ sets and the union is $\underset{\sim}{\Sigma}^1_1$ bounded since $\mathbb{R} \setminus C$ is in Γ. Let ψ be the norm coming from the union of the C_ξ, i.e the norm defined by $\psi(x) =$ the least γ such that $x \in C_\gamma$. Then the argument below applied to Γ will show that ψ is a Γ norm, and then since Γ is closed under $\wedge, \vee$ and since by $4C.11$ of [22] $\check{\Gamma}$ will be bounded in the norm ψ. For every $\alpha < \kappa$, let $A^c_\alpha = C_\gamma \cup B^c_\alpha$. But then the sequence of sets $\{C_\gamma \cup B^c_\alpha\}_{\gamma<\kappa}$ is a $\check{\Gamma}$ bounded union. Now let

$$x <^*_\varphi y \leftrightarrow \exists \beta < \kappa \exists \gamma \leq \beta (x \in A_\alpha \wedge x \in C_\beta \cup B^c_\alpha).$$

Notice that

$$\exists \gamma \leq \beta (x \in A_\alpha \wedge x \in C_\beta \cup B^c_\alpha)$$

defines a $\check{\Gamma}$ set, since $\check{\Gamma}$ is closed under union of lengths less than κ and the union is of length less than $\beta < \kappa$. So let E_β be sets in $\check{\Gamma}$ such that $<^*_\varphi \bigcup_\beta E_\beta$. We need to see that this union is $\underset{\sim}{\Sigma}^1_1$ bounded. Let $S \subseteq <^*_\varphi$ be a $\underset{\sim}{\Sigma}^1_1$ set. Then $S_1 = \{x : \exists y S(x, y)\}$ is also $\underset{\sim}{\Sigma}^1_1$ and $S_1 \subseteq A$, so there is a $\kappa_0 < \kappa$ such that $S_1 \subseteq A_{\kappa_0}$

□

□

2.3. Strong Partition Relations

Let Γ be a Steel pointclass and let $\exists^{\mathbb{R}}\Delta \subseteq \Delta$ and let $\kappa = o(\Delta)$ be the Wadge ordinal of the Steel pointclass Γ and κ is regular. Then we show that κ has the strong partition property, that is $\kappa \longrightarrow (\kappa)^\kappa$. Notice that by [14], there are cofinally many in Θ pointclasses Γ such that $\Delta = \Gamma \cap \check{\Gamma}$ is selfdual and closed under $\exists^{\mathbb{R}}$. As alluded to above, if we let

$$C = \{o(\Delta) : \exists^{\mathbb{R}}\Delta \subseteq \Delta \wedge \Delta \text{ is selfdual}\},$$

then C is a c.u.b set in Θ. These correspond to the places where we are at the base of a projective-like hierarchy. By the Coding Lemma, every $\kappa \in C$ is a cardinal. As noted above, a theorem of Kechris in [13] shows that for $\lambda \leq o(\text{IND})$ where IND is the pointclass of inductive sets, such that $\omega\lambda = \lambda$, the λ^{th} cardinal of C is the λ^{th} Suslin cardinal. Steel uses this to obtain a characterization of $o({}^3E)$, the supremum of the length of the prewellordering of $\mathbb{R}$ recursive in ${}^3E$, where ${}^3E$ is the deterministic quantification over $\mathbb{R}$ (see [27]). More specifically ${}^3E$ is defined as follows. By induction on $n < \omega$ define the objects T^n of type n over ω:

$$T^0 = \omega$$

$$T^{n+1} = \{f : T^n \to \omega : f \text{ is unary}\}$$

Then letting Ψ be an object of type $n+1$ and Ξ an object of type n we have:

$$^{n+2}E(\Psi) = \begin{cases} 0 & \text{if } \exists\Xi(\Psi(\Xi) = 0) \\ 1 & \text{if } \exists\Xi(\Psi(\Xi) \neq 0) \end{cases}$$

For example, let $f : \mathcal{X} \to \omega$ be a partial function. Then we say that f is Γ-recursive if $\text{Graph}(f) =_{def} \{(x, i) : f(x) = i\}$ is in Γ. We then say that Γ is closed under Kleene ${}^3E$, if whenever $f : \mathcal{X} \times \mathbb{R} \to \omega$ is a Γ-recursive partial function, then the relation

$$P(x) \leftrightarrow \forall z(f(x, z) \text{ is defined }) \wedge \exists z(f(x, z) = 0)$$

is in Γ. The precise statement of Steel's result is that $o({}^3E)$ is the least regular limit cardinal in C.

The proof of Steel's conjecture implies a specific boundedness result and this will allows us to prove a new strong partition relation for the ordinals associated to the Steel pointclass.

THEOREM 2.31. *Let Γ be a nonselfdual pointclass, closed under $\forall^{\mathbb{R}}$ and $\vee$ with $PWO(\Gamma)$ and such that $\exists^{\mathbb{R}}\Delta \subseteq \Delta$, then $\delta(\Delta)$ has the strong partition property.*

Using the proof of Steel's conjecture, notice that the Steel pointclass Γ satisfies all the properties of the above theorem. Even though the above theorem directly shows that $o(\Delta)$ has the strong partition property, we outline a direct proof of this fact below. Before we start, we note that every known proof of strong partition properties goes through Martin's theorem which we state below.

We show that for $\kappa \in C$ as above,

$$\kappa \text{ is regular iff } \kappa \text{ has the strong partition property. } (1)$$

In particular $o({}^3E)$ satisfies the relation

$$o({}^3E) \longrightarrow (o({}^3E))^{o({}^3E)}$$

If κ has the strong partition relation then κ must be regular, so the right to left direction of (1) is immediate. In our proof we use the uniform coding lemma for wellfounded relations. We refer to [19] and [14] for a proof of the uniform coding lemma for prewellorderings. This version of the coding lemma is different than the one in [19] and [14] but the proof is basically the same with some modifications.

Theorem 2.32 (Uniform Coding Lemma for wellfounded relations). *Let U be universal for the class $\Sigma_1(Q)$ where Q is a binary predicate symbol. Let Γ be a any pointclass such that $\Delta_1(Q) \subseteq \Gamma$ and $\exists^{\mathbb{R}}\Gamma \subseteq \Gamma$. Let $\preceq$ be a Γ wellfounded relation of length $o(\Delta)$. Then for every relation $R \subseteq \mathbb{R}^2$ such that $R = dom(\preceq)$, there exists $\varepsilon \in \mathbb{R}$ which codes, via U, a $\Sigma_1(\preceq_\alpha)$ choice set $C_\alpha \subseteq \mathbb{R}^2$ for $R_\alpha \subseteq \preceq_\alpha \times \mathbb{R}$ uniformly in $\alpha < o(\Delta)$.*

Theorem 2.33. *Let Γ be a Steel pointclass, i.e $\exists^{\mathbb{R}}\Delta \subseteq \Delta$ and $o(\Delta) = \kappa$ is a regular cardinal and $PWO(\Gamma)$. Then we have $\kappa \longrightarrow (\kappa)^\kappa$*

Proof. We recall Martin's conditions used in showing strong partition properties. It should be noted that this is the only known method of showing weak and strong partition relations under AD.

Let κ be a regular cardinal. We say that κ is reasonable if there is a nonselfdual pointclass Γ such that Γ is closed under $\exists^{\mathbb{R}}$ and a map Φ with $dom(\Phi) = \mathbb{R}$ with the following properties:

(1) $\forall x(\Phi(x) \subseteq \kappa \times \kappa)$

(2) $\forall f : \kappa \to \kappa, \exists x \in \mathbb{R}(\Phi(x) = f)$

(3) $\forall \beta < \kappa, \forall \gamma < \kappa, R_{\beta,\gamma} \in \Delta$, where $x \in R_{\beta,\gamma} \longleftrightarrow \Phi(x)(\beta, \gamma) \wedge \forall \gamma' < \kappa(\Phi(x)(\beta, \gamma') \to \gamma = \gamma')$

(4) Suppose that $\beta < \kappa$ and $A \in \exists^{\mathbb{R}}\Delta, A \subseteq R_\beta = \{x : \exists \gamma < \kappa R_{\beta,\gamma}(x)\}$, then $\exists \gamma_0 < \kappa$ such that $\forall x \in A \exists \gamma < \gamma_0, R_{\beta,\gamma}(x)$.

Our goal is to see $\check{\Gamma}$ will do the job, using that $\exists^{\mathbb{R}}\check{\Gamma} \subseteq \Gamma$.

We define the coding map Φ for all $x \in \mathbb{R}$. Let U be universal for the class $\Sigma_1(Q)$ where Q is a binary predicate symbol. In our case here Q will be interpreted to be a Γ-norm. Then for a formula $X \in \Sigma_1$, we have that

$$X \in \Sigma_1(Q) \leftrightarrow \exists y(Y(x, y) \wedge \forall n Q((y)_n)),$$

where Y is a Σ_1 formula. Then one can define a universal set $U(Q)$ for $\Sigma_1(Q)$ by $U_z(x, y) \leftrightarrow \exists z(S(z, \langle x, y\rangle, w) \wedge \forall n Q((w)_n))$ where S is a universal Σ^1_1 set.

Let A be a Γ-complete set and let φ be a norm on A. Let $A_\alpha = \{x \in A : \varphi(x) \leq \alpha\}$. Consider $\leq^*_\varphi\restriction \alpha = \{(x, y) \in\leq^*_\varphi : \varphi(x) \leq \varphi(y) < \alpha\}$, i.e we restrict to reals of norm less than α. We now code the functions $f : \kappa \to \kappa$ where $\kappa = o(\Delta)$. For every $f : \kappa \to \kappa$ there is $x \in \mathbb{R}$ such that $\forall \alpha < \kappa$, $U_x(\leq^*_\varphi\restriction \alpha)$ codes $f \restriction \alpha$. That is we let

$$U_x(\leq^*_\varphi\restriction \alpha)(y, z) \leftrightarrow \varphi(y) < \alpha \wedge \varphi(z) < \alpha \text{ for } z \in A \text{ and } \varphi(z) = f(\varphi(y)).$$

So we let x codes a function $f : \kappa \to \kappa$ at α if $U_x(\leq^*_\varphi\restriction \alpha)$ satisfies:

(1) $\forall y, \varphi(y) = \alpha \longrightarrow \exists z$ with $U_x(\leq^*_\varphi\restriction \alpha)(y, z)$

(2) $\forall y, y', z, z'$ we have that $U_x(\leq_\varphi\restriction \alpha)(y, z) \wedge U_x(\leq_\varphi\restriction \alpha)(y', z') \wedge \varphi(y) = \varphi(y') = \alpha \longrightarrow \varphi(z) = \varphi(z'))$ holds. So basically we let

$$\Psi(x)(\beta,\gamma) \leftrightarrow \exists y_1, z_1[y_1, z_1 \in A \wedge \varphi(y_1) = \beta \wedge \varphi(z_1) = \gamma \wedge U_x(\leq^*_\varphi\restriction$$

$$\alpha)(y_1, z_1) \wedge \forall y_1', z_1'(\varphi(y_1) = \varphi(y_1') \wedge U_x(\leq^*_\varphi\restriction \alpha)(y_1', z_1') \longrightarrow \varphi(z_1) = \varphi(z_1'))]$$

Now conditions $1, 2$ and 3 follow by the Uniform Coding Lemma and condition 4 follows from the fact that Δ sets being bounded in the norm and from the fact that $\exists^{\mathbb{R}}\Delta \subseteq \Delta$, since

$$\{z : \exists x \in S \exists y \in A(|y| = \alpha \wedge U_x(<^*_\phi\restriction \alpha, y, z))\}$$

is a Δ subset of A.

$\square$

CHAPTER 3

LIGHTFACE SCALES ANALYSIS UNDER AD, GENERALIZATIONS OF THE KECHRIS-MARTIN THEOREM AND CANONICAL T_{2N} TREES

3.1. Context

The notion of a scale is the most important concept in descriptive set theory. Scales allow us to have *definable* choice principles under determinacy in contrast to the fact that AD is inconsistent with AC. Thus using scales, one can establish *definable uniformization* theorems for subsets of $\mathbb{R}^2$. By definable uniformizations we mean if $\undertilde{\Gamma}$ has the scale property and if $\undertilde{\Gamma}$ is closed under universal quantifiers, conjunctions and disjunctions then for every set $A \subseteq \mathbb{R}^2$ there exists $B \subseteq A, B \in \undertilde{\Gamma}$ such that

$$\forall x \in \mathbb{R}[\exists y \in \mathbb{R} A(x,y) \leftrightarrow \exists ! y \in \mathbb{R} B(x,y)].$$

Roughly, what the scale does is allow picking reals which are least to be in the sets. The situation is somewhat similar in some sense to that of the Coding lemma, which provides another definable choice-like principle we can use under AD. As an instance of the work in this section, consider the problem of defining a lightface scale on a universal Π^1_2 set of reals and which doesn't use the theory of sharps. The Martin-Solovay analysis yields a Δ^1_3 scale on a universal Π^1_2 set but this is done under the assumption that for every $x \in \mathbb{R}, x^\#$ exists and thus this analysis relies on the theory of sharps for reals, which is difficult to generalize [1]. The upshot is to define OD scales on OD sets of reals. Closer to us here, the immediate goal is to identify canonical trees which we call T_{2n}. The methods we use here are purely descriptive set theoretical, but notice that we have to use boldface determinacy. This last point is very important since we repeatedly use the Third periodicity Theorem. Without any boldface determinacy, we wouldn't be able to do this. In different work with Sargsyan and Woodin, lightface scales on OD sets of reals are obtained via inner model theory and

[1]We believe it can be generalized using inner model theory. This is relevant to a generalization of the Kechris-Martin theorem using inner model theory. Q-theory plays an important role in such a generalization. This will be the object of a different paper

a strong condensation lemma just from OD-determinacy. Analyzing scales is of importance in the core model induction, since such inductions are organized according to the pattern of appearance of scales.

As a bit of context, recall the following theorem of Steel:

THEOREM 3.1 (Steel). *Every Σ^1_1 set admits a very good scale $\vec{\varphi}$ all of whose norms φ_n are $\omega.(n+1) - \Pi^1_1$, uniformly in n.*

Then by the proof of Moschovakis theorem on the transfer of scales using the game quantifier one obtains:

THEOREM 3.2 (Steel, Moschovakis). *Assume $\Game^{2n-2}\omega.k - \Pi^1_1$ determinacy holds. Then every Π^1_{2n} set admits an excellent scale all of whose norms are $\Game^{2n-1}\omega.(k+1) - \Pi^1_1$ uniformly in k. Therefore, if $\underset{\sim}{\Delta}^1_{2n}$-determinacy holds, then every Σ^1_{2n+1} set of reals admits an excellent scale all of whose norms are $\Game^{2n}\omega.(k+1) - \Pi^1_1$, uniformly in k.*

In this section we will outline a technique which allows us to obtain excellent scales on Π^1_{2n} sets, and therefore on Σ^1_{2n+1} sets without any use of Moschovakis "scale transfer" theorem using the game quantifier.

Recall that obtaining scales and obtaining Suslin representations is the same thing. The Suslin representation of a set of reals A is one of the most important concept in descriptive set theory. Scales give more information on the Suslin representation of a set of reals. In addition to proving definable choice principle under determinacy and giving more information on Suslin representations, another non-trivial use of scales lies in absoluteness and correctness results. For example the Schoenfield tree T on $\omega \times \omega_1$ has a left-most branch in L and since it projects to Σ^1_2 sets, this shows that L is Σ^1_2-correct. In general, under large cardinal hypothesis, one obtains projective absoluteness and $\Sigma_1^{L(\mathbb{R})}$ absoluteness using certain ordinal definable trees (see for instance applications of the Tree Production lemma in [26] to show that the pointclass Hom^* has the scale property).

An important property that Suslin representations have is that of *homogeneity.* We first quickly recall the definition of homogeneity and weak-homogeneity. The notion of homogeneity is due to Kechris, Kunen and Martin. Recall that under AD^+, every tree T on $\omega \times \kappa$ for $\kappa < \Theta$ is homogeneous (Martin, Woodin) and homogeneously Suslin trees are determined (Martin). We begin by recalling the definition of a homogeneous tree. Basically a homogeneous tree looks the same at every section: whenever a sequence $\vec{\alpha}$ in the section of the tree is order-isomorphic to another sequence $\vec{\beta}$ then $\vec{\beta}$ is also in the section of the tree.

Definition 3.3. (homogeneous tree)

A tree T on $\omega \times \kappa$ is said to be homogeneous of there is a family of measures $\langle \mu_s : s \in \omega^\omega \rangle$ satisfying :

(1) Each μ_s is a measure on T_s and $\mu_s(T_s) = 1$,
(2) If t extends s then μ_t projects to μ_s,
(3) For every $x \in \mathbb{R}$, if T_x is illfounded then for any sequence $\{A_n : n \in \omega\}$ of measure one sets with $\mu_{x\restriction n}(A_n) = 1$, there a branch $f \in \kappa^\omega$ such that for all $n, (x \restriction n, f \restriction) \in T$.

T is δ-homogeneous if in addition the measures are δ-complete.

The second clause in the above definition is what makes the tower of measures be countably complete. It is a standard fact that a tower of measures is countably complete if and only if the direct limit of the ultrapowers given by the measures μ_s is wellfounded. We say a tree T is κ-homogeneous if the measures μ_s can be taken to be κ-complete. A set $A \subseteq \mathbb{R}$ is κ-homogeneously-Suslin if $A = p[T]$ for T a κ-homogeneous tree

The second property a tree can have is that of *stability*. This notion is due to Jackson and we define it below. Let T be a tree on $\omega \times \omega \times \kappa$ be homogeneous via the measures $\mu_{s,t}$ on $\kappa^{<\omega}$. So, if we identify the last two coordinates of the tree into a single coordinate by a bijection between $\omega \times \kappa$ and κ, the resulting tree T' on $\omega \times \kappa$ is weakly homogeneous.

Recall that a sequence $A_{s,t}$ of measure one sets with respect to the $\mu_{s,t}$ is said to stabilize the tree T if for all x such that T_x is wellfounded we have that for any measure one sets $B_{x\restriction n,t}$ and for any $t \in \omega^{<\omega}$ with has length n, we have $[f^{\vec{A}}_{x\restriction n,t}]_{\mu_{x\restriction n,t}} \leq [f^{\vec{B}}_{x\restriction n,t}]_{\mu_{x\restriction n,t}}$. Here

$f^A_{x\restriction n,t}(\vec{\alpha})$ is the rank of the tuple $(x \restriction n, t, \vec{\alpha})$ in the tree

$$T_x \restriction \vec{A} = \{(u, \vec{\beta})\colon (x \restriction \mathrm{lh}(u), u, \vec{\beta}) \in T \wedge \forall k \leq n \ (\vec{\beta} \restriction k \in A_{x\restriction k, t\restriction k}\}.$$

We similarly define $f^B_{x\restriction n,t}(\vec{\alpha})$. That is the functions $f^A_{x\restriction n,t}$ are the ranking subfunctions of the canonical ranking function $f_x : T_x \to \mathrm{ORD}$, for x such that T_x is wellfounded, when the tree is restricted to measure one sets.

LEMMA 3.4 (Jackson). *Let T be a stable homogeneous tree as witnessed by measures $\{\mu_s : s \in \omega^{<\omega}\}$ and measure one sets $\{A_s : s \in \omega^{<\omega}\}$. Let T' be the Martin-Solovay tree with $B = p[T']$ constructed from $T^{\vec{A}}$ and μ_s for $s \in \omega^{<\omega}$. Let $\vec{\varphi}$ be the corresponding semi-scale given by for $x \in B$, $\varphi_n(x) = [f^{\vec{A}}_{x\restriction n}]_{\mu_{x\restriction n}}$. Then $\vec{\varphi}$ is a scale.*

Recall that assuming AD^+, for a (weakly) homogeneous tree T, there is a sequence $\vec{A}$ of measure one sets stabilizing the tree T.

THEOREM 3.5 (Jackson). *Every homogeneous tree T on $\omega \times \kappa$, as witnessed by a sequence of measures $\{\mu_s\}$ is stable, for $\kappa < \Theta$ is stable.*

So stability is another property that Suslin representations have and it is a weaker notion than homogeneity. The lemma in the next section is inspired by Jackson's proof of the Kechris-Martin theorem using his theory of descriptions, see [6] for more details on Jackson's proof the Kechris-Martin theorem using. For the original proof of the Kechris-Martin theorem we refer the reader to [8].

3.2. Lightface Sets of Ordinals and Stabilizing the Kunen and Martin Trees

In this section we show the following technical lemma, which tells us that we can stabilize lightface trees by a lightface set of ordinals. The lemma can be generalized to higher levels of the Wadge hierarchy and it will allow us to define lightface scales on sets of reals without having to transfer them using the game quantifier as in the theorem quoted in the previous section. The canonical trees T_{2n} will be trees coming from these lightface scales.

Lemma 3.6 (A., Jackson). *Let T be a tree on $\omega\times\omega\times\omega_1$ which is homogeneous with measures W_1^n (i.e., the n-fold products of the normal measure on ω_1). Assume also that T is Δ^1_1 in the codes. Then there is a c.u.b. $C \subseteq \omega_1$ which stabilizes T and such that C is Δ^1_3 in the codes.*

Proof. Let $U \subseteq \omega \times \omega_1$ be the Kunen tree. If U_x is wellfounded, then let $f_x : \omega_1 \to \omega_1$ be the function $f_x(\alpha) = |U_x \restriction \alpha|$. In this case, let

$$C_x = \{\alpha < \omega_1 : \forall\beta < \alpha \ f_x(\beta) < \alpha\}$$

be the c.u.b. set coded by x. For every c.u.b. $C \subseteq \omega_1$ there is an x with U_x wellfounded and $C_x \subseteq C$.

For $w \in \omega^\omega$, and $\alpha < \omega_1$, we say w is weakly α-good if for all $\beta \leq \alpha$ either $U_w \restriction \beta$ is wellfounded of rank $< \alpha$ or α is in the wellfounded part of $U_w \restriction \beta$. We say w is strongly α-good if for all $\beta \leq \alpha$ we have that $U_w \restriction \beta$ is wellfounded. We say w is $< \alpha$ weakly (strongly) good if for all $\alpha' < \alpha$, w is weakly (strongly) α'-good. Let WG_α be the set of w which are α-weakly good, and SG_α the set of w which are strongly α-good. Likewise define $\mathrm{WG}_{<\alpha}$ and $\mathrm{SG}_{<\alpha}$. These sets are defined with respect to the tree U, and so we also write WG^U_α, SG^U_α. We can also speak of good with respect to the tree T, and so write WG^T_α, SG^T_α. Note that WG^U_α, $\mathrm{WG}^U_{<\alpha}$ are $\utilde{\Delta}^1_1$ (SG_α is $\utilde{\Pi}^1_1$).

Consider now the game G where I plays out w_1, y, and II plays out w_2. II wins the run iff there is an $\eta < \omega_1$ such that one of the following holds:

(1) $w_1 \in \mathrm{WG}^U_{<\eta}$, $y \in \mathrm{WG}^T_{<\eta}$, $w_2 \in \mathrm{SG}^U_\eta$, with either $w_1 \notin \mathrm{WG}^U_\eta$ or $y \notin \mathrm{WG}^T_\eta$, and $w_2 \in \mathrm{SG}^T_\eta$.

(2) $w_1 \in \mathrm{WG}^U_\eta$, $y \in \mathrm{WG}^T_\eta$, $w_2 \in \mathrm{SG}^T_\eta$, and there is a $\gamma \leq \eta$ such that (i) $\forall\beta < \gamma |U_{w_1} \restriction \beta| < \gamma$, (ii) $\forall\beta < \gamma |U_{w_2} \restriction \beta| < \gamma$, (iii) $P_\gamma(w_1, y, w_2)$.

Here $P_\gamma(w_1, y, w_2)$ are, uniformly in γ, $\utilde{\Pi}^1_1$ relations such that if $T_y \restriction \gamma$ is wellfounded and w_1, w_2 satisfy (1) and (2), then $P_\gamma(w_1, y, w_2)$ holds iff $|T_y \restriction (C_{w_2} \cap \gamma)| \leq |T_y \restriction (C_{w_1} \cap \gamma)|$.

Note that this is a Σ^1_2 game for II. So, if II wins G, then II has a Δ^1_3 winning strategy.

Claim 3.7. *II has a winning strategy for G.*

Proof. Let $C \subseteq \omega_1$ be c.u.b. and stabilize T. Let w_2 code a c.u.b. set and such that $C_{w_2} \subseteq C$. Let II play w_2 in G. Suppose I plays w_1, y. If either w_1 or y is not α-weakly good for some $\alpha < \omega_1$, then II wins by clause (1) as w_2 is α-strongly good for all α. So assume w_1, y are α-weakly good for all α. Thus, U_{w_1} and T_y are wellfounded. So, C_{w_1} and C_{w_2} are defined. As C_{w_2} still stabilizes T we have that $[F_y^{C_{w_2}}]_{W_1^1} \leq [F_y^{C_{w_1}}]_{W_1^1}$. It follows that there is an $\alpha < \omega_1$ (in fact, a c.u.b. set) with $\alpha \in C_{w_1} \cap C_{w_2}$ and such that $|T_y \restriction C_{w_2} \cap \alpha| \leq |T_y \restriction C_{w_1} \cap \alpha|$. Thus II has won by clause (2).

Let τ be a Δ^1_3 winning strategy for II. We define a c.u.b. set C^τ which stabilizes T. To do this, we first define inductively a function $b\colon \omega_1 \to \omega_1$. Assume $b(\beta)$ is defined for all $\beta < \alpha$. Let

$$(w_1, y) \in W_\alpha \leftrightarrow [w_1 \in \mathrm{WG}^U_\alpha \wedge y \in \mathrm{WG}^T_\alpha \wedge \neg \exists \gamma \leq \alpha \text{ (II wins by clause (2) at } \gamma)]$$

So, $W_\alpha \in \mathbf{\Sigma}^1_1$. We also easily have that $W_\alpha \neq \emptyset$. If $(w_1, y) \in W_\alpha$ and $w_2 = \tau(w_1, y)$, then w_2 is α-strongly good, that is, $U_{w_2} \restriction \alpha$ is wellfounded. That is, $f_{w_2}(\alpha) = |U_{w_2} \restriction \alpha|$ is defined. By boundedness we then have that

$$b(\alpha) = \sup\{f_{\tau(w_1,y)}(\alpha)\colon (w_1, y) \in W_\alpha\} < \omega_1.$$

This completes the definition of the b function. Let C_b be the set of closure points of b. We claim that C_b stabilizes T. Suppose not, and let C_1, y be such that T_y is wellfounded and $[F_y^{C_1}]_{W_1^1} < [F_y^{C_b}]_{W_1^1}$. Let C_2 be c.u.b. such that $F_y^{C_1}(\alpha) < F_y^{C_b}(\alpha)$ for all $\alpha \in C_2$. Let w_1 code a c.u.b. set such that $C_{w_1} \subseteq C_1 \cap C_2$. Let I play w_1, y against τ. Let $w_2 = \tau(w_1, y)$. We have that U_{w_1}, U_{w_2}, and T_y are wellfounded.

We claim that for all $\alpha < \omega_1$ that $b(\alpha) \geq f_{w_2}(\alpha) = |U_{w_2} \restriction \alpha|$. We show this inductively on α. Assuming this holds below α, we have that $C_b \cap \alpha \subseteq C_{w_2} \cap \alpha$. From the definitions of C_1 and C_2, there cannot be an $\eta \in C_{w_1}$ such that $F_y^{C_b}(\eta) \leq F_y^{C_{w_1}}(\alpha)$. In particular, there cannot be an $\eta \leq \alpha$ in $C_{w_1} \cap C_{w_2}$ for which $F_y^{C_{w_2}}(\eta) \leq F_y^{C_{w_1}}(\alpha)$. That is, there cannot be an $\eta \leq \alpha$ such that II wins by clause (2) at η. Thus, $(w_1, y) \in W_\alpha$. From the definition of the b function we now have that $b(\alpha) \geq f_{w_2}(\alpha)$.

Since $b(\alpha) \geq f_{w_2}(\alpha)$ for all α, we now have that $C_b \subseteq C_{w_2}$. Again from the definitions of C_1 and C_2 we have that there cannot be an $\eta \in C_{w_1}$ such that $F_y^{C_b}(\eta) \leq F_y^{C_{w_1}}(\alpha)$. So, there cannot be an $\eta \in C_{w_1}$ such that $F_y^{C_{w_2}}(\eta) \leq F_y^{C_{w_1}}(\alpha)$. This shows that II has not won by clause (2), and since all the reals are fully good, I has won the run, a contradiction.

So, C_b is a c.u.b. subset of ω_1 which stabilizes T. Since τ is Δ^1_3, it follows that b is Δ^1_3 in the codes, and hence that C_b is Δ^1_3.

$\square$

$\square$

Finally we show that the relation $R(z_1, z_2) \longleftrightarrow z_1, z_2 \in WO \wedge b(|z_1|) = |z_2|$ is Δ^1_3. We have $R(z_1, z_2)$ holds iff the following holds:

(1) $z_1, z_2 \in WO$,

(2) $\exists y \in \mathbb{R}$ and $z \in WO$ with $|z| = |z_1| + 1$ and $|0|_{\prec_z} = |z_1|$ satisfying:

  (a) $\forall n, y_n \in WO$

  (b) the map $n \longmapsto |y_n|$ defines an order preserving map from $\prec_z$ to ω_1,

  (c) $\forall n \in dom(\prec_z)$,

$|y_n| = \{f_{\tau(w_1,y)}(|n|_{\prec_z}) : \forall m \prec_z n[(w_1, y)$ is $|m|_{\prec_z}-$good$\wedge$ II doesn't win by the second clause.$\}$

  (d) $|y_0| = |z_2|$.

So R is $\Sigma^1_2(\tau)$, so it is Δ^1_3, so $rng(b) = C$ is Δ^1_3. This concludes the proof of the lemma.

3.3. The Stable Tree Construction and Lightface Scales on Π^1_{2n} Sets

Before we go into the construction of the canonical trees T_{2n} we need to recall the background theory of the Suslin cardinals which we will need for the coding of ordinals below $\aleph_{\epsilon_0}$ and the theory of descriptions which we need for the construction. As before our base theory is ZF+DC+AD. In this theory, successor cardinals need not be regular. As usual,

$$\underset{\sim}{\delta}^1_n =_{def} \sup\{|\preceq| : \preceq \text{ is a } \underset{\sim}{\Delta}^1_n \text{ prewellordering of } \mathbb{R}\}$$

Recall that by the coding lemma the $\underset{\sim}{\delta}_n^1$ are regular successor cardinals. To see this, first recall that

$$\underset{\sim}{\delta}_n^1 = \sup\{\xi : \xi \text{ is the length of a } \underset{\sim}{\Sigma}_n^1 \text{ wellfounded relation}\},$$

(see theorem 2.13 of [6] for a proof of this fact). Next suppose not and let $f : \gamma \to \underset{\sim}{\delta}_n^1$ a cofinal map with $\gamma < \underset{\sim}{\delta}_n^1$. Let $\preceq$ be a $\underset{\sim}{\Delta}_n^1$ prewellordering of length γ. Let φ be the norm associated to the prewellordering $\preceq$. Then let R be the relation defined by $R(x, w) \leftrightarrow$ w is a code of a $\underset{\sim}{\Sigma}_n^1$ wellfounded relation of length $\varphi(w)$. By the coding lemma let R' be a $\underset{\sim}{\Sigma}_n^1$ choice subrelation of R. Now let U be a $\underset{\sim}{\Sigma}_n^1$ universal set and define the following prewellordering:

$$(x_0, y_0, z_0) \prec (x_1, y_1, z_1) \leftrightarrow (x_0 = x_1 \wedge y_0 = y_1 \wedge R'(x_0, y_0) \wedge U_{y_0}(z_0, z_1)).$$

Then $\prec$ is a wellfounded $\underset{\sim}{\Sigma}_n^1$ relation. Now for any $\xi < \gamma$, if x is such that $\varphi(x) = \xi$ then for any y such that $R'(x, y)$ then the map $z \to (x, y, z)$ embeds U_y into $\prec$. So we have $|U_y| = \xi \leq |\prec|$, and so $|\prec| = \underset{\sim}{\delta}_n^1$. Contradiction! By Kunen, Martin and Solovay, the $\underset{\sim}{\delta}_n^1$ are all measurable cardinals (see theorem 5.2 of [11] for a proof) and by Jackson $\underset{\sim}{\delta}_{2n+1}^1$ satisfy the strong partition property (see [6] for the underlying theory needed to prove this). We define the Suslin cardinals of cofinality ω:

$$\kappa_{2n+1}^1 =_{def} \text{the least } \gamma \text{ s.t for every } A \in \underset{\sim}{\Sigma}_{2n+1}^1 \text{ there exists } T \subseteq \omega \times \gamma \text{ s.t } A = p[T]$$

Below we put these cardinal in context and briefly explain why they are defined.

Recall the following useful theorem of Martin. We refer the reader to theorem 2.15 of [6] for a proof.

THEOREM 3.8 (ZF+AD). *Let $\underset{\sim}{\Gamma}$ be a nonselfdual pointclass closed under $\forall^{\mathbb{R}}, \wedge$ and $\vee$. Then $\underset{\sim}{\Delta} = \underset{\sim}{\Gamma} \cap \check{\underset{\sim}{\Gamma}}$ is closed under unions and intersections of length strictly less than $\underset{\sim}{\delta}(\underset{\sim}{\Gamma})$, where $\underset{\sim}{\delta}(\underset{\sim}{\Gamma}) =_{def} \sup\{\xi : \xi$ is the length of a $\underset{\sim}{\Delta}$ prewellordering of $\mathbb{R}\}$.*

By the scale property on $\underset{\sim}{\Pi}_{2n+1}^1$ and the Kunen-Martin theorem it follows that $(\kappa_{2n+1}^1)^+ = \underset{\sim}{\delta}_{2n+1}^1$. Too see this suppose that $A \in \underset{\sim}{\Sigma}_{2n+1}^1$ is a universal set and let $B \in \underset{\sim}{\Pi}_{2n}^1$ such that $A(x) \leftrightarrow \exists y B(x, y)$. Since the pointclass of κ-Suslin sets, $S(\kappa)$ is closed under $\exists^{\mathbb{R}}$ then if

B is κ-Suslin, the set A is also κ-Suslin. Since the pointclass $\underset{\sim}{\Pi}^1_{2n+1}$ has the scale property then the set B has a $\underset{\sim}{\Delta}^1_{2n+1}$ scale whose norms go onto some $\kappa < \underset{\sim}{\delta}^1_{2n+1}$ since by definition $\underset{\sim}{\delta}^1_{2n+1}$ is the supremum of the $\underset{\sim}{\Delta}^1_{2n+1}$ norms. Let κ^1_{2n+1} the least $\kappa < \underset{\sim}{\delta}^1_{2n+1}$ as above. So B is κ^1_{2n+1}-Suslin and this A is κ^1_{2n+1}-Suslin. Hence the pointclass $\underset{\sim}{\Sigma}^1_{2n+1}$ is contained in $S(\kappa^1_{2n+1})$. By the Kunen=Martin theorem we must then have that $\underset{\sim}{\delta}^1_{2n+1} = (\kappa^1_{2n+1})^+$. From Wadge's lemma and the closure of $\underset{\sim}{\Delta}^1_{2n+1}$ under unions of length less than $\underset{\sim}{\delta}^1_{2n+1}$ we have that $cf(\kappa^1_{2n+1}) = \omega$. To see this, suppose that $cof(\kappa^1_{2n+1}) > \omega$. Then every set $A \in \underset{\sim}{\Sigma}^1_{2n+1}$ can be written as a κ^1_{2n+1} union of sets A_α which are $< \kappa^1_{2n+1}$-Suslin. Since $\underset{\sim}{\Delta}$ is closed under unions of length strictly less than $\underset{\sim}{\delta}^1_{2n+1}$ then $A \in \underset{\sim}{\Delta}^1_{2n+1}$, but A was an arbitrary $\underset{\sim}{\Sigma}^1_{2n+1}$ set. Using this analysis and the coding lemma, it follows that $\underset{\sim}{\Sigma}^1_{2n+1}$ sets are exactly the κ^1_{2n+1} sets, see [6]. By the prewellordering property for $\underset{\sim}{\Pi}^1_{2n+1}$ and since every $\underset{\sim}{\Sigma}^1_{2n+2}$ wellfounded relation is $\underset{\sim}{\delta}^1_{2n+1}$, we have that $(\underset{\sim}{\delta}^1_{2n+1})^+ = \underset{\sim}{\delta}^1_{2n+2}$. We also have the following values for the projective ordinals and the Suslin cardinals of cofinality ω:

(1) $\kappa^1_1 = \aleph_0, \underset{\sim}{\delta}^1_1 = \aleph_1$ and thus $\underset{\sim}{\delta}^1_2 = \aleph_2$,

(2) $\kappa^1_3 = \aleph_\omega, \underset{\sim}{\delta}^1_3 = \aleph_{\omega+1}$ and thus $\underset{\sim}{\delta}^1_4 = \aleph_{\omega+2}$ (Martin and Solovay).

(3) In general (Jackson), we have $\kappa^1_{2n+1} = \aleph_{\underbrace{\omega^{\omega^{\cdots^\omega}}}_{\text{2n+1 tower}}}, \underset{\sim}{\delta}^1_{2n+1} = \aleph_{\underbrace{\omega^{\omega^{\cdots^\omega}}}_{\text{2n+1 tower}}+1}$ and thus $\underset{\sim}{\delta}^1_{2n+2} = \aleph_{\underbrace{\omega^{\omega^{\cdots^\omega}}}_{\text{2n+1 tower}}+2}$

To carry out the construction of the trees T_{2n}, we need to introduce natural families of measures which arise in the context of weak and strong partition properties. We start out by recalling the notion of *uniform cofinality.* The notion has its roots in Martin's proof of the strong partition property of ω_1. Analyzing such functions is central in Jackson's theory of descriptions for proofs of the strong partition property and in the analysis of the trees of uniform cofinality which codes homogeneity measures. We also recall, below, the definitions of trees of uniform cofinality and of the measures coded by the trees of uniform cofinality. These definitions are used extensively in Jackson's analysis of measure in $L(\mathbb{R})$. We won't be working with these trees directly but we need them since they are used in the definitions of level-n complexes which appear in the proof of the generalization of the Kechris-Martin theorem. The reader won't lose much if she/he does not know how the full descriptions are used to analyze the cardinal structure at the projective level in $L(\mathbb{R})$. We will introduce a representative case for the definition of the trees of uniform cofinalities, the reader can see [5] for the general cases.

Recall that under AC, there are no infinite exponent partition relations. Assume AC and suppose that for some infinite cardinal κ, we have that $\kappa \to (\omega)^\omega$. Let $A, B \in [\kappa]^\omega$ and put $A \sim B$ if and only if the set of places where A and B disagree is finite. Then $\sim$ is easily an equivalence relation. By AC pick representatives in each class and define the partition F by $F(A) = 0$ if and only if A disagrees with the representative of its equivalence class an even number of times and $F(A) = 1$ otherwise. But then, there cannot be any $H \subseteq \kappa$ homogeneous set of order-type ω for the partition F since for any such H, at cofinally many place below ω, we can find $A, B \in [H]^\omega$ such that one disagrees with its representatives an even number of times and the other an odd number of times.

Let $\kappa < \delta$ be two regular cardinals. We let μ^δ_κ denote the filter on δ generated by κ-closed c.u.b sets, i.e μ^δ_κ concentrates on points of cofinality κ. μ^δ_κ is defined as follows:

$$\mu^\delta_\kappa = \{X \subseteq \delta : \text{there exists a c.u.b set } C \subseteq \delta \text{ s.t } X \cap \{\gamma < \delta : cf(\gamma) = \kappa\} \subseteq X\}$$

It is a basic result of Kleinberg that if δ has the strong partition property, or just the weak partition property for that matter, it turns out that μ^δ_κ is a normal measure on δ. In addition, for each regular cardinal $\kappa < \delta$ there is a unique normal measure on δ, see [7] for a proof.

Definition 3.9. A function $f : \kappa \to \mathrm{ORD}$ is said to have **uniform cofinality** ω if there is a function $f' : \kappa \times \omega \to \mathrm{ORD}$ which is increasing in the second argument such that for all $\alpha < \kappa, f(\alpha) = \sup_{n<\omega} f'(\alpha, n)$. We say f is of the **correct type** if f is increasing, everywhere discontinuous, i.e $f(\alpha) > \sup_{\beta<\alpha} f(\beta)$ and of uniform cofinality ω. Letting $g : \kappa \to \mathrm{ORD}$, we say $f : \kappa \to \mathrm{ORD}$ is of uniform cofinality g if there is a function f' with domain $\{(\alpha, \beta) : \alpha < \kappa, \beta < g(\alpha)\}$ which is increasing in the second argument and which is such that $f(\alpha) = \sup_{\beta<g(\alpha)} f'(\alpha, \beta)$. If g has constant value γ then we say f has uniform cofinality γ. We say f has type g if f is increasing, everywhere discontinuous and

has uniform cofinality g.

Next we need the definition of the S_1^n measures which come from the strong partition property on ω_1:

Definition 3.10. Let $n \in \omega$ and let $(\omega_1)^n$ be the set of increasing n-tuples from ω_1. We define the wellordering $<_n$ on $(\omega_1)^n$ by:

$$(\alpha_1, ..., \alpha_n) <_n (\beta_1, ..., \beta_n) \leftrightarrow (\alpha_n, \alpha_1, ..., \alpha_{n-1}) <_{lex} (\beta_n, \beta_1, ..., \beta_{n-1})$$

We then let $dom(<_n) = (\omega_1)^n$. Letting π be a permutation of $n+1$ such that $\pi = (n, i_1, .., i_n)$, we say $f : (\omega_1)^n \to \text{ORD}$ is ordered by π if $f(\alpha_1, ..., \alpha_n) \leq f(\beta_1, .., \beta_n)$ iff $(\alpha_{i_1}, ..., \alpha_{i_n}) \leq_{lex} (\beta_{i_1}, ..., \beta_{i_n})$.

Definition 3.11 (Level-2 tree of uniform cofinalities). Let $\mathcal{S}_\infty$ be the set of all permutations of natural numbers. A level-2 tree of uniform cofinalities is a function $\mathcal{R} : T \subseteq \omega^{<\omega} \to \mathcal{S}_\infty$ such that:

(1) $\mathcal{R}(\emptyset) = (1)$, where (1) is just the trivial permutation of one element.

(2) (base case)

For each $(i_1) \in dom(\mathcal{R})$ either:

(a) $\mathcal{R}(i_1) =$ the uniform cofinality ω, in which case (i_1) is a terminal node in $dom(\mathcal{R})$, or

(b) $\mathcal{R}(i_1) = (2, 1)$, where $(2, 1)$ is the unique permutation of length 2 extending $\mathcal{R}(\emptyset)$.

(3) (inductive case)

For each $(i_1, ..., i_n) \in dom(\mathcal{R})$, $\mathcal{R}(i_1, ..., i_{n-1})$ is a permutation of length n beginning with n and either:

(a) $\mathcal{R}(i_1, ..., i_n) =$ the uniform cofinality ω in which case $(i_1, ..., i_n)$ is a terminal node in $dom(\mathcal{R})$, or

(b) $\mathcal{R}(i_1, ..., i_n)$ is a permutation of length $n+1$ beginning with $n+1$ which extends $\mathcal{R}(i_1, ..., i_{n-1})$

DEFINITION 3.12. Let $\mathcal{R}$ be a tree of uniform cofinalities. Then $<_{\mathcal{R}}$ is the lexicographic ordering on tuples of the form $(\alpha_1, i_1, \alpha_2, i_2, ..., \alpha_n, i_n)$ such that $(i_1, ..., i_n) \in dom(\mathcal{R})$ and $(\alpha_1, ..., \alpha_n)$ is order isomorphic to $\mathcal{R}(i_1, ..., i_n)$.

DEFINITION 3.13. A function $f : dom(<_{\mathcal{R}}) \to \omega_1$ is of type $\mathcal{R}$ is the following holds:

(1) $f : dom(<_{\mathcal{R}}) \to \omega_1$ is order preserving,

(2) If $(i_1, ..., i_n)$ is not a terminal node of $dom(\mathcal{R})$, then $f((\alpha_1, i_1, ..., \alpha_n, i_n)) =$

$$\sup\{f((\alpha_1, i_1, ..., \alpha_n, i_n, \beta, 0)) : (\alpha_1, ..., \alpha_n, \beta) \text{ is order isomorphic to } \mathcal{R}(i_1, ..., i_n)\}$$

(3) If $(i_1, ..., i_n)$ is a terminal node of $dom(\mathcal{R})$, then $f((\alpha_1, i_1, ..., \alpha_n, i_n))$ is greater than

$$\sup\{f((\alpha_1, i_1, ..., \alpha_n, i_n, \beta, j)) : \beta < \alpha_n, (i_1, ..., i_n, j) \in dom(\mathcal{R})\}$$

(4) The uniform cofinality of $f((\alpha_1, i_1, ..., \alpha_n, i_n))$ is determined by $\mathcal{R}(i_1, ..., i_n)$ as follows:

   (a) If $\mathcal{R}(i_1, ..., i_n) = \omega$, then $f((\alpha_1, i_1, ..., \alpha_n, i_n))$ has uniform cofinality ω.

   (b) If $\mathcal{R}(i_1, ..., i_n) \neq \omega$, then $f((\alpha_1, i_1, ..., \alpha_n, i_n))$ has uniform cofinality $o.t(\{\beta : (\alpha_1, ..., \alpha_n, \beta) \text{ is order isomorphic to } \mathcal{R}(i_1, ..., i_n)\})$.

Now we can define the measures $M^{\mathcal{R}}$ coded by $\mathcal{R}$. These measures are necessary for the definition of the level-2 complexes. But first we start with the definition of the measures S_1^n.

DEFINITION 3.14. S_1^n is the measure on $\aleph_{n+1}$ induced by the strong partition property on ω_1 and functions $h : dom(<_n) \to \omega_1$ of the correct type:

$$S_1^n(A) = 1 \leftrightarrow \exists C \subseteq \omega_1 \text{ such that } [f]_{W_1^n} \in A \text{ for all } f : dom(<_n) \to C \text{ of the correct type .}$$

DEFINITION 3.15. We define the measure $M^{\mathcal{R}}$ (this is essentially a measure which appears in the homogeneous tree construction for $\underset{\sim}{\Pi}_2^1$ sets) by

$$X \in M^{\mathcal{R}} \leftrightarrow \exists \text{ a c.u.b set } C \subseteq \omega_1 \text{ s.t for every } f : dom(<_{\mathcal{R}}) \to C \text{ of type } \mathcal{R}, [f]_{W_1^n} \in X$$

We now move towards defining $\mathrm{WO}_{\kappa_5^1}$ the set of codes of ordinals up to $\kappa_5^1 = \aleph_{\omega^{\omega^\omega}}$. Once this is done the definition of the set of codes up to $\aleph_{\epsilon_0}$ will be very similar.

Recall that by the weak partition property on $\underset{\sim}{\delta}_3^1$ there are exactly three normal measure which correspond to the three regular cardinals ω, ω_1 and ω_2. Call them μ_1, μ_2 and μ_3 respectively. Since $\underset{\sim}{\delta}_3^1$ satisfies the strong partition property, the ω cofinal measure is such that $j_{\mu_1}(\underset{\sim}{\delta}_3^1) = \underset{\sim}{\delta}_4^1$. The ω_1-cofinal measure μ_2 is such that $j_{\mu_2}(\underset{\sim}{\delta}_3^1) = \aleph_{\omega.2+1}$ and the ω_2-cofinal measure μ_3 is such that $j_{\mu_2}(\underset{\sim}{\delta}_3^1) = \aleph_{\omega^\omega+1}$ (see [6] for a proof that the cardinals $\underset{\sim}{\delta}_4^1, \aleph_{\omega.2+1}$ and $\aleph_{\omega^\omega+1}$ and the only regular cardinals below $\underset{\sim}{\delta}_5^1$, in particular this uses a theorem of Martin stating that if μ is a measure on κ and κ has the strong partition property then $j_\mu(\kappa)$ is also cardinal). W_3^n is the measure on δ_3^1 induced by the weak partition relation on $\underset{\sim}{\delta}_3^1$, functions $f : \aleph_{n+1} \to \delta_3^1$ of the correct type (i.e they have uniform cofinality ω) and the S_1^n induced on $\aleph_{n+1}$ by the strong partition relation on ω_1. Let for $X \subseteq \underset{\sim}{\delta}_3^1$:

$$X \in W_3^n \leftrightarrow \exists C \subseteq \underset{\sim}{\delta}_3^1 \text{ such that } \forall f : \aleph_{n+1} \to C \text{ of the correct type } [f]_{S_1^n} \in X$$

W_3^n is a measure on $\underset{\sim}{\delta}_3^1$ since there exists a $\underset{\sim}{\Delta}_3^1$ coding of subsets of $\aleph_\omega$, that is a map $\pi : \mathbb{R} \to \mathcal{P}(\aleph_\omega)$ and a $\underset{\sim}{\Delta}_3^1$ norm $\varphi : \mathbb{R} \to \aleph_\omega$ such that $\varphi(x) \in \pi(y)$ is a $\underset{\sim}{\Delta}_3^1$ relation, by Jackson, Kunen and Solovay. We use this to see that for $\alpha < \aleph_\omega$, the ultrapower $j_{S_1^n}(\alpha)$ is $\underset{\sim}{\Delta}_3^1$. Then since the relation on the equivalence classes of functions $f : \aleph_{n+1} \to C$ of the correct type is wellfounded, we have that it has length less than $\underset{\sim}{\delta}_3^1$. Let then $C \subseteq \underset{\sim}{\delta}_3^1$ be a c.u.b set and let $f : \aleph_{n+1} \to C$ and $g : \aleph_{n+1} \to C$ be two functions of the correct type. Then we have $[f]_{S_1^n} \leq [g]_{S_1^n} \leftrightarrow \exists$ a S_1^n measure one set A such that $\forall \alpha \in A, f(\alpha) \leq g(\alpha)$. This is then equivalent to $\exists C \subseteq \omega_1$, where C is a c.u.b set such that $\forall h : dom(<_n) \to C$ of the correct type, $[h]_{W_1^n} \in A \wedge f([h]_{W_1^n}) \leq g([h]_{W_1^n})$. Since c.u.b sets of ω_1 can be coded via the Kunen tree T as above, and since the functions f and g can be coded in a $\underset{\sim}{\Delta}_3^1$ way then this statement is at most $\underset{\sim}{\Delta}_3^1$. Therefore since the relation on the equivalence classes of function $f : \aleph_{n+1} \to C$ of the correct type is wellfounded it must have length less than $\underset{\sim}{\delta}_3^1$.

Recall also that $\sup_n j_\mu(\delta_3^1) = \kappa_5^1$, where μ is a measures appearing in the homogeneous tree construction for $\underset{\sim}{\Pi}_3^1$ sets. This is shown using a computation involving level 2 and level

3 descriptions. In fact it can be seen that $j_{W_3^n}(\delta_3^1) \leq \aleph_{\omega^{\omega^n}+1}$. Essentially one needs to use the lowering operator defined on the set of descriptions, then a computation of the rank of the lowering operator yields the result. This is how Jackson computed $\underset{\sim}{\delta}_5^1$ and we refer to [4] for the detail of the computation.

We now outline the plan to construct lightface scales on Π_{2n+2}^1 sets of reals. We first need to define the Jackson tree J_{2n+1}. The tree J_{2n+1} will be a homogeneous tree on $\omega \times \underset{\sim}{\delta}_{2n+1}^1$ which projects to a complete $\underset{\sim}{\Pi}_{2n+1}^1$ set. This tree analyzes the homogeneity measures appearing in the type 2 trees of uniform cofinality $\mathcal{R}$, i.e the homogeneity measures appearing in a the construction of trees projecting to $\underset{\sim}{\Pi}_{2n}^1$ sets. Next from J_{2n+1} one obtains the more general Martin tree T which analyzes functions $f : \underset{\sim}{\delta}_{2n+1}^1 \to \underset{\sim}{\delta}_{2n+1}^1$ with respect to the normal measures on $\underset{\sim}{\delta}_{2n+1}^1$. We show that the Martin tree construction can be modified so as to obtain another Martin tree T which is Δ_{2n+1}^1 in the codes. Once this is done, the generalization of the main technical lemma, shown in section 3.2, applied to this context shows that there is a c.u.b set $C \subseteq \underset{\sim}{\delta}_{2n+1}^1$ which is Δ_{2n+1}^1 in the codes and which stabilizes this modified Martin tree T. Finally the Martin-Solovay construction applied to this modified Martin tree will yield a canonical tree T_{2n+2}. This will allow the construction of Δ_{2n+3}^1 scales on the appropriate sets of reals. Finally an argument from Martin will show that the norms of the scales are $\Game^{2n+1}(\omega n - \Pi_1^1)$.

Theorem 3.16 (Jackson, [6]). *There is a $\underset{\sim}{\Pi}_3^1$ complete set P, a Π_3^1-norm φ such that $\varphi(x) = |x| < \underset{\sim}{\delta}_3^1$ from P onto $\underset{\sim}{\delta}_3^1$ and a homogeneous tree J_3 on $\omega \times \delta_3^1$ for P satisfying the following. There is a c.u.b set $C \subseteq \underset{\sim}{\delta}_3^1$ such that for all $\alpha \in C$, there is a $x \in P$ with $\varphi(x) = \alpha$ and with $J_{3_x} \restriction (\sup_\nu j_\nu(\alpha))$ illfounded, where the supremum ranges over measures appearing in $\mathcal{M}^{R_s}$, the tree of uniform cofinalities, coding measures which appear on a homogeneous tree projecting to WO_2.*

Next consider functions $f : \underset{\sim}{\delta}_3^1 \to \underset{\sim}{\delta}_3^1$ and the Martin tree T on $\omega \times \underset{\sim}{\delta}_3^1$. The Martin tree is the appropriate generalization of the Kunen tree. The Kunen tree on $\omega \times \omega_1$ is used to analyze functions $f : \omega_1 \to \omega_1$. The additional difficulty is to consider all measures below $\underset{\sim}{\delta}_3^1$ which arise from the different cofinalities corresponding the the regular cardinals below

$\underset{\sim}{\delta}^1_3$.

THEOREM 3.17 (Martin,[6]). *There is a tree T on $\omega \times \underset{\sim}{\delta}^1_3$ such that for all $f : \underset{\sim}{\delta}^1_3 \to \underset{\sim}{\delta}^1_3$, there is an $x \in \mathbb{R}$ with T_x is wellfounded and a c.u.b set $C \subseteq \underset{\sim}{\delta}^1_3$ such that for all $\alpha \in C, f(\alpha) < |T_x \restriction \sup_\nu j_\nu(\alpha)|$, where if $cof(\alpha) = \omega$ then we use $|T_x \restriction \alpha|$ and if $cof(\alpha) = \omega_1$, the supremum ranges over the n-fold products, W^n_1, of the normal measure on ω_1 (these occur in the homogeneous tree construction projecting to a $\underset{\sim}{\Pi}^1_1$ set) and if $cof(\alpha) = \omega_2$, the supremum ranges over the measures occurring in the homogeneous tree construction projecting to a $\underset{\sim}{\Pi}^1_2$ set.*

Notice that the Martin tree T is Δ^1_3 in the codes. That is we can find two relations S and T which are Σ^1_3 and Π^1_3 respectively such that

We are now in a position to define the codes of ordinals less than κ^1_5:

DEFINITION 3.18 (The set of codes of ordinals less than κ^1_5). Let then T on $\omega \times \underset{\sim}{\delta}^1_3$ be the Martin tree and define

$$\mathrm{WO}_{\kappa^1_5} = \{\langle z, x_1, ..., x_n\rangle : z \in \mathrm{WO}_\omega \wedge T_{x_i} \text{ is wellfounded } \forall i\}$$

For $y = \langle z, x_1, ..., x_n\rangle \in WO_{\kappa^1_5}$, let $|y| = [f_y]_{W^n_3}$ where $f_y : (\delta^1_3)^n \to \delta^1_3$ is defined by:

$$f_y(\beta_1, ..., \beta_n) = |(T_{x_n} \restriction \sup_\nu j_\nu(\beta_n)(\delta_{n-1})|, \text{ where },$$

$$\delta_{n-1} = |(T_{x_{n-1}} \restriction \sup_\nu j_\nu(\beta_{n-1})(\delta_{n-2})|, ...$$

$$\delta_1 = |(T_{x_1} \restriction \sup_\nu j_\nu(\beta_1)(\delta_0)|, \text{ and } \delta_0 = |z|_{\mathrm{WO}_\omega}$$

In the above we use the appropriate measure ν according to which cofinality the ordinal β_j has, for $1 \leq j \leq n$, in view of Martin's theorem. So for every $\alpha < \kappa^1_5, \exists y \in \mathrm{WO}_{\kappa^1_5}$ such that $\alpha = [f_y]_{W^n_3}$ for some $n \in \omega$. Notice that $\mathrm{WO}_{\kappa^1_5}$ is $\underset{\sim}{\Pi}^1_4$. Also notice that we could have defined $\mathrm{WO}_{\aleph_{\omega^{\omega^n}}}$ for each $n \in \omega$ and then taken the unions of all these sets of codes to obtain $\mathrm{WO}_{\kappa^1_5}$.

In general we define $\mathrm{WO}_{\kappa^1_{2n+3}}$ in a similar manner. Let W^n_{2n+1} the $cof(\gamma)$-cofinal measure on $\underset{\sim}{\delta}^1_{2n+1}$, where γ is the largest regular cardinal strictly less than $\underset{\sim}{\delta}^1_{2n+1}$. The Martin tree T in this case will be a tree on $\omega \times \underset{\sim}{\delta}^1_{2n+1}$ and we'll consider functions $f : \underset{\sim}{\delta}^1_{2n+1} \to \underset{\sim}{\delta}^1_{2n+1}$, except this time there will be a lot more normal measures, all corresponding to the regular cardinals below $\underset{\sim}{\delta}^1_{2n+1}$.For each cofinality the appropriate measure has to be plugged in the Martin tree construction to analyze functions $f : \underset{\sim}{\delta}^1_{2n+1} \to \underset{\sim}{\delta}^1_{2n+1}$.

Definition 3.19 (The set of codes of ordinals less than κ^1_{2n+3}).

$$\mathrm{WO}_{\kappa^1_{2n+3}} = \{\langle z, x_1, ..., x_m\rangle : z \in \mathrm{WO}_{\kappa^1_{2n+1}} \wedge T_{x_i} \text{ is wellfounded } \forall i\}$$

For $y = \langle z, x_1, ..., x_m\rangle \in \mathrm{WO}_{\kappa^1_{2n+3}}$, let $|y| = [f_y]_{W^m_{2n+1}}$, for some $m \in \omega$, where, letting T on $\omega \times \underset{\sim}{\delta}^1_{2n+1}$ be the Martin tree, $f_y : (\delta^1_{2n+1})^m \to \delta^1_{2n+1}$ is defined by:

$$f_y(\beta_1, ..., \beta_m) = |(T_{x_m} \upharpoonright \sup_\nu j_\nu(\beta_m)(\delta_{m-1})|, \text{ where },$$

$$\delta_{m-1} = |(T_{x_{m-1}} \upharpoonright \sup_\nu j_\nu(\beta_{m-1})(\delta_{m-2})|, ...$$

$$\delta_1 = |(T_{x_1} \upharpoonright \sup_\nu j_\nu(\beta_1)(\delta_0)|, \text{ and } \delta_0 = |z|_{\mathrm{WO}_{\kappa^1_{2n+1}}}$$

Again everything below κ^1_{2n+3} is coded and $\mathrm{WO}_{\kappa^1_{2n+3}}$ is a $\underset{\sim}{\Pi}^1_{2n+2}$ set of reals. The coding can be generalized up to the first inaccessible cardinal in $L(\mathbb{R})$.

Next, to apply the technical lemma proved above we first need to obtain a lightface linear ordering version of the Martin tree mentioned above. More specifically we will show the following:

Lemma 3.20. *There is a function $s \to T(s)$ which assigns to each $s \in \omega^{<\omega}$ a wellordering of a subset of $\underset{\sim}{\delta}^1_{2n+1}$ with the following properties. If t extends s then $T(s) \subseteq T(s)$. For $x \in \mathbb{R}$, let $T(x) = \bigcup_n T(x \upharpoonright n)$, so $T(x)$ is a linear order. Then for any function $f : \underset{\sim}{\delta}^1_{2n+1} \to \underset{\sim}{\delta}^1_{2n+1}$ there is an $x \in \mathbb{R}$ such that $T(x)$ is a wellordering and a c.u.b set $C \subseteq \delta^1_{2n+1}$ such that and for all $\alpha \in C, f(\alpha) < |T(x) \upharpoonright \sup_\nu j_\nu(\alpha)|$, where the supremum ranges over each normal measures on $\underset{\sim}{\delta}^1_{2n+1}$ generated by each regular cardinal $\gamma < \underset{\sim}{\delta}^1_{2n+1}$, depending on $cof(\alpha) = \gamma$. Moreover, the map $s \to T(s)$ is Δ^1_{2n+1} in the codes. That is are Σ^1_{2n+1} and Π^1_{2n+1} relations S and R such that for all $x \in WO_{\kappa^1_{2n-1}}$ we have*

PROOF. Fix a bijection $\pi : (\underset{\sim}{\delta}^1_{2n+1})^{<\omega} \to \underset{\sim}{\delta}^1_{2n+1}$ such that for all $\alpha_0, ..., \alpha_n < \kappa^1_{2n+1}$ we have $\pi(\alpha_0, ..., \alpha_n) < \kappa^1_{2n+1}$. For $s \in \omega^{<\omega}$, let T be the Martin tree and let $T(s)$ be the wellordering defined by:

$$\alpha T(s)\beta \leftrightarrow \pi^{-1}(\alpha), \pi^{-1}(\beta) \in T_s \wedge (\pi^{-1}(\alpha) <_{BK,T_s} \pi^{-1}(\beta))$$

For $x \in \mathbb{R}$, let $T(x) = \bigcup_n T(x \restriction n)$. Then by the definition of the Brouwer-Kleene order, $T(x)$ is a linear ordering and $T(x)$ is a wellordering if and only if T_x is wellfounded. Let $f : \underset{\sim}{\delta}^1_{2n+1} \to \underset{\sim}{\delta}^1_{2n+1}$. let $C \subseteq \underset{\sim}{\delta}^1_{2n+1}$ be the c.u.b set of ordinals closed under π. Then $\kappa^1_{2n+1} \in C$. For $\kappa^1_{2n+1} \leq \alpha$, let $l(\alpha)$ be the greatest element of C which is less than or equal to α. Define $f'(\alpha) = \sup\{f(\beta) : l(\beta) = l(\alpha)\}$. Let $x \in \mathbb{R}$ be such that T_x is wellfounded and for all $\omega \leq \alpha, f'(\alpha) < |T_x \restriction \alpha|$. We show the following claim:

CLAIM 3.21. *For every $\omega < \alpha$, we have $f(\alpha) < |T(x) \restriction \alpha|$.*

PROOF. Notice that we have $T_x \restriction l(\alpha) \subseteq \pi^{-1}{}''T(x) \restriction \alpha$. Hence $f(\alpha) \leq f'(l(\alpha)) < |T_x|l(\alpha)| \leq |T(x) \restriction \alpha|$. We can choose π so that it is Δ^1_{2n+1} in the codes.

□

The above claim finishes the proof of the lemma.

□

Using the Martin tree T_3 on $\omega \times \omega \times \underset{\sim}{\delta}^1_3$, we now define T_4 on $\omega \times \kappa^1_5$ for Π^1_4 complete sets of reals using the Martin-Solovay construction. Let $C \subseteq \underset{\sim}{\delta}^1_3$ be a Δ^1_5 c.u.b set of $\underset{\sim}{\delta}^1_3$ stabilizing the tree T_3, by the main technical lemma (see below for the statement). Let $A \subseteq \mathbb{R}$ be a complete Π^1_4 set. Then for some $B \in \Pi^1_3$ we have that:

$$A(x) \leftrightarrow \neg\exists y B(x, y) \leftrightarrow \neg\exists y \exists f (x, y, f) \in [T_3].$$

Then define T_4 as follows:

$$(s, \vec{\alpha}) \in T_4 \leftrightarrow \exists f_s : T^C_{3_s} \to \delta^1_3 \text{ such that } \vec{\alpha} = (\alpha_1, ..., \alpha_{lh(s)}) \text{ where } \alpha^C_i = [f^C_{s\restriction i}]_{W^i_3}, \forall i \leq lh(s)$$

We then have that $A = p[T_4]$. Also notice that T_4 is a tree on $\omega \times \kappa_5^1$. In the general case, one can construct the Jackson tree J under AD. For instance the following theorem of Jackson when combined with a result of Martin and Steel gives the general construction:

THEOREM 3.22 (Jackson, [6]). *Let $\lambda < \kappa$ be regular cardinals and $\undertilde{\Gamma}$ be a pointclass closed under $\forall^{\mathbb{R}}, \wedge, \vee$. Assume that:*

(1) *There is a $\undertilde{\Delta}$ coding of the ordinals less than λ, that is there is a $\undertilde{\Delta}$ set $C \subseteq \mathbb{R}$ and a map $\varphi : C \to \gamma < \lambda$ such that the relations $(x_1, x_2 \in C \wedge \varphi(x_1) \leq \varphi(x_2))$ and $(x_1, x_2 \in C \wedge \varphi(x_1) < \varphi(x_2))$ are both in $\undertilde{\Delta}$,*

(2) *There is a homogeneous tree U which projects to C and such that for all $x \in C, \varphi(x) \leq \psi(x) < \lambda$ where $\vec{\psi_n}$ is the semi-scale from U,*

(3) *There is a map $F : z \to A_z \subseteq \lambda \times \kappa$ for $z \in \mathbb{R}$, satisfying:*

   (a) *For every $f : \lambda \to \kappa \exists z A_z = f$,*

   (b) *The relation $P'(z, x) \leftrightarrow (x \in C \wedge \exists! \beta A_z(\varphi(x), \beta))$ is in $\undertilde{\Gamma}$,*

   (c) *For all $\alpha < \lambda, \beta < \kappa, P_{\alpha,\beta} = \{z : \forall \alpha' \leq \alpha \exists \beta' \leq \beta(A_z(\alpha', \beta') \wedge \forall \beta''(A_z(\alpha', \beta'') \longrightarrow \beta' = \beta''))\}$ is in $\undertilde{\Delta}$.*

(4) *Every $\undertilde{\Gamma}$ set admits a homogeneous tree on $\omega \times \kappa$ with κ-complete measures,*

(5) *Every $\undertilde{\Delta}$ set is α-Suslin for some $\alpha < \kappa$. Also, if $A \subseteq P \overset{-}{=} \{z : \forall x \in C P'(x, z)\}$ is in $\exists^{\mathbb{R}} \undertilde{\Delta}$, then $\sup\{\varphi(z) : z \in A\} < \kappa$, where for $z \in P$, z is the supremum of the range of the function $A_z : \lambda \to \kappa$.*

Then there is a tree J on $\omega \times \kappa$ such that $p[J] = P$ and a c.u.b set $D \subseteq \kappa$ such that for all $\alpha \in D$ with $cf(\alpha) = \lambda$, there is a $z \in P$ with $\varphi(z) = \alpha$ and $J_z \restriction (\sup_\nu j_\nu(\alpha))$ illfounded, where the supremum ranges over measures ν for the tree U.

Recall that is $\undertilde{\Gamma}$ is the Steel pointclass then Sep($\check{\undertilde{\Gamma}}$), so Red ($\undertilde{\Gamma}$), so there are disjoint $\undertilde{\Gamma}$ sets U, V which code disjoint $\undertilde{\Gamma}$ sets $A = U_x$ and $B = V_x$. $\undertilde{\Delta}$ is said to be uniformly closed under $\exists^{\mathbb{R}}$ of the relations:

$$R(x, z) \leftrightarrow \forall z, w(U_x(z, w) \vee V_x(z, w)) \wedge \exists w U_x(z, w)$$

$$S(x, z) \leftrightarrow \forall z, w(U_x(z, w) \vee V_x(z, w)) \wedge \forall w U_x(z, w)$$

are in $\underset{\sim}{\Gamma}$

THEOREM 3.23 (Martin-Steel,see [6]). *Let $\underset{\sim}{\Gamma}$ be a nonselfdual pointclass and let A be a $\underset{\sim}{\Gamma}$ complete set of reals. Assume that both A and A^c are Suslin. Let $B = \{\sigma : \forall y \sigma(y) \in A\}$. Then B is $\forall^{\mathbb{R}}\underset{\sim}{\Gamma}$-complete and B admits a scale $\vec{\varphi}$ whose corresponding tree T coming from the scale is homogeneous. If $\vec{\varphi}$ is a $\underset{\sim}{\Gamma}$ very good scale on A and either $\underset{\sim}{\Gamma}$ is closed under $\exists^{\mathbb{R}}$ or $\underset{\sim}{\Delta}$ is uniformly closed under $\exists^{\mathbb{R}}$, then $\vec{\varphi}$ is a $\forall^{\mathbb{R}}\underset{\sim}{\Gamma}$ scale. If $\underset{\sim}{\Gamma}$ is closed under $\forall^{\omega}, \cup_{\omega}$ and $\cap$, then the measures in T are κ complete, where $\kappa = \delta(\underset{\sim}{\Delta})$.*

Therefore the above theorem of Jackson can be extended using the Martin-Steel theorem for any $\kappa < \underset{\sim}{\delta}^2_1$ which is a regular Suslin cardinal. In particular we'll need the following in the projective hierarchy.

THEOREM 3.24 (Jackson). *There is a $\underset{\sim}{\Pi}^1_{2n+1}$ complete set P, a Π^1_{2n+1} norm φ such that $\varphi(x) = |x| < \underset{\sim}{\delta}^1_{2n+1}$ from P onto $\underset{\sim}{\delta}^1_{2n+1}$ and a homogeneous tree J_{2n+1} on $\omega \times \delta^1_{2n+1}$ for P satisfying the following. There is a c.u.b set C $\underset{\sim}{\delta}^1_{2n+1}$ such that for all $\alpha \in C$, there is a $x \in P$ with $\varphi(x) = \alpha$ and with $J_{2n+1_x} \restriction (\sup_{\nu} j_{\nu}(\alpha))$ illfounded, where the supremum ranges over measures appearing in $\mathcal{M}^{R_s}$, the tree of uniform cofinalities, coding measures which appear on a homogeneous tree projecting to $WO_{\kappa^1_{2n-1}}$, where κ^1_{2n-1} is the Suslin cardinal of cofinality ω such that $(\kappa^1_{2n-1})^+ = \underset{\sim}{\delta}^1_{2n-1}$ and $(\kappa^1_{2n-1})^{++} = \underset{\sim}{\delta}^1_{2n}$.*

PROOF. see [6]

□

Now let $A \subseteq \mathbb{R}$ be a complete Π^1_{2n+2} set and let T_{2n+1} be the Martin tree on $\omega \times \omega \times \underset{\sim}{\delta}^1_{2n+1}$. Let $C \subseteq \underset{\sim}{\delta}^1_{2n+1}$ be a Δ^1_{2n+3} in the codes c.u.b set stabilizing the Martin tree T_{2n+1} Then for some $B \in \Pi^1_{2n+3}$ we have that:

$$A(x) \leftrightarrow \neg\exists y B(x,y) \leftrightarrow \neg\exists y \exists f (x,y,f) \in [T_{2n+1}].$$

Using the tree T_{2n+1} on $\omega \times \delta^1_{2n+1}$, define the tree T_{2n+2} on $\omega \times \kappa^1_{2n+3}$ as follows:

$(s, \vec{\alpha^C}) \in T_{2n+2} \leftrightarrow \exists f_s : T^C_{2n+1_s} \to \delta^1_{2n+1}$ such that $\vec{\alpha^C} = (\alpha^C_1, ..., \alpha^C_{lh(s)})$ where $\alpha^C_i = [f^C_{s \restriction i}]_{W^i_{2n+1}}, i \leq lh(s)$

Then T_{2n+2} is a tree on $\omega \times \kappa^1_{2n+3}$ and we have that $A = p[T_{2n+2}]$

Lemma 3.25. *Let T be a tree on $\omega \times \omega \times \delta^1_{2n+1}$ which is homogeneous with measures W^n_{2n+1}, i.e., the n-fold products of the normal measure on δ^1_{2n+1}. Assume also that T is Δ^1_{2n+1} in the codes. Then there is a c.u.b. $C \subseteq \delta^1_{2n+1}$ which stabilizes T and such that C is Δ^1_{2n+3} in the codes.*

Proof. Just as the corresponding lemma in the case of ω_1 above, with the necessary modifications to make the proof work.

□

We outline the construction of lightface scales on Π^1_4 sets. The same method, using the appropriate generalization of the technical lemma, will yield scales on Π^1_{2n+2} sets of reals.

Let A be a Π^1_4 complete set of reals, for $x, y \in A$ we let

$$\varphi_n(x) \leq \varphi_n(y) \leftrightarrow [f^C_{x\restriction n}]_{W^n_3} \leq [f^C_{y\restriction n}]_{W^n_3},$$

where $C \subseteq \underset{\sim}{\delta}^1_5$ is a Δ^1_5 in the codes c.u.b set stabilizing the Martin tree. Without stabilizing the Martin tree, this is a semi-scale but the stability argument will show that this actually is a scale. By the technical lemma above, the definability of $\vec{\varphi}$ comes out at Δ^1_5 and $\forall n \in \omega, \varphi_n \in \Game^3(\omega n - \Pi^1_1)$ since the prewellordering of $j_{W^n_3}(\delta^1_3)$ is $\Game^3(\omega n - \Pi^1_1)$. In general Δ^1_{2n+3} scales $\vec{\varphi}$ on Π^1_{2n} sets such that ϕ_n is $\Game^{2n+1}(\omega n - \Pi^1_1)$, since by Martin's argument the prewellordering of the equivalence classes of $j_{W^n_{2n+1}}(\delta^1_{2n+1})$ is $\Game^{2n+1}(\omega n - \Pi^1_1)$ in the codes.

Lemma 3.26. *Let A be a universal Π^1_4 set of reals. Let $f_{x\restriction n} : (T_3)_{x\restriction n} \to \underset{\sim}{\delta}^1_3$ be the canonical ranking function, for every $n \in \omega$. For $x, y \in A$, let $\varphi_n(x) = [f^C_{x\restriction n}]_{W^n_3}$ and let*

$$\varphi_n(x) \leq \varphi_n(y) \leftrightarrow [f^C_{x\restriction n}]_{W^n_3} \leq [f^C_{y\restriction n}]_{W^n_3},$$

where $C \subseteq \underset{\sim}{\delta}^1_3$ be Δ^1_5 in the codes c.u.b set which stabilizes the tree T_3. Then $\vec{\varphi}$ is a Δ^1_5 scale and $\forall n \in \omega, \varphi_n \in \Game^3(\omega n - \Pi^1_1)$

Proof. (Sketch)

This follows by modifying a generalization of an argument of Martin as in [20]. We sketch the argument. Let player I and player II play the game G where I plays reals $\varepsilon, x_\beta, z_0$, where ε codes a c.u.b subset $C \subseteq \underset{\sim}{\delta}^1_3$ and x_β code ordinals less than $\underset{\sim}{\delta}^1_3$, for
$\beta < \omega.(n+1)$. Player II plays out reals y_β, z_1, for $\beta < \omega.(n+1)$, which also code ordinals less than $\underset{\sim}{\delta}^1_3$ using the coding defined above of ordinals. z_0, z_1 will be codes for functions $f : (\underset{\sim}{\delta}^1_3)^n \to \underset{\sim}{\delta}^1_3$ via the "nesting" construction using the Martin tree as above. If a player fails to code an ordinal, then player I wins. Define then

$$\gamma_i = \sup\{\max\{|x_{\omega.i+j}|, |y_{\omega.i+j}|\} : j \in \omega\}$$

Player I wins if and only if

$$\forall^* \vec{\alpha} \in C^n_\varepsilon, f^{C_\varepsilon}_{z_0}(\vec{\alpha}) \leq f^{C_\varepsilon}_{z_1}(\vec{\alpha}).$$

Then the game G is $\Game(\omega.n)-\Pi^1_3 = \Game^3(\omega.n)\Pi^1_1$ and we are done. Therefore the prewellordering of equivalence classes in the ultrapower $j_{W^n_3}(\underset{\sim}{\delta}^1_3)$ is $\Game^3(\omega.n)\Pi^1_1$.

□

We next show that the trees defined above T_{2n} are homogeneous. Let $x \in \mathbb{R}$ such that $x \in p[T_{2n}]$ and let A_n be a sequence of measure one sets with respect to W^n_{2n-1}. Let C_j be clubs of $\underset{\sim}{\delta}^1_{2k-1}$ defining W^j_{2n-1} measure one sets such that $C_j \subseteq A_j$. We let $C = \bigcap C_n$. Then $(J_{2n-1})_x$ is wellfounded since J_{2n-1} projects to the complement of a Σ^1_{2n}. Let $f :<_{BK(J_{2n-1})_x)} \to C$ be an order preserving function from the Brouwer-Kleene order on $(J_{2n-1})_x$ to C such that for every $n \in \omega, f :<_{BK((J_{2n-1})_{x\restriction n}} \to C$ is of the correct type. Let $[f]_{W^i_{2n-1}} = \alpha_i$. Then the sequence $(\alpha_1, ..., \alpha_n)$ is in A_n by the strong partition property on $\underset{\sim}{\delta}^1_{2n-1}$.

We now outline a more general version of the canonical trees T_{2n} which are which can directly be shown to be homogeneous with respect to the measures coded by the trees of uniform cofinality. The construction is outlined in [6] and we generalize it to all trees T_{2n}. The construction also rests on the Martin-Solovay construction.

Let $\mathcal{Q}$ be a type $2n-1$ trees of uniform cofinalities. Define $(s, \vec{\alpha}) \in T_{2n}$ if and only if there is a function $f : dom(\prec^{\mathcal{Q}_s}) \to \delta^1_{2n+1}$ of type Q_s such that $[f] \restriction lh(s) = \vec{\alpha}$.

Letting $(i_1, ..., i_k) \in \omega^{<\omega}$ the k^{th} element of $\omega^{<\omega}$ in an enumeration of $\omega^{<\omega}$ and letting $p_j = \pi_{s\restriction j,(i_1,...,i_j))}$be the permutation associated to $(s \restriction j, (i_1, ..., i_j))$, we set $\alpha_i = [f^{\langle p_1,i_1,...,p_k,i_k}]_{W^k_{2n+1}}$ for every $i < lh(s)$, where $f^{\langle p_1,i_1,...,p_k,i_k\rangle}$ means $f^{\langle p_1,i_1,...,p_k,i_k\rangle}(\alpha_1, ..., \alpha_n) = f(\langle \alpha_1, i_1, ..., \alpha_n, i_n\rangle)$ and

$f(\langle \alpha_1, i_1, ..., \alpha_n, i_n\rangle) = \sup\{f(\vec{s}) : \vec{s} \preceq^{\mathcal{Q}} \langle \alpha_1, i_1, ..., \alpha_n, i_n\rangle\}$. As above, by the strong partition property on $\underset{\sim}{\delta}^1_{2n+1}$, the trees T_{2n} are homogeneous.

3.4. Closure of Π^1_{2n+3} under Existential Ordinal Quantification up to κ^1_{2n+3}

In this section the aim is to show Jackson's theorem which says that the pointclasses Π^1_{2n+3} is closed under existential ordinal quantification up to κ^1_{2n+3}. Again we assume AD throughout this section. In the proof that the pointclass Π^1_{2n+3} is closed under existential quantification up to κ^1_{2n+3} we need a coding of ordinals up to κ^1_{2n+3}. This is done via the *Martin tree* and canonical measures below. We will follow Jackson's proof of the Kechris-Martin theorem in the case Π^1_3 case.

Definition 3.27. A relation $R \subseteq \mathbb{R} \times \mathrm{WO}_{\kappa^1_{2n+3}}$ is invariant in the codes if

$$\forall x, w_1, w_2 (w_1, w_2 \in \mathrm{WO}_{\kappa^1_{2n+3}} \wedge |w_1| = |w_2| \wedge R(x, w_1) \longrightarrow R(x, w_2))$$

We can just then write $R(x, \alpha)$ for $\alpha < \kappa^1_{2n+3}$ instead of

$$\exists w \in \mathrm{WO}_{\kappa^1_{2n+3}} (|w| = \alpha \wedge R(x, w))$$

Theorem 3.28 (Jackson, Kechris, Martin). *Let $R \subseteq \mathbb{R} \times WO_{\kappa^1_{2n+3}}$ be Π^1_{2n+3} and invariant in the codes. Then*

$$P(x) \leftrightarrow \exists w \in WO_{\kappa^1_{2n+3}} R(x, w)$$

is also Π^1_{2n+3}

Proof. We first show that the pointclass Π^1_{2n+3} is closed under quantification up to $\aleph_1$ by the usual Solovay boundedness argument:

Lemma 3.29. *Let $S \subseteq WO$ be Σ^1_{2n+3} in the codes and assume that S is bounded in WO, i.e* $\sup\{|w| : w \in S\} = \alpha_0 < \omega_1$. *Then $\exists w^* \in \Delta^1_{2n+3} \cap WO(|w^*| > \alpha_0)$.*

Proof. Let

$$S(w) \leftrightarrow \exists z B(w,z),$$

where B is Π^1_{2n+2}. Consider the game where I plays the reals w_1, z and II plays w_2. The payoff condition if given by player II wins iff $w_2 \in \mathrm{WO}$ and $(B(w_1,z) \to |w_2| > |w_1|)$. Notice that this is a Σ^1_{2n+2} game for player II and II wins the game, so let τ be a winning strategy for II. By the third periodicity theorem, τ is Δ^1_{2n+3}. But now notice that $\tau(\mathbb{R}) = A \subseteq \mathrm{WO}$ is $\Sigma^1_1(\tau)$, so there is a $\Delta^1_1(\tau)$ real w^* such that $w^* \in \mathrm{WO}$ with

$$|w^*| > \sup\{|w| : w \in A\} \geq \sup\{|w| : w \in S\}.$$

Since $\tau \in \Delta^1_{2n+3}$ then $w^* \in \Delta^1_{2n+3}$.

$\square$

Lemma 3.30. *Let $S \subseteq WO_{2n+1}$ be Σ^1_{2n+3} in the codes and assume that S is bounded in WO_{2n+1}, i.e $\sup\{|w| : w \in S\} = \alpha_0 < \omega_1$. Then $\exists w^* \in \Delta^1_{2n+3} \cap WO_{2n+1}(|w^*| > \alpha_0)$.*

Proof. see [2] $\square$

Just as in [6], as a consequence of Solovay's boundedness argument and Harrington and Kechris results, we have the following lemma which follows from the closure of Π^1_{2n+3} under existential quantification up to κ^1_{2n+1}:

Lemma 3.31. *Let $R \subseteq \mathbb{R} \times WO_{\kappa^1_{2n+1}}$ be Σ^1_{2n+3} and invariant in the codes. Then*

$$P(x) \leftrightarrow \forall^*_{W^1_{2n+1}} \alpha R(x,\alpha)$$

is also Σ^1_{2n+3}

Proof. see [2] and [6], in particular one uses Harringto-Kechris boundedness properties. $\square$

Recall that $<^n$ denotes the ordering on n-tuples of ordinals $(\alpha_1, ..., \alpha_n)$ where $\alpha_1 < ... < \alpha_n$. We can define the ordering $<^n_{2n+1}$ on $(\underset{\sim}{\delta}^1_{2n+1})^n$: $<^n_{2n+1}$ is defined by

$$(\alpha_1, ..., \alpha_n) <^n_{2n+1} (\beta_1, ..., \beta_n) \text{ iff } (\alpha_n, \alpha_1 ..., \alpha_{n-1}) <_{lex} (\beta_n, \beta_1, ..., \beta_{n-1}),$$

where $<_{lex}$ is the lexicographic ordering.

Definition 3.32. W^n_{2n+1} is the measure on $\underset{\sim}{\delta}^1_{2n+1}$ induced by the weak partition relation on $\underset{\sim}{\delta}^1_{2n+1}$, function $f : dom(S^{2^n-1,n}_{2n-1}) \to \underset{\sim}{\delta}^1_{2n+1}$ of the correct type and the measure $S^{2^n-1,n}_{2n-1}$. $S^{1,n}_{2n+1}$ is the measure induced by the strong partition relation on $\underset{\sim}{\delta}^1_{2n+1}$, functions

$g : dom(<^n_{2n+1}) \to \underset{\sim}{\delta}^1_{2n+1}$ of the correct type and the n-fold product of the ω-cofinal normal measure on $\underset{\sim}{\delta}^1_{2n+1}$. For $l \geq 2, S^{l,m}_{2n+1}$ is the measure induced by the strong partition relation on $\underset{\sim}{\delta}^1_{2n+1}$, function $g : \underset{\sim}{\delta}^1_{2n+1} \to \underset{\sim}{\delta}^1_{2n+1}$ of the correct type and the measure μ on $\underset{\sim}{\delta}^1_{2n+1}$. μ is the measure induced by the weak partition relation on $\underset{\sim}{\delta}^1_{2n+1}$, functions

$f : dom(\nu^m) \to \underset{\sim}{\delta}^1_{2n+1}$ of the correct type and the measures ν^m. ν^m is the $(l-1)$st measure in the list $W^m_1, S^m_1, W^m_3, ..., S^{2^n-1,m}_{2n-1}$

Also need level-n complexes. In particular we will use the level-$2n+2$-complexes, but we introduce the definition for every $n \in \omega$.

Definition 3.33 (Level-n pre-descriptions and level-n descriptions). Let W^m_n be a measure and let $K_1, ..., K_k$ be a sequence of measures, where each $K_j = S^{m_j}_{n-2}$ or $K_j = W^{m_j}_{n-2}$. Then a level-n pre-description defined relative to the sequence $K_1, ..., K_k$ is an expression of the form (d) or $(d)^s$, where $d \in \mathcal{D}^m(K_1, ..., K_k)$ is a level-$n-1$ description. Then we denote the set of level-n pre-description defined with respect to the sequence of measures $K_1, ..., K_k$ by $\mathcal{D}'(W^m_n, K_1, ..., K_k)$

(1) (Condition D, wellfoundedness and well-definiteness requirement) We say a level-n pre-description $(d) \in \mathcal{D}'(W^m_n, K_1, ..., K_k)$ satisfies condition D if for almost all $h_1, ..., h_k, (d; \vec{h})$ is the equivalence class of a function $f : (\underset{\sim}{\delta}^1_{n-2})^m \to \underset{\sim}{\delta}^1_{n-2}$ of the correct type. We also say $(d)^s$ satisfies condition D if for almost all $h_1, ..., h_k, (d; \vec{h})$ is a supremum of ordinals represented by f of the correct type.

(2) A level-n description is a level-n pre-description which satisfies condition D. We let $\mathcal{D}(W^m_n, K_1, ..., K_k)$ denoted the set of level-n descriptions.

Definition 3.34. A level-n complex is a sequence of the form

$$\mathcal{C} = \langle \mathcal{S}; x_0, .., x_k; d_0, ..., d_k; K_1, .., K_k \rangle$$

where $\mathcal{S}$ is a level-n tree of uniform cofinalities, $x_i \in \mathbb{R}$ are such that the sections of the higher level Martin tree T_{x_i} are wellfounded, $d_0, ..., d_k$ are extended level-n descriptions with d_i defined relative to the tree of uniform cofinalities $\mathcal{S}$ and the sequences of measures $K_1, .., K_k$, where $K_1, .., K_k$ are canonical measures in the list $W_1^m, S_1^m, W_3^m, ..., S_{n-1}^{n-1,m}$ with $l \leq n-1$.

Recall Jackson's Δ^1_{2n+1} coding of functions $z \longrightarrow F_z \subseteq \underset{\sim}{\delta}^1_{2n+1}$ and the general measures W^n_{2n+1} on $\underset{\sim}{\delta}^1_{2n+1}$. Recall that each z codes countably many z_n, each of which codes reals σ_n, w^1_n, w^2_n and a partial level-n complex. Need the following properties of the coding:

Lemma 3.35. *Consider the relation R_0, R_1, R_2, R_3 defined by:*

$$R_0(z) \leftrightarrow \forall \beta \exists \gamma F_z(\beta, \gamma)$$

$$R_1(z, y) \leftrightarrow y \in WO_{\kappa^1_{2n-1}} \wedge \exists \gamma F_z(|y|, \gamma)$$

$$R_2(z, y) \leftrightarrow y \in WO_{\kappa^1_{2n-1}} \wedge \forall \beta \leq |y| \exists y F_z(\beta, \gamma)$$

$$R_3(z, x, y) \leftrightarrow x, y \in WO_{\kappa^1_{2n-1}} \wedge \forall \beta \leq |x| \exists \gamma \leq |y| F_z(\beta, \gamma)$$

Then R_0 is Π^1_{2n+2} and R_1, R_2 are Π^1_{2n-1}. R_3 is Δ^1_{2n-1} in the codes for x, y, that is there are two relations $C \in \Sigma^1_{2n-1}$ and $D \in \Pi^1_{2n-1}$ such that for all z and $x, y \in WO_{\kappa^1_{2n-1}}$,

$$R_3(z, x, y) \leftrightarrow C(z, x, y) \leftrightarrow D(z, x, y)$$

Proof. For example using the level-n complex $\mathcal{C}$ and the Martin tree and the coding of c.u.b sets using the Martin tree T one computes that:

$$R_1(z, y)$$

$$\leftrightarrow$$

$$y \in \mathrm{WO}_{\kappa^1_{2n-1}} \wedge \exists n [w^1_n, w^2_n \in \mathrm{WO}_{\kappa^1_{2n-1}} \wedge |w^1_n|, |w^2_n| < |y| \wedge \exists \beta_{k-1} < ... < \beta_0 \leq |y| \exists \gamma_{k-1}, ..., \gamma_1 < |y|$$

$$\exists \delta_{k-1}, ..., \delta_1 < |y| (\beta_{k-1} > \max(|w^1_n|, |w^2_n|) \wedge \forall i \beta_i \in C_{\sigma_n} |(T_{x_{k-1}} \restriction \beta_{k-1})(|w^1_n|)| = \gamma_{k-1}$$

$$\wedge |(T_{x_{k-2}} \restriction \beta_{k-2})(\gamma_{k-1})| = \gamma_{k-2} \wedge ... \wedge |(T_{x_0} \restriction \beta_0)(\gamma_1)| = |y| \wedge |(T_{x_{k-1}} \restriction \beta_{k-1})(|w^2_n|)| = \delta_{k-1}$$

$$\wedge |(T_{x_{k-2}} \restriction \delta_{k-2})(\delta_{k-1})| = \delta_{k-2} \wedge ... \wedge |(T_{x_0} \restriction \beta_0)(\delta_1)| = |y| \wedge \forall n' \in \omega (w^1_{n'}, w^2_{n'} \in \mathrm{WO}_{\kappa^1_{2n-1}} \wedge |w^1_{n'}|, |w^2_{n'}| < |y|$$

$$\wedge \exists \beta'_{k'-1} < ... < \beta'_0 \leq |y| \exists \gamma'_{k'-1}, ..., \gamma'_1 < |y|$$

$\exists \delta'_{k'-1}, ..., \delta'_1 < |y| (\beta'_{k'-1} > \max(|w^1_{n'}|, |w^2_{n'}|) \wedge \forall i \beta'_i \in C_{\sigma_{n'}} |(T_{x'_{k'-1}} \upharpoonright \beta'_{k'-1})(|w^1_{n'}|)| = \gamma_{k'-1}$

$\wedge |(T_{x'_{k'-2}} \upharpoonright \beta_{k'-2})(\gamma_{k'-1})| = \gamma_{k'-2} \wedge ... \wedge |(T_{x'_0} \upharpoonright \beta'_0)(\gamma'_1)| = |y| \wedge |(T_{x'_{k'-1}} \upharpoonright \beta_{k'-1})(|w^2_{n'}|)| = \delta'_{k'-1}$

$\wedge |(T_{x'_{k'-2}} \upharpoonright \delta'_{k'-2})(\delta'_{k'-1})| = \delta'_{k'-2} \wedge ... \wedge |(T_{x'_1} \upharpoonright \beta'_1)(\delta'_2)| = \delta'_1)] \to (|(T_{x'_0} \upharpoonright \beta'_0)(\delta'_1)| = |(T_{x_0} \upharpoonright \beta_0)(\delta_1)|)))$

□

F_z is a function will abbreviate $R_0(z)$ and $F_z(|y|)$ will abbreviate $R_1(z, y)$

Lemma 3.36. *The relation*

$$Q(x, y) \leftrightarrow (x \in WO_{\underset{\sim}{\delta}^1_{2n+1}} \wedge F_y \text{ is a function} \wedge |x| = [F_y]_{W^1_{2n+1}})$$

is Δ^1_{2n+3}.

Proof. Let $T \subseteq \omega \times \underset{\sim}{\delta}^1_{2n+3}$ be the Martin tree. For $\sigma \in \mathbb{R}$ we define a basis for c.u.b subsets of $\underset{\sim}{\delta}^1_{2n+3}$. Let $C_\sigma = \{\alpha : \alpha$ is a limit ordinal $, \forall \beta < \alpha, T_\sigma \upharpoonright \beta$ is wellfounded of rank $< \alpha\}$. Since the Martin tree $T \subseteq \omega \times \underset{\sim}{\delta}^1_{2n+3}$ analyzes functions $f : \underset{\sim}{\delta}^1_{2n+3} \to \underset{\sim}{\delta}^1_{2n+3}$, and in particular analyzes the function $\rho : C_\sigma \to C$, defined by $\rho(\alpha) =$ the least $\gamma \in C$ s.t $\gamma > \alpha$, where C is a c.u.b subset of $\underset{\sim}{\delta}^1_{2n+3}$, then for every $C \subseteq \underset{\sim}{\delta}^1_{2n+3}$ c.u.b, there is a $\sigma \in \mathbb{R}$ such that C_σ is a c.u.b subset of C. Now the computation can be finished as follows: we have $Q(x, y) \leftrightarrow \exists \sigma (T_\sigma$ is wellfounded $\wedge \forall w \in \mathrm{WO}_{\kappa^1_{2n+1}}(|w| \in C_\sigma \to \exists z \in \mathrm{WO}_{\kappa^1_{2n+1}}(f_x(|w|) = |z| \wedge F_y(|w|, |z|)))$. But now by Solovay's boundedness argument and Harrington/Kechris (see above), we have that $Q \in \Sigma^1_{2n+3}$. Similarly $Q^c \in \Sigma^1_{2n+3}$.

□

Next we show a presentation theorem for Π^1_{2n+2} subsets of $\mathbb{R}^2$ in terms of wellfounded tree. Let $\mathcal{T}$ be a tree on $\omega \times \underset{\sim}{\delta}^1_{2n+1}$. Let $\preceq_x$ denote the Brouwer-Kleene order on $\mathcal{T}_x$. Recall that $\mathcal{T}_x$ is wellfounded if and only if $\preceq_x$ is a wellorder. Let $\alpha < \underset{\sim}{\delta}^1_{2n+1}$. Then α is represented in the wellfounded part of $\mathcal{T}_x \upharpoonright \beta$ if there is a sequence $s \in \mathcal{T}_x \upharpoonright \beta$ such that $\preceq_x^{\upharpoonright \beta, s} \cong \alpha$, where $\preceq_x^{\upharpoonright \beta, s}$ is the initial segment of the Brouwer-Kleene order on $\mathcal{T}_x \upharpoonright \beta$ determined by s.

Lemma 3.37. *Let* $R \subseteq \mathbb{R}^2$ *be* Π^1_{2n+2}. *Then there is a tree* $\mathcal{T}$ *on* $\omega \times \underset{\sim}{\delta}^1_{2n+1}$ *such that:*

(1) $\mathcal{T}$ *is* Δ^1_{2n+1} *in the codes,*

(2) *For any* $x, y \in \mathbb{R}$,

$(R(x,y) \leftrightarrow \mathcal{T}_{\langle x,y\rangle}$ *is w.f* $\leftrightarrow \forall \alpha < \underset{\sim}{\delta}^1_{2n+1}(\alpha$ *is represented in the w.f.p of* $\mathcal{T}_{\langle x,y\rangle} \restriction \alpha)$,

(3) *The relation* $S(x,y,z) \leftrightarrow (z \in WO_{\kappa^1_{2n+1}} \wedge |z|$ *is represented in the w.f.p of* $\mathcal{T}_{\langle x,y\rangle} \restriction |z|)$ *is* Δ^1_{2n+1} *in the* WO_{2n+1} *codes for* z.

Proof. This is proved just as in [6], except instead of using the Schoenfield tree construction, one uses the Martin-Solovay tree construction to carry out the proof. □

Next need to show the following main lemma which is central for the result. It shows the boundedness result which goes in establishing that the pointclass Π^1_{2n+3} is closed under existential quantification up κ^1_{2n+3}.

Lemma 3.38. *Let* $W \subseteq WO_{\kappa^1_{2n+1}}$ *be* Σ^1_{2n+1}, *invariant in the codes, and code a bounded initial segment of* κ^1_{2n+1}. *Then there is a* Δ^1_{2n+1} *function* $F \subseteq WO_{\kappa^1_{2n-1}} \times WO_{\kappa^1_{2n-1}}$ *which is invariant in the codes, and defines a total function* $F : \underset{\sim}{\delta}^1_{2n-1} \to \underset{\sim}{\delta}^1_{2n-1}$ *such that* $[F]_{W^1_{2n-1}} > |x|$ *for all* $x \in W$.

Proof. Define the following relation W':

$$W'(x) \leftrightarrow \exists x \in \mathrm{WO}_{\kappa^1_{2n+1}}[W(x) \wedge (x \text{ codes a function } F_w : \underset{\sim}{\delta}^1_{2n-1} \to \underset{\sim}{\delta}^1_{2n-1}) \wedge (|x| = [F_w]_{W^1_{2n-1}}).$$

Then by the above lemma, $W' \in \Sigma^1_{2n-1}$. In addition W' is invariant in the codes in the sense that if w, w' code functions $F_w, F_{w'}$ such that $[F_w]_{W^1_{2n-1}} = [F_{w'}]_{W^1_{2n-1}}$ and $W'(w)$ holds then $W'(w')$ holds. Let $W'(w) \leftrightarrow \exists y R(w,y)$ where $R \in \Pi^1_{2n}$. As in above we let $\mathcal{T}$ be a tree on $\omega \times \underset{\sim}{\delta}^1_{2n-1}$, so that $R(w,y) \leftrightarrow \mathcal{T}_{\langle w,y\rangle}$ is wellfounded .

Say a real w is α-good if $F_w(\alpha)$ is defined and say w is $\leq \alpha$-good and α is represented in the wellfounded part of $\mathcal{T}_{\langle w,y\rangle} \restriction \alpha$.

Consider the integer game G where I plays out reals w_1, y and II plays out w_2 and II wins the run iff there exists an $\eta_0 < \underset{\sim}{\delta}^1_{2n-1}$ such that either:

(1) $\forall \eta < \eta_0 (w_1, y), w_2$ are η-good, (w_1, y) is not η_0-good and w_2 is η_0-good, or

(2) $\forall \eta \leq \eta_0 (w_1, y), w_2$ are η-good and $F_{w_1}(\eta_0) < F_{w_2}(\eta_0)$.

Using the above lemmas the game G is Σ^1_{2n} for player II. II easily wins the game by playing any w^* coding a function $F_{w^*} : \underset{\sim}{\delta}^1_{2n-1}$ such that $[F_{w^*}]_{W^1_{2n-1}} > \sup\{|x| : x \in W\}$. Notice here that the coding of functions is the full descriptions coding given by the complex $\mathcal{C}$. Thus, by the third periodicity theorem, II has a Δ^1_{2n+1} winning strategy τ.

Define the function $b : \underset{\sim}{\delta}^1_{2n-1} \to \underset{\sim}{\delta}^1_{2n-1}$ inductively as follows. Let $b(\eta_0)$ be the maximum of $(\sup_{\eta<\eta_0} b(\eta)) + 1$ and

$$\sup\{F_{\tau(w_1,y)}(\eta_0) : \forall\eta < \eta_0[(w_1, y) \text{ is } \eta \text{ -good } \wedge F_{w_1}(\eta) = b(\eta)]\}$$

The following is now shown by induction on η_0:

Lemma 3.39. (1) *$b(\eta_0)$ is well-defined and $b(\eta_0) < \underset{\sim}{\delta}^1_{2n+1}$.*

(2) *If (w_1, y) is $\leq \eta_0$-good and $\forall\eta \leq \eta_0 F_{w_1}(\eta) = b(\eta)$, then $\forall\eta \leq \eta_0 F_{w_2}(\eta) \leq F_{w_1}(\eta)$, where $w_2 = \tau(w_1, y)$.*

Proof. Suppose the claim holds for all $\eta < \eta_0$. If (w_1, y) is η-good for all $\eta < \eta_0$ and $\forall\eta < \eta_0 F_{w_1}(\eta) = b(\eta)$, then by (b) and by induction then $F_{w_2}(\eta_0)$ is defined where $w_2 = \tau(w_1, y)$ since otherwise II would lose the run of the game G. Define the set

$$B_{\eta_0} = \{(w_1, y) : \forall\eta < \eta_0[(w_1, y) \text{ is } \eta \text{ -good } \wedge F_{w_1}(\eta) = b(\eta)]\},$$

then B_{η_0} is $\underset{\sim}{\Delta}^1_{2n-1}$ since it is Δ^1_{2n-1} in any real in the appropriate coding set coding η_0 and $b \restriction \eta_0$ by boundedness. Since the coding $z \to F_z$ is reasonable, i.e it satisfies Martin's condition for proving partition relations, this gives that $b(\eta_0)$ is well-defined. The second item now follows from the definition of $b(\eta_0)$. □

Next need to show that $[b]_{W^1_{2n-1}} > |x| \forall x \in W$. If not, then by the invariance and initial segment properties of W', there is a $w_1 \in W'$ such that $F_{w_1} = b$. Let y be such that $R(w_1, y$ holds and let I play (w_1, y) against τ, producing a real $w_2 = \tau(w_1, y)$. Since $\forall\eta_0 < \underset{\sim}{\delta}^1_{2n-1}(w_1, y)$ is η_0 -good , then by induction using (b) in the lemma above, it is true that $\forall\eta_0 < \underset{\sim}{\delta}^1_{2n-1} F_{w_2}(\eta_0)$ is defined and $F_{w_2}(\eta_0) \leq F_{w_1}(\eta_0)$, a contradiction to II winning the game G.

Finally, a computation just like in [6] shows that the relation

$$F(z_1, z_2) \leftrightarrow z_1, z_2 \in \mathrm{WO}_{\kappa^1_{2n-1}} \wedge b(|z_1|) = |z_2|$$

is Δ^1_{2n+1} and then we can compute that $F \in \Sigma^1_{2n}(\tau)$ so $F \in \Delta^1_{2n+1}$. $\square$

We now show that Π^1_{2n+3} pointclasses are closed under existential quantification up to $\underset{\sim}{\delta}^1_{2n+2}$. This is can regarded as the base case of the generalization of the Kechris-Martin on our way to κ^1_{2n+3}, extending the results of Harrington and Kechris. So let $R(x,\gamma) \subseteq \mathbb{R} \times \underset{\sim}{\delta}^1_{2n+2}$ be Π^1_{2n+3} and invariant in he codes. Define

$$R'(x,\gamma) \leftrightarrow \gamma < \underset{\sim}{\delta}^1_{2n+2} \wedge \exists \gamma_0 (\gamma_0 \leq \gamma \wedge R(x,\gamma_0))$$

Then R' is invariant in the codes and we claim that R' is Π^1_{2n+3}. But notice that we can write R' as follows:

$R'(x,w) \leftrightarrow w = \langle \varepsilon, \varepsilon_1 \rangle \in \mathrm{WO}_{\delta^1_{2n+2}} \wedge \exists \varepsilon^* \in \mathrm{WO}_{\delta^1_{2n+1}} (\forall^* \alpha < \delta^1_{2n+1} |(T_{\varepsilon_1} \restriction \alpha)(|\varepsilon^*|)| \leq |(T_{\varepsilon_1} \restriction \alpha)(|\varepsilon|)| \wedge \forall z \in \mathrm{WO}_{\delta^1_{2n+2}} (|z| = |\langle \varepsilon^*, \varepsilon_1 \rangle| \rightarrow R(x,z)))$. So by Harrington and Kechris and closure of Π^1_{2n+3} under measure quantification, we have that $R' \in \Pi^1_{2n+3}$. So assume w.l.o.g that R is closed upwards in the codes.

Next we use a standard coding of $\Delta^1_{2n+1}(x)$ subsets of $\mathbb{R} \times \mathbb{R}$, uniformly in x. Let $Q \subseteq \mathbb{R}^3$ be Π^1_{2n+1} and such that for every $\Pi^1_{2n+1}(x)$ set $A \subseteq \mathbb{R}^2$ there is a real y, $y \in \Sigma^0_1(x)$ such that $A = Q_x$. Let $Q'_0(x,y,z) \leftrightarrow Q(x_1,y,z)$. Let Q_0, Q_1 in Π^1_{2n+1} reduce Q'_0, Q'_1. Say x codes a Δ^1_{2n+1} set if $\forall y, z (Q_0(x,y,z) \vee Q_1(x,y,z))$, so that x codes the $\Delta^1_{2n+1}(x)$ set $D_x = \{(y,z) : Q_0(x,y,z)\}$.

Now let $P(x) \leftrightarrow \exists w \in \mathrm{WO}_{\kappa^1_{2n+1}} R(x,w)$ where $R \in \Pi^1_{2n+1}$ is invariant and closed upwards in the codes. By the boundedness lemma one can compute that:

$$P(x) \leftrightarrow \exists y \in \Delta^1_{2n+1}(x)((y \text{ codes a } \Delta^1_{2n+1} \text{ relation } D_y \subseteq \mathbb{R}^2) \wedge D_y \subseteq \mathrm{WO}_{\kappa^1_{2n-1}} \times \mathrm{WO}_{\kappa^1_{2n-1}} \wedge$$
$$D_y \text{ is invariant in the codes }) \wedge (D_y \text{ defines a total function from } \underset{\sim}{\delta}^1_{2n-1} \text{ to } \underset{\sim}{\delta}^1_{2n-1}) \wedge \forall w \in$$
$$\mathrm{WO}\kappa^1_{2n+1}(\forall^*_{W^1_{2n-1}} \alpha < \underset{\sim}{\delta}^1_{2n-1}(\alpha, f_w(\alpha)) \in D_y) \rightarrow R(x,w)))$$

Notice that the statement

$$\varphi = D_y \text{ defines a total function from } \underset{\sim}{\delta}^1_{2n+1} \text{ to } \underset{\sim}{\delta}^1_{2n+1}$$

is a Π^1_{2n+3} statement since
$\forall x, z_1, z_2 \in \mathrm{WO}_{\delta^1_{2n+1}}(D_y(x, z_1) \wedge D_y(x, z_2) \to |z_1| = |z_2|) \wedge (\forall x \in \mathrm{WO}_{\delta^1_{2n+1}} \exists z \in \mathrm{WO}_{\delta^1_{2n+1}} (\forall z' \in \mathrm{WO}_{\delta^1_{2n+1}}(|z'| = |z| \to D_y(x, z')))$ is Π^1_{2n+3}. This completes the base case.

Now for the general case, let $\gamma = \aleph_{\underbrace{\omega^{\omega^{\cdots^{\omega}}}}_{\text{m tower}}+1} < \underset{\sim}{\delta}^1_{2n+3}$. Then let

$$P(x) \leftrightarrow \exists w \in \mathrm{WO}_\gamma R(x, w),$$

where R is Π^1_{2n+3} is invariant in the code w. recall that for a code $w \in \mathrm{WO}_\gamma$, we have the corresponding coded function $f_w : (\underset{\sim}{\delta}^1_{2n+1})^n \to \underset{\sim}{\delta}^1_{2n+1}$ defined W^1_{2n+1} almost everywhere and the function represents the ordinal $|w|$. By the main theorem of the theory of descriptions at the level n, there is a function $g : \underset{\sim}{\delta}^1_{2n+1} \to \underset{\sim}{\delta}^1_{2n+1}$ such that $\forall^*_{W^n_{2n+1}}(\alpha_1, ..., \alpha_n) f_w(\vec{\alpha}) < g(\alpha_n)$. can let g be f_y where $y \in \mathrm{WO}_{\underset{\sim}{\delta}^1_{2n+2}}$. Then we have, for $\xi = \aleph_{\underbrace{\omega^{\omega^{\cdots^{\omega}}}}_{\text{m-1 tower}}+1} < \underset{\sim}{\delta}^1_{2n+3}$ that
$P(x) \leftrightarrow \exists y = \langle \varepsilon, \varepsilon_1 \rangle \in \mathrm{WO}_\xi (\forall^*_{W^{n-1}_{2n+1}} \alpha_1, ..., \alpha_{n-1} f_y(\vec{\alpha}) \preceq_{T_{\varepsilon_1}} |\varepsilon| \wedge \forall w \in \mathrm{WO}_\gamma (\forall^*_{W^n_{2n+1}} \alpha_1, ..., \alpha_n f_w(\vec{\alpha}) = |(T_{\varepsilon_1} \restriction \alpha_n)(f_y(\vec{\alpha}))|) \to R(x, w)))$. By induction on the heights of towers of ω appearing in the images of $\underset{\sim}{\delta}^1_{2n+3}$ by ultrapowers of the appropriate measures, this shows the result. So $P \in \Pi^1_{2n+3}$. □

Corollary 3.40. *For every $n \in \omega$, the pointclasses $\underset{\sim}{\Pi}^1_{2n+3}$ are closed under unions of length strictly less $\underset{\sim}{\delta}^1_{2n+3}$. Similarly, the pointclasses $\underset{\sim}{\Sigma}^1_{2n+3}$ are closed under intersections of length strictly less than $\underset{\sim}{\delta}^1_{2n+3}$.*

Proof. We show the corollary for the pointclasses $\underset{\sim}{\Pi}^1_{2n+3}$. Then the result for $\underset{\sim}{\Sigma}^1_{2n+3}$ will be immediate. So let $\{A_\xi\}_{\xi<\gamma}$ for $\gamma < \underset{\sim}{\delta}^1_{2n+3}$ be a sequence of $\underset{\sim}{\Pi}^1_{2n+3}$ sets. Recall by Solovay we have that $\underset{\sim}{\delta}^1_{2n+3} = u_{\underset{\sim}{\delta}^1_{2n+3}}$, then we may assume that $\xi = \kappa^1_{2n+3}$ since $(\kappa^1_{2n+3})^+ = \underset{\sim}{\delta}^1_{2n+3}$. Define $f : \kappa^1_{2n+3} \to \mathcal{P}(\mathbb{R})$ by

$$f(\alpha) = \{x : x \text{ is a } \underset{\sim}{\Pi}^1_{2n+3}\text{-code of } A_\xi\}.$$

By the coding lemma, let $g : \kappa^1_{2n+3} \to \mathcal{P}(\mathbb{R})$ be a nonempty choice subfunction for f, i.e $\forall \xi < \kappa^1_{2n+3}, g(\xi) \subseteq f(\xi)$ and the relation

$$P(y, z) \leftrightarrow y \in \mathrm{WO}_{\kappa^1_{2n+3}} \wedge z \in g(|y|)$$

is $\underset{\sim}{\Sigma}^1_{2n+3}$. Let B_x be the $\underset{\sim}{\Pi}^1_{2n+3}$ set coded by x. Then we have

$$w \in \bigcup_{\xi<\kappa^1_{2n+3}} A_\xi \leftrightarrow \exists y \in \mathrm{WO}_{\kappa^1_{2n+3}} \forall z(P(y,z) \to w \in B_z)$$

and this is $\underset{\sim}{\Pi}^1_{2n+3}$.

$\square$

3.5. Companion Theorems, Generalized Kleene Theorems for Π^1_{2n+3} and Theory of Descriptions

In this section we record theorems which follow from the above structural analysis of the pointclasses Π^1_{2n+2} and Π^1_{2n+3}. The proofs are generalizations of the theory at the level of the pointclass Π^1_3. We first gather all basic notions needed for the theorems of this section, see [32] for a use of these notions in the more general context of *ordinal definability*. We restate for the reader's convenience the notions as defined in [32]. The notion of a companion structure originated in Moschovakis work on elementary induction on abstract structures, see [**?**].

A structure $(M, \in, R_1, ..., R_n)$, where $R_1, ..., R_n$ are relations on M is said to be admissible if nonempty, transitive, closed under pairing and union, and satisfies Δ_0-separation and Δ_0-collection axiom schemas.

Definition 3.41. (The companion structure) For every $n \in \omega$, we define the companion of Π^1_{2n+3} to be a structure $\mathcal{M} = (M, \in, R_1, ..., R_1)$ which satisfies the following:

(1) M is a transitive set and there is some $A \subseteq \mathbb{R}$ such that $A \in M$
(2) $\mathcal{M}$ is admissible
(3) $\mathcal{M}$ is projectible on A: there is a $\Delta^{\mathcal{M}}_1$ partial surjection $A \to M$
(4) $\mathcal{M}$ is resolvable: there is a $\Delta^{\mathcal{M}}_1$-sequence $(M_\alpha : \alpha < \mathrm{ORD}^M)$ such that $M = \cup_\alpha M_\alpha$
(5) Π^1_{2n+3} is the pointclass of all $\Sigma^{\mathcal{M}}_1$ relations.

Moschovakis has shown that companions to the pointclasses Π^1_{2n+3}, for every $n \in \omega$ are unique. The following provides a characterization Π^1_{2n+3} in terms of definability over

T_{2n+2}. The characterization of pointclasses in terms of constructible models has its roots in the following theorem of Spector-Gandy:

THEOREM 3.42 (Spector-Gandy). *A set of reals is* Π^1_1 *if and only if it is* Σ_1 *over* $L_{\omega_1^{CK}}[x]$.

THEOREM 3.43 (Companion theorem for Π^1_{2n+3}). *Assume* $AD^{L(\mathbb{R})}$ *and let* κ *be the least admissible above* κ^1_{2n+3}. *Then a set* $A \subseteq \mathbb{R}$ *is* Π^1_{2n+3} *if and only if* $A(x) \leftrightarrow L_\kappa[T_{2n+2}, x] \vDash \varphi(x)$, *where* $\varphi \in \Sigma_1$.

REMARK 3.44. Notice that every Π^1_{2n+3} set is of the form $L_\kappa[T_{2n+2}, x] \vDash \varphi(x)$, where $\varphi \in \Sigma_1$. This is because T_{2n+2} projects to a universal Π^1_{2n+2} set of reals. The converse holds by the generalization of the Kechris-Martin theorem.

By Moschovakis, notice that the least $\kappa > \kappa^1_{2n+3}$, as in the above, is the same as $\kappa^{L[T_{2n+2}]}$, i.e the closure ordinal of positive elementary induction on $\mathcal{M}$ or the supremum of the hyperelementary in $L[T_{2n+2}]$ prewellorderings of $L[T_{2n+2}]$.

We make the following conjecture. We refer to section 4 for the meaning of the terms involved in the conjecture. The conjecture shares similarities with the mouse set conjecture. Sargsyan informed us that it is possible the conjecture below should follow from the mouse set conjecture

CONJECTURE 3.45 ($AD_\mathbb{R}$). Assume there is no (ω, ω_1)-iterable mouse with a superstrong cardinal. Let $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ be a Π^1_1-like pointclass (possibly closed under real quantifiers). Then a set of reals A is in Γ if and only for $x \in A$, there a mouse $\mathcal{M}$ such that A is Σ_1 over $\mathcal{M}(x)$.

As usual, one can show that under determinacy, the structure $L_\kappa[T_{2n+2}, x]$ has only countably many reals. However we show the result directly by characterizing the set of reals in $L_\kappa[T_{2n+2}, x]$. This set of reals will be Q_{2n+3} and it is countable. The set Q_{2n+1} is defined by

$$Q_{2n+1} = \{x : x \text{ is } \Delta^1_{2n+1} \text{ in a countable ordinal}\}$$

By $\mathcal{Q}$-theory, recall that Q_{2n+1} is a countable set of reals and it is the largest Π^1_{2n+1}-bounded set of reals and largest countable Π^1_{2n+1} set of reals. This means that for every $P(x,y) \in \Pi^1_{2n+1}$, where y can be taken to range over an arbitrary perfect product space $\mathcal{Y}$ in general, the set

$$R(y) \longleftrightarrow \exists x \in Q_{2n+1} P(x,y)$$

is also Π^1_{2n+1} and there are no sets C such that $Q \subsetneq C$ and

$$R(y) \longleftrightarrow \exists x \in C P(x,y)$$

is still Π^1_{2n+1}, for $P \in \Pi^1_{2n+1}$. This trivially implies that Q_{2n+1} is a Π^1_{2n+1} set of reals. A less obvious fact is that Q_{2n+1} is contained in C_{2n+1} the largest thin Σ^1_{2n+1} set of reals. It should also be noted, and we come back to this aspect on the next section, that Q_{2n+3} is the set of reals of $\mathcal{M}^{\#}_{2n+1}$, the unique ω-sound, ω_1-iterable premouse such that $\rho_\omega(\mathcal{M}^{\#}_{2n+1}) = \omega$ with $2n+1$ Woodin cardinals and which is active (this is due to Steel and Woodin, see [24]). Using this result, it can be seen that Q_{2n+3} contains no non-trivial Π^1_{2n+3} singletons and from this one can see that $\mathcal{M}^{\#}_{2n+1}$ is the least non-trivial Π^1_{2n+3} singleton. We refer the reader to [15] for more of these specific sets of reals.

We now prove the following characterization of the set of reals of $L_\kappa[T_{2n+2}]$ using the generalization of the Kechris-Martin theorem above. Recall, as before, for a scale $\vec{\varphi}$ on a set $A \subseteq \mathbb{R}$ we have the tree from the scale defined by

$$((n_0, ..., n_i), (\alpha_0, ..., \alpha_i)) \in T \leftrightarrow \exists x \in A \forall k \leq i (n_k = x(k) \wedge \varphi_k(x) = \alpha_k)$$

We also let $Q_{2n+3}(x)$ be the relativization of Q_{2n+3} to the real parameter x.

Theorem 3.46. *Assume AD and let κ be the least admissible above κ^1_{2n+3}. Let $x \in \mathbb{R}$. Then*

$$Q_{2n+3}(x) = L_\kappa[T_{2n+2}, x] \cap \mathbb{R}.$$

In the next section we will actually show that the models $L[T_{2n+2}]$ are unique, that is they are independent of the choice of universal Π^1_{2n+2} set and of the choice of scale $\vec{\varphi}$ on that universal set. We will show the above theorem in a sequence of lemmas.

PROOF. We start by showing that $Q_{2n+3} \subseteq L_\kappa[T_{2n+2}] \cap \mathbb{R}$. Recall that by $\mathcal{Q}$-theory, assuming $\underset{\sim}{\Delta}^1_{2n+2}$-determinacy, there is a Π^1_{2n+2} set of reals $P \subseteq \mathbb{R} \times \mathbb{R}$ such that if $P' = \{y : P(x,y)\}$, we have that $Q_{2n+3}(y) = \{z : \forall x \in P'(z \text{ is recursive in } x)\}$. In addition Q_{2n+3} is the largest Σ^1_{2n+3}-hull, i.e we can find a Π^1_{2n+2} set of reals P such that $Q_{2n+3} = Hull_{2n+3}(P)$. To see this let $S = \{y : \forall x \in Q_{2n+3}(x \text{ is recursive in } y)\}$. Then S is a Σ^1_{2n+3} set and we have

$$Q_{2n+3} \subseteq \{x : \forall y \in S(x \text{ is recursive in y}) \ \} \subseteq Hull_{2n+3}(S).$$

But then let $P \in \Pi^1_{2n+2}$ be such that $S(y) \leftrightarrow \exists \varepsilon P(y, \varepsilon)$. Then we obtain

$$Q_{2n+3} \subseteq \{x : \forall z \in P(x \text{ is recursive in } z)\} \subseteq Hull_{2n+3}(P)$$

and we're done since Q_{2n+3} is the largest Π^1_{2n+3}-bounded set of reals. In what follows, we may as well assume we have no real parameter, so we let $y = 0$.

Let $z \in Q_{2n+3}$. Let $\rho_{2n+3} = \omega_1^{L(C_{2n+3})}$. Let $\varphi : C_{2n+3} \to \rho_{2n+3}$ be the norm associated to the Δ^1_{2n+3} good wellordering $<$ of C_{2n+3}, by which we mean that for every $x \in C_{2n+3}$, the set $\{y : y \leq x\}$ is countable and there are relations S and T in Σ^1_{2n+3} and Π^1_{2n+3} such that

$$x \in C_{2n+3} \leftrightarrow ((\{(\varepsilon)_n : n \in \omega\} = \{y : y < x\} \leftrightarrow S(\varepsilon, x) \leftrightarrow T(\varepsilon, x)).$$

Then if $\varphi(z) = \alpha$ then we have for all $w \in \text{WO}$ such that $|w| = \alpha$,

$$z(m) = n \leftrightarrow \forall \varepsilon \in Q_{2n+3}(\varphi(\varepsilon) = |w| \to \varepsilon(m) = n).$$

This last clause is equivalent to $\exists u P(m, n, u, w)$, where $P \in \Pi^1_{2n+2}$, as $Q_{2n+3} = Hull_{2n+3}(P)$. Now fix $w_0 \in \text{WO}$ such that $|w_0| = \alpha$ and for each m, n such that $z(m) = n$ let $u_{m,n}$ be the witness to $P(m, n, u, z)$. Since Π^1_{2n+2} sets are κ^1_{2n+3}-Suslin, then one can find a Σ_1 formula Ξ, involving ordinal parameters $< \kappa^1_{2n+3}$ such that

$$z(m) = n \leftrightarrow \Xi(m, n, u_{m,n}, \vec{\alpha}, w_0).$$

Since $L_\kappa[T_{2n+2}]$ is an admissible structure then $z \in L_\kappa[T_{2n+2}]$.

Next we show that $L_\kappa[T_{2n+2}] \cap \mathbb{R} \subseteq Q_{2n+3}$. It is enough to show that $L_\kappa[T_{2n+2}] \cap \mathbb{R}$ is a Π^1_{2n+3} set and then by determinacy and maximality of Q_{2n+3}, we have that $L_\kappa[T_{2n+2}] \cap \mathbb{R}$ is countable and thus $L_\kappa[T_{2n+2}] \cap \mathbb{R} = Q_{2n+3}$

Lemma 3.47. *Let κ be the least admissible ordinal above κ^1_{2n+3}, then $L_\kappa[T_{2n+2}] \cap \mathbb{R}$ is Π^1_{2n+3}.*

Proof. We compute the complexity of the statement $x \in L_\kappa[T_{2n+2}]$, where $x \in \mathbb{R}$. We may assume without loss of generality that $T_{2n+2} \subseteq \kappa^1_{2n+3}$, since we can use a coding function to identify ordinals. We then have

$$x \in L_\kappa[T_{2n+2}] \leftrightarrow \exists \xi < \kappa^1_{2n+3}, \exists \gamma < \xi \text{ s.t } x \in L_\xi[T_{2n+2} \cap \gamma].$$

This is now equivalent to asserting: $\exists \mathcal{M}, E, \alpha, \beta < \kappa^1_{2n+3}$ s.t $\mathcal{M} \subseteq \kappa^1_{2n+3} \wedge E \subseteq \mathcal{M} \times \mathcal{M} \wedge \alpha, \beta \in \mathcal{M} \wedge \mathcal{M} \vDash$ "$V = L[\beta] + \mathrm{ZFC}^-$" $\wedge (\mathcal{M}, E)$ is wellfounded $\wedge (\mathcal{M}, E) \vDash$ "$\alpha \in \mathrm{ORD} \wedge \beta \subseteq \alpha$"$\wedge$ if π is the transitive collapse of $(\mathcal{M}, E)$ then $\pi(\beta) = T_{2n+2} \cap \alpha \wedge x \in \pi$"$\mathcal{M}$. By the coding lemma subsets of κ^1_{2n+3} are $\underset{\sim}{\Delta}^1_{2n+3}$, so we can transform quantification over subsets of κ^1_{2n+3} into quantification over reals (by coding these subsets by Δ^1_{2n+3} sets of reals). By the generalization of Kechris-Martin and bounded quantification, this is Π^1_{2n+3}.

□

□

In terms of representation theorems, we have the following:

Theorem 3.48. *A set $A \subseteq \mathbb{R}$ is Π^1_{2n+3} if and only if it is absolutely inductive over the structure $\langle \kappa^1_{2n+3}, <, R \rangle$. Furthermore $Q_{2n+3} = \boldsymbol{HYP}(\widehat{\kappa^1_{2n+3}})$*

We explain what R is in the above statement. Define an embedding j_ξ as follows for $\xi < \underset{\sim}{\delta}^1_{2n+3}$. Consider the uniform indiscernibles u_ξ for $\xi < \underset{\sim}{\delta}^1_{2n+3}$. Recall by Solovay that $\underset{\sim}{\delta}^1_n = u_{\underset{\sim}{\delta}^1_n}$ for every $n \in \omega$. We consider the shift map:

$s_\xi(u_\gamma) = u_\gamma$ if $\gamma < \xi$ and $s_\xi(u_\gamma) = u_{\gamma+1}$ if $\gamma \geq \xi$. Then we extend s_ξ to an embedding $j_\xi : \kappa^1_{2n+3} \to \kappa^1_{2n+3}$ by letting:

$$j_\xi(f_x(u_{\gamma_1}, ..., u_{\gamma_n})) = f_x(s_\xi(u_{\gamma_1}), ..., s_\xi(u_\gamma)),$$

where f_x is $f_x : \underset{\sim}{\delta}^1_{2n+1} \to \underset{\sim}{\delta}^1_{2n+1}$ coded by x as in the coding above. Now let R be the following relation:

$$R(\xi, \alpha, \beta) \leftrightarrow \xi < \underset{\sim}{\delta}^1_{2n+3} \wedge j_\xi(\alpha) = \beta$$

Then the structure $\widehat{\kappa^1_{2n+3}}$ is defined as $\langle \kappa^1_{2n+3}, <, R\rangle$.

THEOREM 3.49. *A set of reals is* Π^1_{2n+3} *if and only if it is* Π^1_1 *over* $\mathcal{Q}_{2n+3}$ *where* $\mathcal{Q}_{2n+3} = \langle \kappa^1_{2n+3}, <, \{u_\xi : \xi < \kappa^1_{2n+3}\}\rangle$.

Now considering the canonical trees T_{2n} defined earlier using the theory of descriptions we obtain the following:

THEOREM 3.50 (Kleene theorem for Π^1_{2n+3}). *A set of reals is* Π^1_{2n+3} *if and only if it is absolutely inductive over the structure* $\mathcal{Q}^+_{2n+3}$*, where* $\mathcal{Q}^+_{2n+3} = (\mathcal{Q}_{2n+3}, T_{2n})$.

The results of section 4 suggest that the structure of the projective hierarchy can be analyzed using directed system of mice instead of using the lightface theory. The intuition is that the theory of Π^1_3 sets for example, needs to existence of a Woodin cardinal, whereas the theory of Π^1_1 sets only requires to look at L. In general, the theory of Π^1_{2n+3} sets requires looking at mice $\mathcal{M}$ with $2n+1$ Woodin cardinals. We will look at this relationship between mice with Woodin cardinals and the projective hierarchy in section 4. The hope is to obtain clues on how to prove Kechris-Martin like theorems using inner model theory by characterizing the models $L[T_{2n}]$ using inner model theory. Neeman has shown the Kechris-Martin theorem using inner model theoretic tools however his proof is hard to generalize, see [23]. Instead of approximating the $L[T_{2n}]$ in mice with Woodin cardinals, we would like to obtain a direct characterization of the $L[T_{2n}]$ using mice with Woodin cardinals.

CHAPTER 4

THE UNIQUENESS OF THE $L[T_{2N}]$ MODELS AND INNER MODEL THEORETIC ANALYSIS

4.1. Analysis of the Model $L[T_{2n}]$

Next we consider constructibility over the trees T_{2n}. The models $L[T_{2n}]$ are not known to be independent from the universal sets and the scales the tree T_{2n} may depend on. The only result in this direction is due to Hjorth who shows in [3] that $L[T_2]$ is unique. In [1], Becker and Kechris have shown that the following:

THEOREM 4.1. *Assume projective determinacy and let P be a Π^1_{2n+1} complete set of reals P. Let $\vec{\varphi}$ be a regular Π^1_{2n+1} scale on P. Let $T_{2n+1}(P,\vec{\varphi})$ be the tree constructed from the scale $\vec{\varphi}$, then the model $L[T_{2n+1}(P,\vec{\varphi})]$ is independent of the choice of P and $\vec{\varphi}$ on P.*

What Becker and Kechris actually show is a bit more: given the same assumptions as above, every Σ^1_{2n+2} (in the codes provided by the 0^{th} norm of the scale) subset of $\underset{\sim}{\delta}^1_{2n+1}$ is in the model $L[T_{2n+1}]$. We state the theorem below.

THEOREM 4.2 (Becker, Kechris, see [1]). *Let Γ be an ω-parametrized pointclass such that $\Delta^0_2 \subseteq \Gamma$, closed under recursive substitutions and under $\wedge$. Let A be a Γ-complete set of reals, let $\vec{\varphi} = \langle \varphi_n : n \in \omega \rangle$ be a regular $\exists^{\mathbb{R}}\Gamma$ scale on A and consider the 0^{th} norm $\varphi_0 : A \twoheadrightarrow \kappa$. Then for any $X \subseteq \kappa$ which is $\exists^{\mathbb{R}}\Gamma$ in the codes given by φ_0 then $X \in L[T(A,\vec{\varphi})]$*

Since every tree T_{2n+1} coming from a universal Π^1_{2n+1} set P and a regular Π^1_{2n+1} scale $\vec{\varphi}$ on P can be computed to be Σ^1_{2n+2} in the codes by the Coding lemma, this establishes that $L[T_{2n+1}]$ is unique. Steel has shown that the $L[T_{2n+1}] = H_{2n+1}$ are extender models. Recall that H_{2n+1} is the model $L[P_{\vec{\rho},\delta}]$ where $P_{\vec{\rho},\delta}$ is a subset of $\omega \times \underset{\sim}{\delta}^1_{2n+1}$ defined by

$$P_{\vec{\rho},\delta}(n,\alpha) \leftrightarrow \exists x(x \in P_{2n+1} \wedge \rho(x) = \alpha \wedge G(n,x)),$$

where G is a good universal set for $\exists^{\mathbb{R}}\Pi^1_{2n+1} = \Sigma^1_{2n+2}$, $\vec{\rho}$ a $\underset{\sim}{\Pi}^1_{2n+1}$ scale on P. In particular they're constructible models over a specific direct limit of a directed system of mice, see [30].

Here, we aim at generalizing Hjorth proof that $L[T_2]$ is unique. The main difference is that we are not using the theory of sharps as in Hjorth's proof but Jackson's theory of descriptions. We first briefly recall the set up from Becker and Kechris and some previous partial results on the problem of the independence of $L[T_{2n}]$.

Definition 4.3. Let κ^1_{2n+1} be the Suslin cardinal of cofinality ω under AD, i.e $(\kappa^1_{2n+1})^+ = \underset{\sim}{\delta}^1_{2n+1}$

Let P be a complete Π^1_{2n} set of reals and let $\vec{\varphi}$ a regular Δ^1_{2n+1} scale on P. Let $\varphi_n : P \twoheadrightarrow \kappa_n$ and let $\kappa = \sup_n \kappa_n$. Then $\vec{\varphi}$ is *nice* if $\kappa = \kappa^1_{2n+1}$ and the norms φ_n satisfy the following bounded ordinal quantification condition:

If $A(x, y)$ is Σ^1_{2n+1} then the following is also Σ^1_{2n+1}

$$R(n, z, x) \leftrightarrow z \in U \wedge \forall w \in U(\varphi_n(w) \leq \varphi_n(z) \rightarrow A(x, y))$$

Notice that for $n = 1$ this is essentially the Kechris-Martin theorem. For $n > 1$ the existence of nice scales relies on Jackson's generalization of the Kechris-Martin theorem. With the following theorem of Becker and Kechris, the $L[T_{2n}]$ models are independent of the choice of any Π^1_{2n} complete set $A \subseteq \mathbb{R}$ and any nice scale $\vec{\varphi}$:

Theorem 4.4 (Becker and Kechris). *Assume AD. Let A be a complete Π^1_{2n} set of reals and let $\vec{\varphi}$ be a nice Δ^1_{2n+1} scale on A. Then the model $L[T_{A,\vec{\varphi}}]$ is independent of the choice of A and $\vec{\varphi}$*

Let P be a complete Π^1_{2n} complete set of reals and let $\vec{\varphi}$ be a regular Δ^1_{2n+1} scale on P. Let κ_n be such that $\varphi_n : P \twoheadrightarrow \kappa_n$. Let then $\kappa = \sup\{\kappa_n : n \in \omega\}$. Then we have that $\kappa^1_{2n+1} \leq \kappa$. Using the scale $\vec{\varphi}$, one can define the following coding of ordinals less than κ: let

$$P^* = \{(n, x) : n \in \omega \wedge x \in P\},$$

where (n, x) denotes the new real $(n, x(0), x(1), x(2), ...)$. For $(n, x) \in P^*$, define $\varphi^*((n, x)) = \varphi_n(x)$. We will abuse the notation and drop the parenthesis around the real (n, x) when we plug in inside the norm φ^*. For κ some ordinal, we say that $X \subseteq \kappa$ is Γ in the codes provided by (P^*, φ^*) if $\{(n,x) \in P^* : \varphi^*(n,x) \in X\}$ is in the pointclass Γ.

The above theorem then relies on the following result of Becker and Kechris:

THEOREM 4.5 (Becker, Kechris). *Assume AD. Let $X \subseteq \kappa^1_{2n+1}$ and X is Σ^1_{2n+1} in the codes provided by (P^*, φ^*). Then $X \in L[T(P, \vec{\varphi})]$, where P is a complete Π^1_{2n} set of reals and $\vec{\varphi}$ is a Δ^1_{2n+1} regular scale on P.*

To see this, let P be a complete Π^1_{2n} set of reals and let $\vec{\varphi}$ be a regular Δ^1_{2n+1} scale on P. Consider P^* as above and let ψ be the scale defined by $\psi_0(n,x) = \varphi_n(x)$ and $\psi_{k+1}(n,x) = \varphi_k(x)$. Then we have that $X \in L[T(P^*, \vec{\psi})]$. We then need to see that the tree $T(P^*, \vec{\psi}) \in L[T(P, \vec{\varphi})]$. But we can compute membership in $T(P^*, \vec{\psi})$ as follows:

$$(a_0, ..., a_n), (\alpha_0, ..., \alpha_n) \in T(P^*, \vec{\psi}) \leftrightarrow \exists (b_0, ..., b_k), (\beta_0, ..., \beta_k) \in T(P, \vec{\varphi})(a_0 \leq l \wedge n+1 \leq l \wedge a_1 = b_0 \wedge ... \wedge a_n = b_{n-1} \wedge \alpha_0 = \beta_{a_0} \wedge \forall j (k \leq j \leq n \to \alpha_j = \beta_{j-1})).$$

Throughout the proof, we will then use the 0^{th} norm ψ_0 associated to any scale $\vec{\varphi}$ as defined above and we will denote it by $\psi_{0,\vec{\varphi}}$. The goal is to show that the models $L[T_{2n}]$ are independent of the choice of an arbitrary scale not just a nice scale. We will follow Hjorth proof to show that an arbitrary scale can be analyzed in the model $L[T_{2n}]$ by a nice scale.

The problem is to use generalizations of the Kechris-Martin theorem for the appropriate pointclasses in the proof. The Kechris-Martin theorem, and its generalizations, significantly simplify the descriptive set theoretical complexity of certain computations involved in the proof, which allows certain sets to be computed in the models $L[T_{2n}]$. For example we now have that $\forall \kappa < \kappa^1_{2n+1} \undertilde{\Sigma}^1_{2n+1}$ is still $\undertilde{\Sigma}^1_{2n+1}$.

We recall what it means to be a regular scale:

DEFINITION 4.6. Let $\undertilde{\Gamma} \subseteq \mathcal{P}(\mathbb{R})$ be a pointclass and let $A \in \undertilde{\Gamma}$. Then a regular $\undertilde{\Gamma}$-scale is a sequence $\vec{\varphi} = \langle \varphi_n : n \in \omega \rangle$ of onto maps $\varphi_n : A \twoheadrightarrow \kappa_n$, for $\kappa_n \in \mathrm{ORD}$, satisfying the following properties:

(1) Whenever $\{x_i\} \subseteq A$ is a sequence of reals such that $x_i \to x$ and $\varphi_n(x_i) \to \gamma_n$ for every n as $i \to \omega$, then $x \in A$ and we have the *lower semi continuity* property:

$\varphi_n(x) \leq \gamma_n$.

(2) The following norm relations, $\leq^*_{\varphi_n}$ and $<^*_{\varphi_n}$ are in $\underset{\sim}{\Gamma}$, for every n:

$$x \leq^*_{\varphi_n} y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \varphi_n(x) \leq \varphi_n(y)))$$

$$x <^*_{\varphi_n} y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \varphi_n(x) < \varphi_n(y)))$$

Also recall that starting from a regular scale $\vec{\varphi}$, we have the tree T derived from the scale which is defined as follows

$$(s, \vec{\alpha}) \in T_{\vec{\varphi}} \longleftrightarrow \exists x(x \upharpoonright lh(s), \varphi_0(x) = \alpha_0, ..., \varphi_{lh(s)-1}(x) = \alpha_{lh(s)-1})$$

It is then straightforward to show that $A = p[T_{\vec{\varphi}}]$ where $A \subseteq \mathbb{R}$ is the set on which the scale $\vec{\varphi}$ is. For example, if $x \in p[T_{\vec{\varphi}}]$ then use the properties of the scale to obtain $x \in A$. Notice that the tree T is on $\omega \times \kappa$ where $\kappa = \sup\{\kappa_n : n \in \omega\}$ and thus κ has to be a Suslin cardinal of cofinality ω.

Next we recall the definition of our Δ^1_{2n+3} scales, $\vec{\varphi}$ on Π^1_{2n+2} sets which we defined in the previous sections using the appropriate measures and using stability arguments. For $x, y \in A$ and $A \in \Pi^1_{2n+2}$, we let

$$\varphi_n(x) \leq \varphi_n(y) \leftrightarrow [f^C_{x\upharpoonright n}]_{W^n_{2n+3}} \leq [f^C_{y\upharpoonright n}]_{W^n_{2n+3}}.$$

where C is a c.u.b subset of $\underset{\sim}{\delta}^1_{2n+3}$ stabilizing the Martin tree at the level of Π^1_{2n+3} which is Δ^1_{2n+3} in the codes.

Below we state the generalized version of the Kechris-Martin theorem that we need here. Although we assume AD in the statements of the following theorems, it should be noted that their proofs only require local determinacy hypothesis.

THEOREM 4.7. *Assume AD+$V = L(\mathbb{R})$. Let X be a $\Pi^1_{2n+1}(x)$ subset of $\mathbb{R} \times \omega$. Suppose that $\exists \gamma < \kappa^1_{2n+1}$ such that for all $x \in \mathbb{R}$, for all $m \in \omega$, whenever $[f_x]_{W^m_{2n+1}} = \gamma$ then $(x, m) \in X$, for $f : (\underset{\sim}{\delta}^1_{2n-1})^m \to \underset{\sim}{\delta}^1_{2n-1}$. Then there exists a $x_0 \in \Delta^1_{2n+1}(y)$ and an $n_0 \in \omega$ such that for all $x \in \mathbb{R}$ and all $m \in \omega$, whenever $[f_x]_{W^m_{2n+1}} = [f_{x_0}]_{W^{n_0}_{2n+1}}$ then $(x, m) \in X$.*

Theorem 4.8. *Assume AD. Let X be a Σ^1_{2n+1} subset of $\mathbb{R}\times\mathbb{R}\times\omega$. Then the set*

$$\{x\in\mathbb{R}:\forall\gamma<\kappa^1_{2n+1}\exists y\in\mathbb{R}\exists k\in\omega([f_y]_{W^k_{2n+1}}=\gamma\wedge(x,y,k)\in X\}$$

is also Σ^1_{2n+1}.

Definition 4.9. Let Γ be a pointclass such that $\Sigma^0_1\subseteq\Gamma$. Let $z\in\mathbb{R}$. We define the relativization $\Gamma(z)$ of Γ by: $P\subseteq\mathbb{R}$ is in $\Gamma(z)$ if there exists a set $Q\subseteq\mathbb{R}^2$ in Γ such that,

$$P(x)\longleftrightarrow Q(z,x).$$

In particular $\Sigma^0_1(z)$ is the pointclass of semirecursive in z sets.

Definition 4.10. Let φ be a norm on $\mathbb{R}$. We say P is invariant in x if for all $x,x'\in\mathbb{R}$ and for all $y\in\mathbb{R}$,

$$\varphi(x)=\varphi(x')\longrightarrow[P(x,y)\leftrightarrow P(x',y)]$$

Definition 4.11. Let $\vec{\varphi}$ be a regular scale on a set $A\subseteq\mathbb{R}$ such that $\varphi_n:A\to\kappa_n$. We say that a set $X\subseteq\mathbb{R}$ is relatively Π^1_{2n+3} invariant in the codes given by the 0^{th} norm ψ_0 if there exists a set $Y\subseteq\mathbb{R}^2$ in Π^1_{2n+3} such that

$$x\in X\longleftrightarrow\forall x_1,...,x_n\in A\forall k\forall i\leq n(\psi_0(k,x_i)=\alpha_{k,i}\wedge(\langle x_1,...,x_n\rangle,x)\in Y)$$

Similarly a set $X\subseteq\mathbb{R}$ is relatively Σ^1_{2n+3} invariant in the codes given by the 0^{th} norm ψ_0 if there exists a set $Y\subseteq\mathbb{R}^2$ in Σ^1_{2n+3} such that

$$x\in X\longleftrightarrow\forall x_1,...,x_n\in A\forall k\forall i\leq n(\psi_0(k,x_i)=\alpha_{k,i}\wedge(\langle x_1,...,x_n\rangle,x)\in Y)$$

One can of course also let $X\subseteq\mathbb{R}^n$ and $Y\subseteq\mathbb{R}^{n+1}$ in the above definitions.

We have the following result of Solovay, see [9],

Theorem 4.12 (Solovay). *Assume AD. Let $\vec{\varphi}$ be a regular Δ^1_{2n+3} scale on a a Π^1_{2n+2} set $A\subseteq\mathbb{R}$. Fix $x_1,...,x_n\in A$. Let Λ be the pointclass of sets of reals which are relatively Π^1_{2n+3} invariant in the codes given by ψ_0. Then, $PWO(\Lambda)$.*

Recall that a pointclass Γ is ω-parametrized if there exists a $U \subseteq \omega \times \mathbb{R}$ which is universal for Γ subsets of $\mathbb{R}$.

LEMMA 4.13 (Kechris). *Assume AD. Let $\vec{\varphi}$ be a regular Δ^1_{2n+3} scale on a a Π^1_{2n+2} set $A \subseteq \mathbb{R}$. Fix $x_1, ..., x_n \in A$. Let Λ be the pointclass of sets of reals which are relatively Π^1_{2n+3} invariant in the codes given by ψ_0. Then Λ is ω-parametrized.*

Also we will repeatedly use in the proof the fact due to Kechris that, under $\mathrm{Det}(\Gamma)$, every prewellordering in $\exists^{\mathbb{R}}\Gamma$ does not have a perfect set of inequivalent element. (since there is no $\exists^{\mathbb{R}}\Gamma$ wellordering of $\mathbb{R}$ under $\mathrm{Det}(\Gamma)$ and since by a result of Kechris, every set in $\Game\Gamma$ has the property of Baire, see [10]). This only requires local determinacy hypothesis, although we just work under AD.

We will also use the following nice determinacy transfer result due to Kechris and Solovay, see [16]:

THEOREM 4.14 (Kechris, Solovay). *Assume ZF+DC. Let Γ be a pointclass such that $\Delta^0_2 \subseteq \Gamma$ and Γ is a Spector pointclass. Then we have that*

$$\mathit{Det}(\Delta) \longrightarrow \mathit{Det}(\Gamma)$$

PROOF. See [16] $\square$

COROLLARY 4.15. *Assume ZF+DC. Let Γ be a pointclass such that $\Delta^0_2 \subseteq \Gamma$ and Γ is a Spector pointclass. Then we have that*

$$\mathit{Det(HYP)} \longrightarrow \mathit{Det(IND)}$$

COROLLARY 4.16. *Suppose $V \models \mathit{Det}(\Pi^1_{2n})$. Let $\mathcal{M}$ be an inner model of ZF such that $ORD \subseteq \mathcal{M}$ and such that $\mathcal{M} \prec_{\Sigma^1_{2n+1}} V$. Then,*

$$\mathcal{M} \models \mathit{Det}(\Pi^1_{2n})$$

Notice that assuming $\mathrm{Det}(\underset{\sim}{\Delta}^1_{2n})$, $\mathcal{M}$ is an inner model of ZF such that $\mathrm{ORD} \subseteq \mathcal{M}$ and such that $T_{2n+1} \in \mathcal{M}$, where T_{2n+1} is a tree on $\omega \times \underset{\sim}{\delta}^1_{2n+1}$ which projects to a universal set U and which comes from a regular Π^1_{2n+1} scale $\vec{\varphi}$ on U, we have that

$$\mathcal{M} \prec_{\Sigma^1_{2n+1}} V.$$

LEMMA 4.17 (Woodin). *Suppose $V \vDash Det(\Pi^1_{2n})$. Let x be a Cohen generic real over V. Then,*

$$V \prec_{\Sigma^1_{2n+2}} V[x]$$

PROOF. Let T_{2n+2} be the tree coming from the Kechris-Martin scale on $\omega \times \omega \times \kappa^1_{2n+3}$ such that for some Σ^1_{2n+3} set A, $pp[T] = A$ and for some Π^1_{2n+2} set B, $p[T] = B$ and

$$A = \{x : \exists x \in \mathbb{R}((x, y) \in B)\}.$$

Let τ be a term in the forcing language for Cohen forcing. Let $\kappa^1_{2n+3} < \kappa$ be least such that $L_\kappa[T_{2n+2}, \tau]$ is admissible (i.e satisfies KP [1]).

If x is Cohen generic over V, then $L[T_{2n+2}, \tau, x]$ is still admissible. But then by absoluteness of wellfoundedness $V[x] \vDash p[T_{2n+2}] \subseteq B$. Since $L_\kappa[T_{2n+2}, \tau, x]$ is admissible, if $V[x] \vDash \forall y((y, \tau_G(x)) \notin B)$ then for all $z \in B$ such that $(z, \tau_G(x)) \in p[T_{2n+2}]$, the fact that $(T_{2n+2})_z$ is wellfounded will be witnessed in $L_\kappa[T_{2n+2}, \tau, x]$.

But since there are only countably many reals in the model $L_\kappa[T_{2n+2}, \tau, x]$, since $L_\kappa[T_{2n+2}, \tau, x] \cap \mathbb{R} = Q_{2n+3}(x, z)$, which is countable by $\mathcal{Q}$-theory, with τ coded by a real z, we can let x' such that $x' \in V$ and such that x' is Cohen generic over $L_\kappa[T_{2n+2}, \tau]$. Pick x' below a condition p which is such that

$$p \Vdash \text{the tree of attempts to build } y \text{ with } (y, \tau[x]) \in p[T_{2n+2}] \text{ is wellfounded}$$

Then we have that

$$L_\kappa[T_{2n+2}, \tau] \vDash p \Vdash \text{the tree of attempts to build } y \text{ with } (y, \tau[x]) \in p[T_{2n+2}] \text{ is wellfounded}$$

and so

$$V \vDash \text{the tree of attempts to build } y \text{ with } (y, \tau[x]) \in p[T_{2n+2}] \text{ is wellfounded}$$

[1]KP is Kripke-Platek set theory. It is weaker than ZFC, has no power set axiom with separation and collection are limited to $\Sigma_0 (= \Delta_0 = \Pi_0)$ formulae

$\square$

THEOREM 4.18. *Assume AD. Let $y \in \mathbb{R}$ and let ρ be a $\Pi^1_{2n+3}(y)$ norm on some set of reals. Let A be a complete $\Pi^1_{2n+2}(y)$ set of reals and let $\vec{\varphi}$ be a regular $\Delta^1_{2n+3}(y)$ scale. Suppose that for all $B \in \Sigma^1_{2n+3}(y)$, the following set*

$$\{x \in \mathbb{R} : \forall x_1, ..., x_n \in A, \exists y_1, ..., y_n \forall k \forall i \leq n(\psi_0(k, y_i) = \psi_0(k, x_i), (\langle y_1, ..., y_n \rangle, x) \in B)\}$$

is also $\Sigma^1_{2n+3}(y)$.

Then for every $x \in \mathbb{R}$, there exists a sequence $\{x_k\} \subseteq A$ such that for $\psi_0(k, x_i) = \alpha_i$, for every $i \leq n$ and there exists a set $D \subseteq \mathbb{R}$ which is relatively $\Delta^1_{2n+3}(y)$ invariant in the codes given by the 0^{th} norm ψ_0 satisfying the following properties:

(1) $x \in D$,

(2) $D \subseteq \{z \in \mathbb{R} : \rho(z) = \rho(x)\}$.

PROOF. We let $y = 0$ since the case with a real parameter y is exactly the same. We will establish the theorem with a series of claims.

First we show the following claim which follows from the separation property of the pointclass of sets which are relatively Σ^1_{2n+3} invariant in the codes given by the 0^{th}-norm ψ_0.

CLAIM 4.19. *Suppose B is relatively Σ^1_{2n+3} invariant in the codes given by the 0^{th}-norm ψ_0. Suppose that*

$$\forall w, z \in B \text{ we have that } \rho(w) = \rho(z)$$

Then there exists a set B^*, such that $B \subseteq B^*$, B^* is relatively Δ^1_{2n+3} invariant in the codes given by the 0^{th}-norm ψ_0 and

$$\forall w, z \in B^* \text{ we have that } \rho(w) = \rho(z)$$

PROOF. Consider the set

$$C = \{w \in \mathbb{R} : \exists z \in B(\rho(w) \neq \rho(z))\}$$

Then the set C is relatively Σ^1_{2n+3} invariant in the codes given by ψ_0 since B is also in that pointclass. Also $C \cap B = \emptyset$. Recall that, under ZF for a nonselfdual pointclass the prewellordering property of a pointclass implies the separation property of the dual pointclass. So choose a set B^* which is relatively Δ^1_{2n+3} invariant in the codes given by ψ_0 such that $B \subseteq B^*$ and such that $C \cap B^* = \emptyset$.

$\square$

We define the set A_0 as follows:

A_0 is the set of all $\in \mathbb{R}$ such that $\forall x_1, ..., x_n \in A$, $\forall \alpha_{k,i}$, if $\psi_0(k, x_i) = \alpha_{k,i}$, where $i \leq n$, then for every D which are relatively Δ^1_{2n+3} in ψ_0 the codes given by we have either

(1) $x \notin D$, or

(2) $\exists w, z \in D(\rho(w) \neq \rho(z))$

Assume that A_0 is nonempty. Then notice that $A_0 \in \Sigma^1_{2n+3}$, since $\vec{\varphi}$, and hence $\vec{\psi}$ is a Δ^1_{2n+3} scale on A, and since we can obtain, uniformly in the codes give by the 0^{th} norm ψ_0 a code for the set D, say from a universal relatively Π^1_{n+3} invariant in the codes given by ψ_0 set and since this pointclass also has the prewellordering property uniformly in the codes given by ψ_0.

Claim 4.20. *If $A_1 \subseteq A_0$ and $A_1 \neq \emptyset$ is relatively Σ^1_{2n+3} in the codes given by ψ_0, then $\exists w, z \in A_1$ such that $\rho(w) \neq \rho(z)$.*

Proof. Suppose that $\forall w, z \in A_1$, we have that $\rho(w) = \rho(z)$, then let $A_1 \subseteq A_2$ such that A_2 is relatively Δ^1_{2n+3} in the codes given by ψ_0 and $\forall w, z \in A_2$, we have $\rho(w) = \rho(z)$. But now notice that $A_2 \cap A_0 = \emptyset$, by definition of A_0 and then we must have $A_1 = \emptyset$. Contradiction!

$\square$

Now we define the following partial order $\mathbb{P}$:

$$\mathbb{P} = \{B \subseteq \mathbb{R} : B \neq \emptyset, B \subseteq A_0, \exists \{x_i\}_{i \leq n} \subseteq A \psi_0(k, x_i) = \alpha_{k,i} \text{ and } B \text{ is rel. } \Sigma^1_{2n+3} \text{ inv. in } \psi_0\}$$

For $B_0, B_1 \in \mathbb{P}$, we let

$$B_0 \leq_{\mathbb{P}} B_1 \longleftrightarrow B_0 \subseteq B_1.$$

Notice that by assumption $\mathbb{P} \neq \emptyset$.

Let V_λ a large enough rank initial segment of V such that $V_\lambda \models \mathrm{ZFC}^-$. Let $X \prec V_\lambda$ be a countable elementary substructure of V_λ and let M be the transitive collapse of X. Let $\mathbb{Q} = \mathbb{P} \cap M$ and let $\leq_{\mathbb{Q}} = \leq_{\mathbb{P}} \cap \mathbb{Q} \times \mathbb{Q}$.

If G is $\mathbb{Q}$-generic over V, we let x_G be the real introduced by forcing with $\mathbb{Q}$. We also let $\dot{G}$ be a name for the $\mathbb{Q}$ generic G.

Claim 4.21. $(A_0, A_0) \Vdash \rho(x_{\dot{G}_0}) \neq \rho(x_{\dot{G}_1})$.

Proof. Suppose that there are conditions $B_0 \subseteq A_0$ and $B_1 \subseteq A_0$ such that

$$(B_0, B_1) \Vdash \rho(x_{\dot{G}_0}) = \rho(x_{\dot{G}_1})$$

Let

$$B_0^* = B_0 \times B_0 \cap \{(w, z) : \rho(w) \neq \rho(z)\}$$

Then since $\mathbb{Q}$ is countable, we have by elementarity of M that $B_0^* \in M$. Also $B_0^* \neq \emptyset$ by the above claim. Let

$$\mathbb{Q}' = \{B \subseteq \mathbb{R}^2 : B \in M, B \neq \emptyset, B \text{ is rel. } \Sigma^1_{2n+3} \text{ inv. in the codes } \alpha_{k,i} \text{ given by } \psi_0(k, x_i)\}$$

Let (K, G) be $\mathbb{Q}' \times \mathbb{Q}$ generic over V such that $K \subseteq B_0^* \wedge G \subseteq B_1$. Let

$$G^0 = \{B_0 \subseteq \mathbb{R} : \{(w, z) \in B_0^* : z \in B_0\} \in H\}$$

and let

$$G^1 = \{B_1 \subseteq \mathbb{R} : \{(w, z) \in B_1^* : z \in B_0\} \in H\}$$

Notice that (G^0, G) and (G^1, G) are both $\mathbb{P} \times \mathbb{P}$ generic over V [2]. Also since $B_0 \in G^0$, $B_0 \in G^1$ and $B_1 \in G$ we have that

$$\rho(x_{G^0}) = \rho(x_G) \text{ and } \rho(x_{G^1}) = \rho(x_G)$$

Since A is a complete Π^1_{2n} set, any Π^1_{2n} set $X \subseteq \mathbb{R}^2$ which projects to $(\leq^*_\rho)^c$ is such that $X \leq_W A$. Let ε be a real coding the function Wadge reducing X to A. Then this fact continues to hold in $V[H, G]$ with $\varepsilon \in V[H, G]$. In addition, by absoluteness of

[2] One can use a genericity argument to show this

wellfoundedness we have that $V[H,G] \vDash p[T_{2n+2}] \subseteq A$. Let $\bar{\varepsilon} = \pi^{-1}(\varepsilon)$, so that $\bar{\varepsilon}$ codes the Wadge reduction inside M. Since π naturally lifts to generic extensions. By genericity of G^0, G^1, we then have reals x_{G^0} and x_{G^1} such that

$$\rho(x_{G^0}) \neq \rho(x_{G^1}).$$

But then $\rho(x_{G^0}) = \psi(x_G)$ and $\rho(x_{G^1}) = \rho(x_G)$ yet $\rho(x_{G^0}) \neq \rho(x_{G^1})$ in $V[H,G]$. Since $\mathbb{Q} \times \mathbb{P}$ is countable then $V[H,G]$ is equivalent to $V[x]$ for x a Cohen real. Contradiction!

$\square$

To finish the proof of the theorem, we use the following basic lemma from forcing theory:

LEMMA 4.22. *Let z be a Cohen real. Then there is a perfect set F in $V[x]$ such that for every $F' \subseteq F$, $F' = \{z_0, ..., z_j\}$ finite, we have z_j is generic over $V[z_0, ..., z_{j-1}]$.*

PROOF. Consider the following poset:

$$\mathbb{P} = \{(T,k) : T \subseteq 2^{<\omega}, ht(T) = k\}$$

We also let

$$(T,k) \leq (S,l) \longleftrightarrow S \subseteq T \wedge l \leq k.$$

Any $\mathbb{P}$-generic$/V$ adds a perfect tree U. Let G be $\mathbb{P}$-generic over V. Let $z_0, ..., z_j \in U$ be in $V[G]$. Let $(T,k) \in V$ such that for branches $f_0, ..., f_j \in [T]$ we have $f_0 \subseteq z_0, ..., f_j \subseteq z_j$. Notice that there are densely many conditions $(S,l) \leq (T,k)$ for which there exists a conditions (R,m) such that for branches $f_0^0, ..., f_j^0 \in [R]$ we have $f_0 \subseteq f_0^0, ..., f_j^0 \subseteq f_j$ and $N_{f_0^0} \times ... \times N_{f_j^0} \cap X = \emptyset$ for some nowhere dense set X. But since G is generic, it has one such condition. So $(z_0, ..., z_j) \notin X$,and it is a sequence of Cohen reals, so z_j is generic over $V[z_0, ..., z_{j-1}]$.

$\square$

So let z be a Cohen real and let F be a perfect set, in $V[z]$, of $\mathbb{R}$-many Cohen reals x_f, $f \in 2^\omega$ such that if $f \neq g$ there exists G_f and G_g satisfying the following:

(1) (G_f, G_g) are mutually V-generic below (A_0, A_0) for $\mathbb{P} \times \mathbb{P}$

(2) $x_{G_f} = f, x_{G_g} = g$ and $\rho(x_f) \neq \rho(x_g)$.

But F is in V, since the second clause above is Σ^1_{2n+2} and since $V \prec_{\Sigma^1_{2n+2}} V[z]$. But ρ was supposed to be a Π^1_{2n+3} norm. Contradiction!

□

Corollary 4.23. *Assume AD. Let ρ be a $\Pi^1_{2n+3}(y)$ norm on some set of reals. Then $\forall x \in \mathbb{R}, \exists \{\alpha_k\} \subseteq (\kappa^1_{2n+3})^{<\omega}, \exists D$ which is relatively Δ^1_{2n+3} in the codes given by some scale $\vec{\varrho}$ such that*

(1) $x \in D$

(2) $D \subseteq \{z \in \mathbb{R} : \rho(z) = \rho(x)\}$.

Proof. Since we don't have the assumption on the norms of the scale $\vec{\varrho}$ as in the above theorem, we use the Kechris-Martin theorem. Then the set A_0 defined in the above claims is Σ^1_{2n+3} by the Kechris-Martin theorem. If $f_x : (\underset{\sim}{\delta}^1_{2n+1})^k \to \underset{\sim}{\delta}^1_{2n+1}$ and $f_y : (\underset{\sim}{\delta}^1_{2n+1})^j \to \underset{\sim}{\delta}^1_{2n+1}$ are two functions coded by the "nesting" defined for generalized Martin tree, and if $[f_x]_{W^k_{2n+1}} = [f_y]_{W^j_{2n+1}}$ and if $\psi_{0,\varrho_k}(x) = \alpha_{0,k}, \psi_{0,\varrho_j}(x) = \beta_{0,j}$ then the pointclass of relatively Δ^1_{2n+3} invariant in the codes given by ψ_{0,ϱ_k} for some $\{x_i\}_{i \leq k}$ and the pointclass of relatively Δ^1_{2n+3} invariant in the codes given by ψ_{0,ϱ_j} for some $\{x_i\}_{i \leq j}$ are the same. So one can always find new codes in ψ_0 for some sequence of real such that the corollary holds.

□

Corollary 4.24. *Assume AD. Let ρ be a $\Pi^1_{2n+3}(y)$ norm on some set of reals. Then $\forall x \in \mathbb{R}, \exists j \in \omega, \exists \alpha < \kappa^1_{2n+3}$ such that there exists a $D \subseteq \mathbb{R}$ such that*

(1) $\exists y \in \mathbb{R}([f_y]_{W^j_{2n+3}} = \alpha)$

(2) $\forall y \in \mathbb{R}([f_y]_{W^j_{2n+3}} = \alpha \longrightarrow D$ *is invariantly* $\Delta^1_{2n+3}(y))$

(3) $x \in D$

(4) $D \subseteq \{z \in \mathbb{R} : \rho(z) = \rho(x)\}$.

So basically D is Δ^1_{2n+3} in the equivalence classes functions $f : (\underset{\sim}{\delta}^1_{2n+1})^{<\omega} \to \underset{\sim}{\delta}^1_{2n+1}$

4.2. The Main Theorem on the Uniqueness of $L[T_{2n}]$

We assume AD again throughout this section. We start with the following basic lemma from $\mathcal{Q}$-theory:

LEMMA 4.25 ([15]). *Assume AD. Then there exists a non trivial Π^1_{2n+3} singleton, i.e a $y_{2n+3} \in \mathbb{R}$ such that $\{y_{2n+3}\} \in \Pi^1_{2n+3}$ and $y_{2n+3} \notin \Delta^1_{2n+3}$.*

Next, we aim to see that any Π^1_{2n+3} subset of κ^1_{2n+3} is uniformly $\Delta^1_{2n+3}(y_{2n+3})$.

LEMMA 4.26. *Assume AD. Let $A \subseteq \mathbb{R}$ be a universal Π^1_{2n+3} set (recall that Π^1_{2n+3} is ω-parametrized). Suppose that $\{y_{2n+3}\} = A_t$, for some $t \in \omega$, and $y_{2n+3} \notin \Delta^1_{2n+3}$. Suppose ψ is a Π^1_{2n+3} norm on the set A.*

Then $\forall \alpha < \kappa^1_{2n+3}, \forall k, l \in \omega$, we have

$$\forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \to A(k,w)) \leftrightarrow \exists z \in \mathbb{R}, \exists j \in \omega[[f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\boldsymbol{d}(k,j,l),z)) < \psi(t, y_{2n+3}),$$

where $\boldsymbol{d} : (\omega)^3 \to \omega$ is a recursive function such that for all $z \in \mathbb{R}$ and for all $k, j, l \in \omega$,

$$A(\boldsymbol{d}(k,j,l)), z)) \leftrightarrow \forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = [f_z]_{W^j_{2n+1}} \to A(k,w))$$

PROOF. Notice that our hypothesis on $\mathbf{d}$ immediately gives that

$$\exists z \in \mathbb{R}, \exists j \in \omega[[f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\mathbf{d}(k,j,l),z)) < \psi(t, y_{2n+3}) \longrightarrow \forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \longrightarrow A(k,w))$$

Suppose the conclusion of the lemma fails. Then there must be $l \in \omega$ and $\alpha < \kappa^1_{2n+3}$ such that for all $z \in \mathbb{R}$, $\forall j \in \omega$, whenever we have that $[f_z]_{W^j_{2n+1}} = \alpha$ then

$$A(\mathbf{d}(k,j,l),z)) \wedge \psi(t, y_{2n+3}) \leq \psi((\mathbf{d}(k,j,l),z))$$

But now this implies that

$$\{y_{2n+3}\} \in \Delta^1_{2n+3}(z),$$

by assumption. This then gives that

$$y_{2n+3} \in \Delta^1_{2n+3}(z)$$

and

$$\forall z \in \mathbb{R}, \forall j \in \omega([f_z]_{W^j_{2n+1}} = \alpha \longrightarrow \exists y \in \Delta^1_{2n+3}(z)(A(t,y)))$$

By notice that by restricted quantification, we have that

$$B(z) \longleftrightarrow \exists y \in \Delta^1_{2n+3}(z)(A(t,y))$$

is also Π^1_{2n+3} and by Kechris-Martin we have

$$\exists x \in \Delta^1_{2n+3} \text{ such that } \exists y \in \Delta^1_{2n+3}(x)(A(t,y))$$

and hence

$$\exists y \in \Delta^1_{2n+3}(A(t,y))$$

Contradiction!

$\square$

LEMMA 4.27. *Assume AD. Let A be a universal Π^1_{2n+3} set of reals and let $\boldsymbol{d}$ be as above. Let $M \prec_{\Sigma^1_{2n+3}} V$ be a transitive inner model of ZF+DC such that $ORD \subseteq M$. Then $\exists y \in M \cap \mathbb{R}, \exists t \in \omega$ such that $A(t,y)$ and for all $\alpha < \kappa^1_{2n+3}$, for all $k, l \in \omega$, we have that*

$$\forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \to A(k,w)) \leftrightarrow \exists z \in \mathbb{R}, \exists j \in \omega[f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\boldsymbol{d}(k,j,l),z)) < \psi(t, y_{2n+3})$$

PROOF. By assumption, M satisfies Π^1_{2n+2}-determinacy. So

$$M \vDash \forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \to A(k,w)) \leftrightarrow \exists z \in \mathbb{R}, \exists j \in \omega[f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\mathbf{d}(k,j,l),z)) < \psi(t, y_{2n+3})$$

Also by assumption and since $M \vDash$ "$A(k,w)$ holds" then we have that $A(k,w)$ really holds. So have that

$$\exists z \in \mathbb{R}, \exists j \in \omega[f_z]_{W^j_{2n+1}} = \alpha \wedge \psi((\mathbf{d}(k,j,l),z)) < \psi(t, y_{2n+3})$$

implies that

$$\forall w \in \mathbb{R}([f_w]_{W^l_{2n+1}} = \alpha \to A(k,w))$$

Now suppose that there is an $l \in \omega, \exists \alpha < \kappa^1_{2n+3}$ such that $\forall z \in \mathbb{R} \forall j \in \omega$ whenever $[f_z]_{W^j_{2n+1}} = \alpha$ then we have that $\psi(t, y_{2n+3}) \leq \psi((\mathbf{d}(k,j,l),z))$. Since this is a $\Pi^1_{2n+3}(y_{2n+3})$ statement about α, by Kechris-Martin $\exists x \in \Delta^1_{2n+3}(y_{2n+3})$ and $t \in \omega$ such that $[f_x]_{W^t_{2n+1}} = \alpha$. But then

x is definable in M thus $x \in M$. Since $M \vDash \psi(\mathbf{d}(k,t,j),x) < \psi(k,w)$ by assumption. But we have $M \prec_{\Sigma^1_{2n+3}} V$. Contradiction!

$\square$

Finally in the next last two lemmas we use the fact that every Π^1_{2n+3} subset of κ^1_{2n+3} is uniformly $\Delta^1_{2n+3}(y_{2n+3})$ to compute any Δ^1_{2n+3} scale $\vec{\varrho}$ in a nice scale $\vec{\varphi}$.

Lemma 4.28. *Assume AD. Let P and Q be two universal $\Pi^1_{2n+2}(y_{2n+3})$ sets of reals. Let $\vec{\varphi}$ be a $\Delta^1_{2n+3}(y_{2n+3})$ scale on P and $\vec{\rho}$ a $\Delta^1_{2n+3}(y_{2n+3})$ scale on Q. Consider the trees from the scales $T_{2n+2}(P,\vec{\varphi})$ and $T_{2n+2}(Q,\vec{\rho})$. Suppose that for every $B \in \Sigma^1_{2n+3}(y_{2n+3})$, the following set*

$$\{x \in \mathbb{R} : \forall x_1,...,x_n \in P_0, \exists y_1,...,y_n(\psi_{0,\vec{\varphi}}(k,y_i) = \psi_{0,\vec{\varphi}}(k,x_i), \forall k \leq n, (\langle y_1,...,y_n\rangle, x) \in B)\}$$

is also $\Sigma^1_{2n+3}(y_{2n+3})$. Then $T_{2n+2}(Q,\vec{\rho}) \in L[T(\vec{\varphi}), y_{2n+3}]$.

Proof. Since we're assuming AD, all relevant pointclass are ω-parametrized, in particular, the pointclass of sets which are relatively Σ^1_{2n+3} invariantly in the codes is ω-parametrized uniformly in the codes given by $\psi_{0,\vec{\varphi}}$. So we can find a set $U \subseteq \omega\times\mathbb{R}\times\mathbb{R}$ which is $\Pi^1_{2n+3}(y_{2n+3})$ and such that

(1) $\forall x_1,...,x_n, \forall w_1,...,w_n \in P, \forall k \in \omega, \forall l \forall i \leq n$
$(\psi_{0,\vec{\varphi}}(l,x_i) = \psi_{0,\vec{\varphi}}(l,w_i) \longrightarrow \{x \in \mathbb{R} : (x,\langle x_i\rangle,k) \in U\} = \{x \in \mathbb{R} : (x,\langle w_i\rangle,k) \in U\}$

(2) $\forall x_1,...,x_n \in P$ whenever $\psi_{0,\vec{\varphi}}(l,x_i) = \kappa_{l,i}$ and W is relatively Π^1_{2n+3} invariant in the codes $\kappa_{0,0},...,\kappa_{l,i}$, then $\exists k \in \omega$ such that $W = \{x \in \mathbb{R} : (x,\langle x_i\rangle,k) \in U\}$

Let $\vec{\kappa}$ denote the sequence of ordinals $\kappa_{0,0},...,\kappa_{l,i}$. Now let $U_{\vec{\kappa},k}$ denote projection of U onto the first coordinate, i.e the set

$$\{x \in \mathbb{R} : (x,\langle x_i\rangle,k) \in U\}.$$

Next consider the set

$$\mathcal{U}_n = \{(\vec{\kappa},k) : U_{\vec{\kappa},k} \text{ is rel } \Delta^1_{2n+3} \text{ inv. }, U_{\vec{\kappa},k} \neq \emptyset, \forall x,y \in U_{\vec{\kappa},k}(\psi_{0,\vec{\rho}}(l,x_0) = \psi_{0,\vec{\rho}}(l,y_0), \forall l \leq n)\}$$

This is basically the set of codes of sections of relatively Δ^1_{2n+3} in the codes sets of reals but we just require that they're invariant in the norm being analyzed by the Kechris-Martin norm. Also we have that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$. For any $(\vec{\kappa}, k)$ and $(\vec{\gamma}, j)$, we define $(\vec{\kappa}, k) \leq_n (\vec{\gamma}, j)$ if and only if for every $x \in U_{\vec{\kappa},k}$ and for every $y \in U_{\vec{\gamma},j}$, $\psi_{0,\vec{\rho}}(n, x) \leq \psi_{0,\vec{\rho}}(n, y)$. But by Becker and Kechris, we have that $(\mathcal{U}_n, \leq_n)$ is in $L[T(\vec{\varphi}), y_{2n+3}]$ since the prewellordering $\leq_n$ is $\Sigma^1_{2n+3}(y_{2n+3})$ in the codes and since that sets $\mathcal{U}_n$ are also $\Sigma^1_{2n+3}(y_{2n+3})$ in the codes. By 5.18, we can also find a code $(\vec{\kappa}, k) \in \mathcal{U}_n$ for every $n \in \omega$, for every $x \in Q$, $x \in U_{\vec{\kappa},k}$, since these are exactly the codes of relatively Δ^1_{2n+3} in the codes sets of reals. Next for each $n \in \omega$, let $\varrho_n : \mathcal{U}_n \to \zeta_n$ be the norm associated to the prewellordering $\leq_n$ defined above:

$$\text{for any codes } (\vec{\kappa}, k) \text{ and } (\vec{\gamma}, j) \text{ in } \mathcal{U}_n, \varrho_n((\vec{\kappa}, k)) < \varrho_n((\vec{\gamma}, j)) \text{ iff } (\vec{\kappa}, k) <_k (\vec{\gamma}, j)$$

Notice that for every $n \in \omega$, $\zeta_n < \kappa^1_{2n+3}$. By Becker and Kechris, the sequence of norms $\vec{\varrho}$ is in $L[T(\vec{\varphi}), y_{2n+3}]$. Since $T(\vec{\rho})$ is the set

$$\{\vec{\alpha} \in \mathrm{ORD}^{<\omega} : \exists n \in \omega, lh(\vec{\alpha}) = n, \exists (\vec{\kappa}, k) \in \mathcal{U}_n \text{ such that } \forall l \leq n, \varrho_n((\vec{\kappa}, k)) = u(n)\}$$

then $T(\vec{\rho}) \in L[T(\vec{\varphi}), y_{2n+3}]$ and we are done.

□

We finally conclude with the last lemma which finishes the proof that the models $L[T_{2n+2}]$ are unique.

Lemma 4.29. *Assume AD. Let P and Q be two universal Π^1_{2n+2} set of reals. Let $\vec{\varphi}$ be a Δ^1_{2n+3} scale on P and $\vec{\rho}$ be a Δ^1_{2n+3} scale on Q. Consider the trees from the scales $T(\vec{\varphi}) = T_{2n+2}(P, \vec{\varphi})$ and $T(\vec{\rho}) = T_{2n+2}(Q, \vec{\rho})$ as usual. Then $L[T(\vec{\varphi})] = L[T(\vec{\rho})]$*

Proof. By the previous lemma, we just have to show that $T(\vec{\rho}) \in L[T(\vec{\varphi})]$. By lemma 4.28, we only need to see that if $y \in \mathbb{R}$ is such that for $L[T(\vec{\varphi})] \prec_{\Sigma^1_{2n+3}} V$, $y \in L[T(\vec{\varphi})] \cap \mathbb{R}$ and satisfies the conclusion of lemma 4.27, then for all sets B which are $\Sigma^1_{2n+3}(y)$, then

$$\{x \in \mathbb{R} : \forall x_1, ..., x_n(x_i \in P \to \exists y_1, ..., y_n(\psi_{0,\vec{\varphi}}(k, y_i) = \psi_{0,\vec{\varphi}}(k, x_i), \forall i \leq n, \forall k, (\langle y_1, ..., y_n\rangle, x) \in B)\}$$

is also $\Sigma^1_{2n+3}(y)$. By the proof we give in the next section of the fact that $L[T_{2n+2}] = L[\mathcal{M}^{\#}_{2n+1,\infty}]$, y can be considered to be y^0_{2n+3}, the least non-trivial Π^1_{2n+3} singleton.

Next we define a Π^1_{2n+3} norm Φ for which the above lemma applies, by setting $\Phi(x) = \Phi(y)$ if and only if either

(1) $x = \langle x_i\rangle, y = \langle y_i\rangle, \forall i \leq n$, for some $n \in \omega$, and $\forall i \leq n, x_i \in P \wedge y_i \in P \wedge \psi_{0,\vec{\varphi}}(k, x_i) = \psi_{0,\vec{\varphi}}(k, y_i)$, or

(2) $x \neq \langle x_i\rangle$ and either for every $i \leq n$, $x_i \notin P$ or there exists an $i \in \omega$ such that $x_i \notin P$ and $y \neq \langle y_i\rangle$ and either $i \leq n$, $y_i \notin P$ or there exists an $i \in \omega$ such that $y_i \notin P$

Next we fix a set $U \subseteq \mathbb{R} \times \mathbb{R} \times \omega$ such that

(1) For all $j, l \in \omega$ for all $w, z \in \mathbb{R}$ and for all $t \in \omega$

$$[f_x]_{W^l_{2n+1}} = [f_y]_{W^j_{2n+1}} \to \{z : A(z, l^\frown x, t)\} = \{z : A(z, j^\frown y, t)\},$$

(2) $U \in \Pi^1_{2n+3}$,

(3) For every $\alpha < \kappa^1_{2n+3}$, whenever $W = \{z : \forall x([f_x]_{W^l_{2n+1}} = \alpha \to V(z, x)\}$ where $V \in \Pi^1_{2n+3}$, then there is $t \in \omega$, $y \in \mathbb{R}$ and $j \in \omega$ such that $W = \{z : U(z, j^\frown y, t)\}$.

For $t \in \omega$, $\alpha < \kappa^1_{2n+3}$ and $[f_x]_{W^l_{2n+1}} = \alpha$ we consider as in lemma 4.27, the projection of U onto the first coordinate:

$$U_{\alpha,t} = \{z \in \mathbb{R} : U(z, l^\frown x, t)\}.$$

By lemma 4.27, the assumption on y^0_{2n+3} implies that for $B \in \Pi^1_{2n+3}$, we have that

$$\{(x, l) : \forall (y, j) \in \mathbb{R} \times \omega([f_x]_{W^l_{2n+1}} = [f_y]_{W^j_{2n+1}} \to B(y, j))\}$$

is $\Sigma^1_{2n+3}(y^0_{2n+3})$. We now fix a set $B \in \Sigma^1_{2n+3}(y^0_{2n+3})$.

Let X be the set of all $z \in \mathbb{R}$ such that for all $\alpha < \kappa^1_{2n+3}$ and for all t_1:

(1) Either for all $t_2 \in \omega, U_{\alpha,t_1} \neq_\Phi U_{\alpha,t_2}$, or

(2) there are $x, y \in U_{\alpha,t_2}$ which are not Φ-equivalent, or

(3) $U_{\alpha,t_2} = \emptyset$, or

(4) There exists an $x \in U_{\alpha,t_2}$ such that $x = \langle x_i \rangle, \forall i < \omega, x_i \in P \wedge \exists y = \langle y_i \rangle$ such that $\Phi(x) = \Phi(y)$ and $B(y, z)$, or

(5) There is an $x \in U_{\alpha,t_2}$ such that either $x \neq \langle x_i \rangle$ for all x_i or $x = \langle x_i \rangle$ and for some $i \in \omega$, $x_i \notin P$.

Claim 4.30. *X is $\Sigma^1_{2n+3}(y^0_{2n+3})$*

Proof. We check that the clauses (1) through (5) are at most $\Sigma^1_{2n+3}(y^0_{2n+3})$. Clause (1) is $\Sigma^1_{2n+3}(y^0_{2n+3})$ since the pointclass Π^1_{2n+3} has the prewellordering property. Taking the existential quantifier in clause (2) outside the conjunction of clauses (1) and (2), shows that $(1) \vee (2)$ is also $\Sigma^1_{2n+3}(y^0_{2n+3})$. The same holds for $(1) \vee (3), (1) \vee (4)$ and $(1) \vee (5)$. By the generalization of the Kechris-Martin theorem, X is now $\Sigma^1_{2n+3}(y^0_{2n+3})$.

□

This last claim now finishes the proof of the lemma:

Claim 4.31. *We have that*

$$X = \{z \in \mathbb{R} : \forall x_1, ..., x_n \in P_0, \exists y_1, ..., y_n(\psi_{0,\vec{\varphi}}(k, y_i) = \psi_{0,\vec{\varphi}}(k, x_i), \forall k \forall i \leq n, (\langle y_1, ..., y_n \rangle, x) \in B\}$$

Proof. Let $x_1, ..., x_n \in P_0$ and let $\psi_{0,\vec{\varphi}}(k, x_i) = \alpha_{k,i}$ for all $k \in \omega$ and $i \leq n$ then by corollary 4.24, there exists $\alpha < \kappa^1_{2n+1}$ and $t_2 \in \omega$ such that

(1) U_{α,t_2} is Δ^1_{2n+3} in any code w which codes a function $f : (\underset{\sim}{\delta}^1_{2n+1})^{<\omega} \to \underset{\sim}{\delta}^1_{2n+1}$ via the "nesting" of the Martin tree and which equivalence class gives α and

(2) $x = \langle x_i \rangle \in U_{\alpha,t_2}$ and

(3) For every $y \in U_{\alpha,t_2}$, we have $y = \langle y_i \rangle$ with $\psi_{0,\vec{\varphi}}(k, y_i) = \alpha_{k,i}$, and so we have $\Phi(y) = \Phi(x)$.

Hence if the defining condition of the set

$$\{z \in \mathbb{R} : \forall x_1, ..., x_n \in P_0, \exists y_1, ..., y_n(\psi_{0,\vec{\varphi}}(k, y_i) = \psi_{0,\vec{\varphi}}(k, x_i)), \forall k \leq n, (\langle y_1, ..., y_n \rangle, x) \in B\}$$

fails, then U_{α,t_2} witnesses that $z \in \mathbb{R} \notin X$. Conversely, if $z \notin X$ then clause (4) above must fail and thus

$$z \notin \{z \in \mathbb{R} : \forall x_1, ..., x_n \in P_0, \exists y_1, ..., y_n(\psi_{0,\vec{\varphi}}(k, y_i) = \psi_{0,\vec{\varphi}}(k, x_i), \forall k \leq n, (\langle y_1, ..., y_n\rangle, x) \in B\}.$$

$\square$

$\square$

This completes the proof of the main theorem. In the next two section, we show that the models $L[T_{2n}]$ are constructible models over direct limits associated to directed systems of mice and that $L_\kappa[T_{2n}]$, where κ is the least admissible above κ^1_{2n+1} is a mouse. This provides a counterpart to Steel's result which says that the $H_\Gamma = L[T_\Gamma] = L[\mathcal{M}_\infty]$, where $\mathcal{M}_\infty$ is the HOD limit of all Γ correct and Γ-properly small iterates $\mathcal{M}_{2n}$, are extender models for Γ a Π^1_1-like pointclass, see [30]. In the special case where $\Gamma = \Pi^1_3$ then $H_{\Pi^1_3} = L[T_3] = \mathcal{M}^+_\infty|\kappa$, where κ is the least strong to the bottom Woodin cardinal $\delta_{0,\infty}$ and $\mathcal{M}^+_\infty$ is the HOD limit of all iterates of $\mathcal{M}_2$, the minimal proper class inner model containing two Woodin cardinals. It turns out that $\kappa = \underset{\sim}{\delta}^1_3$ and

$$L[T_3] \models \underset{\sim}{\delta}^1_3 \text{ is the least } < \delta_{0,\infty}\text{strong cardinal in HOD.}$$

These results hold at all Π classes which are scaled. At the level of Π classes where we do not have the scale property the situation is a bit different as we show below. We will define all the notions below before showing the results.

4.3. $L[T_{2n}]$ and Direct Limit Associated to Mice

In this section the goal is to show that $L[T_{2n+2}] = L[\mathcal{M}^{\#}_{2n+1,\infty}]$. It should be true that directed system of mice provide a complete structural analysis of $L(\mathbb{R})$ and we try to illustrate this point of view in this section, We'll use ideas of Sargsyan and Steel to show the main theorem below. We are grateful to Sargsyan for showing us and explaining to us the proof below.

The following theorem is a central theorem in descriptive inner model theory. It jumpstarted the analysis of HOD's of models of determinacy.

THEOREM 4.32 (Steel [28])). *$AD^{L(\mathbb{R}}$ implies that $HOD^{L(\mathbb{R})}$ is a core model below Θ. In $L(\mathbb{R})$ every regular cardinal below Θ is measurable.*

The following very useful theorem is due to Woodin. It characterizes the Suslin cardinals of cofinality ω of $L(\mathbb{R})$ in HOD:

THEOREM 4.33 (Woodin). *Assume $V = L(\mathbb{R}) = AD$. For every $n \in \omega, \kappa^1_{2n+3}$ is the least cardinal δ of HOD such that*

$$\mathcal{M}_{2n}(HOD|\delta) \models \text{“}\delta \textit{ is a Woodin cardinal”}$$

In general, Woodin has characterized all the Suslin cardinals of $L(\mathbb{R})$ as exactly the cardinal cutpoints of $\mathrm{HOD}^{L(\mathbb{R})}$.

THEOREM 4.34 (Main Theorem). *[A., Sargsyan]*

Assume $AD^{L(\mathbb{R})}$. Then the $L[T_{2n+2}]$ are the models $L[\mathcal{M}^{\#}_{2n+1,\infty}]$.

We need to record all the notions involved in the computation. Given a set of reals A, $\Game A$ is defined as follows:

$$x \in \Game A \leftrightarrow \exists n_0 \forall n_1 \exists n_2 \forall n_3 ...(x, \{(i, n_i) : i \in \omega\}) \in A$$

Notice that this is the same as saying :

$$\Game A = \{x : \text{I has a winning strategy in } G_{A_x}\}$$

Let $\mathcal{M}$ be a premouse. For $\alpha < o(\mathcal{M})$, we let $\mathcal{M}||\alpha$ be $\mathcal{M}$ cutoff at α and the last predicate indexed at α is kept. $\mathcal{M}|\alpha$ is $\mathcal{M}||\alpha$ without its last predicate. We say that α is a cutpoint if there are no extenders on the extender sequence of $\mathcal{M}$ such that $\alpha \in (cp(E), lh(E)]$. We say α is a strong cutpoint is there are np extender on the extender sequence of $\mathcal{M}$ such that $\alpha \in [cp(E), lh(E)]$.

If $\mathcal{M}$ is an n-sound premouse then a (n, θ)-iteration strategy for $\mathcal{M}$ is a winning strategy for player II in the iteration game $G_n(\mathcal{M}, \theta)$ and a n-normal iteration tree on $\mathcal{M}$ is a play of the iteration game in which II has not yet lost, i.e all the models are wellfounded. Let for η be a limit ordinal. If b is a branch of an iteration tree $\mathcal{T}$ such that b drop only finitely often then $\mathcal{M}_b^{\mathcal{T}}$ is the direct limit along the branch b. We also let $\delta(\mathcal{T}) = \sup_{\alpha<\eta} lh(E_\alpha$. We let $\mathcal{M}(\mathcal{T}) = \cup_{\alpha<\eta}\mathcal{M}_\alpha \restriction lh(E_\alpha)$. If $\alpha \leq_T \beta$ and $(\alpha,\beta]_T \cap D = \emptyset$ then the iteration embedding exists, i.e we have

$$i_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$$

DEFINITION 4.35. Let $\mathcal{T}$ be an n-normal iteration tree of limit length on an n-sound premouse $\mathcal{M}$ and let b be a cofinal branch of $\mathcal{T}$. Then $\mathcal{Q}(b,\mathcal{T})$ is the shortest initial segment $\mathcal{Q}$ of $\mathcal{M}_b^{\mathcal{T}}$, if one exists, such that $\mathcal{Q}$ projects strictly across $\delta(\mathcal{T})$ or defines a function witnessing $\delta(\mathcal{T})$ if not Woodin via extenders on the sequence of $\mathcal{M}(\mathcal{T})$.

Next we need the Dodd-Jensen property which is implicit, especially in reference to showing below that we have scale instead of just semi-scale. The property says that iteration maps are minimal. The main use of the Dodd-Jensen property is in showing that HOD limits exist.

DEFINITION 4.36. Suppose $\mathcal{M}$ is a mouse and Σ is a (ω_1, ω_1+1)-iteration strategy for $\mathcal{M}$. Σ has the Dodd-Jensen property of whenever $\mathcal{N}$ is an iterate of $\mathcal{M}$ via Σ and $\pi : \mathcal{M} \to \mathcal{S} \trianglelefteq \mathcal{N}$ is a fine-structural embedding then

(1) The iteration fro $\mathcal{M}$ to $\mathcal{N}$ doesn't drop,
(2) $\mathcal{S} = \mathcal{N}$ and,
(3) if $i : \mathcal{M} \to \mathcal{N}$ is the iteration embedding given by Σ then for every α, $i(\alpha) \leq \pi(\alpha)$.

DEFINITION 4.37. (C_Γ) For a a countable transitive set we let

$$C_\Gamma(a) = \{b \subseteq a : b \in OD(a)\} = \mathcal{P}(a) \cap Lp^\Gamma(a)$$

where $Lp^\Gamma(a)$ is the union of all a premice projecting to a having an ω_1 iteration strategy in Γ.

let Γ_n be such that $C_{\Gamma_n}(x) = \mathbb{R}^{\mathcal{M}_n(x)}$. So we'll let Γ_ω be $(\Sigma_1^2)^{L(\mathbb{R})}$.

DEFINITION 4.38. Let Γ_n be as above. $\mathcal{N}$ is called Γ_n-suitable if there is a δ such that $\mathcal{N} = Lp^{\Gamma_n}(\mathcal{N} \mid \delta)$ and

(1) $\mathcal{N} \vDash \delta$ is Woodin

(2) For every $\eta < \delta$,

 (a) If η is a cutpoint of $\mathcal{N}$ then $Lp^{\Gamma_n}(\mathcal{N} \mid \eta) \trianglelefteq \mathcal{N}$

 (b) $Lp^{\Gamma_n}(\mathcal{N} \mid \eta) \vDash \eta$ is not Woodin, and

 (c) If η is a strong cutpoint of $\mathcal{N}$, then $Lp^{\Gamma_n}(\mathcal{N} \mid \eta) = \mathcal{N} \mid (\eta^+)^{\mathcal{N}}$

We write $\delta^{\mathcal{N}}$ for the unique such δ.

Given an iteration tree $\mathcal{T}$ on a suitable mouse $\mathcal{N}$, $\mathcal{T}$ is correctly guided if for every limit $\alpha < \mathrm{lh}(\mathcal{T})$, if b if the branch of $\mathcal{T} \restriction \alpha$ chosen by $\mathcal{T}$ and $Q(b, \mathcal{T} \restriction \alpha)$ exists then $Q(b, \mathcal{T} \restriction \alpha) \trianglelefteq Lp(\mathcal{N}(\mathcal{T} \restriction \alpha)$. $\mathcal{T}$ is said to be short if either $\mathcal{T}$ has a last model or there is a wellfounded branch b such that $\mathcal{T}^\frown\{\mathcal{N}_b^{\mathcal{T}}\}$ is correctly guided. $\mathcal{T}$ is maximal if $\mathcal{T}$ is not short. Notice that maximal trees can't be normally continued since every initial segment of a normal tree is short.

Definition 4.39. Let $\mathcal{N}$ be suitable. then $\mathcal{N}$ is short tree iterable iff whenever $\mathcal{T}$ is a short tree on $\mathcal{N}$ then:

(1) If $\mathcal{T}$ has a last model then it can be freely extended by one more ultrapower, that is every putative normal tree $\mathcal{U}$ extending $\mathcal{T}$ and having length $\mathrm{lh}(\mathcal{T}) + 1$ has a wellfounded last model, and

(2) If $\mathcal{T}$ has limit length and $\mathcal{T}$ is short , then $\mathcal{T}$ has a cofinal wellfounded branch.

Definition 4.40. Let $k < \omega$ and let $\mathcal{N}$ be suitable. We say $(\langle \mathcal{T}_i : i < k \rangle, \langle \mathcal{N}_i : i \leq k \rangle)$ is a finite full stack on $\mathcal{N}$ if

(1) $\mathcal{N}_0 = \mathcal{N}$,

(2) $\forall i < k, \mathcal{N}_{i+1}$ is a pseudo normal iterate of $\mathcal{N}_i$ as witnessed by $\mathcal{T}_i$.

As usual for a suitable mouse $\mathcal{N}$ we let

$$\gamma_s^{\mathcal{N}} = \sup(Hull^{\mathcal{N}}(s^-) \cap \delta^{\mathcal{N}}),$$

$$\mathrm{Th}_s^{\mathcal{N}} = \{(\varphi, t) : t \in (\delta^{\mathcal{N}} \cup s^-)^{<\omega} \wedge L[\mathcal{N} | \max(s)] \vDash \varphi(t)\},$$

and

$$H_s^{\mathcal{N}} = Hull^{\mathcal{N}}(\gamma_s^{\mathcal{N}} \cup \delta^{\mathcal{N}})$$

We say $\mathcal{N}$ is n-iterable if whenever $\mathcal{T}$ is a normal tree on $\mathcal{N}$ there is a correct branch b of $\mathcal{T}$ such that $i_b(s_n) = s_n$, where s_n is the sequence of the first n uniform indiscernibles, then $i_b \restriction H_{s_n}^{\mathcal{N}}$ is independent of the branch b. We let $i^n_{\mathcal{N},\mathcal{Q}}$ be the iteration embedding which fixes the s_n and call it the n-iterability embedding.

Next we recall the notion of Π^1_n iterability for mice with n Woodin cardinals. This notion is a strengthening of the notion of Π^1_2 iterability and the basic theory can be found in [24]. Π^1_n iterability will be sufficient for comparison of mice with the appropriate number of Woodin cardinals which can be embedded in the background. However the definition of Π^1_n iterability is asymetrical in the case where n is even or odd, reflecting the periodicity phenomenon from descriptive set theory. The definition is slightly easier in the case n is odd. Fortunately, we only need the definition in the case n is odd (Notice that this is the same as Π_n-iterability, where n is even, following Steel's notation, since $\Pi_n^{HC} = \Pi^1_{n+1}$)

Definition 4.41. A premouse $\mathcal{M}$ is n-small if and only if whenever κ is the critical point of an extender of the extender sequence of $\mathcal{M}$ then $J_\kappa^{\mathcal{M}} \not\models$ there are n Woodin cardinals .

Now let $\mathbb{C}$ be the sequence of models $\langle \mathcal{N}_\xi : \xi < \Omega \rangle$ built using a full background extender construction as in [30]. Suppose there is a ξ which is least such that $\mathcal{N}_\xi$ is not n-small. Then $\mathcal{N}_\xi$ has a top extender witnessed the existence of n Woodin cardinals so $\mathcal{N}_\xi$ is active. We then define $\mathcal{M}_n^\# = \mathcal{C}_\omega(\mathcal{N}_\xi)$. Then $\mathcal{M}_n$ is defined by iterating the top extender of $\mathcal{M}_n^\#$ (i.e the top extender) out of the ordinals and letting $\mathcal{M}_n = \mathcal{M}_b^{\mathcal{T}}$. Both $\mathcal{M}_n^\#$ and $\mathcal{M}_n$ are ω-sound and $\mathcal{M}_n$ and all its levels are n-small. We also have that $\rho_\omega(\mathcal{M}_n^\#) = \omega$ so that $\mathcal{M}_n^\#$ is a real.

Let $\mathcal{M}$ be a countable premouse. We define a weak iteration game as in [24], $\mathcal{G}(\mathcal{M}, n)$. The game $\mathcal{G}(\mathcal{M}, n)$ has n rounds. At the first round, we consider $\mathcal{M}$. At round k, the game starts with $\mathcal{M}_k$ and it is played as follows. Player I plays an ω-maximal, countable, putative iteration tree $\mathcal{T}$ on $\mathcal{M}_k$. Player II either accept the tree $\mathcal{T}$ or plays a maximal wellfounded branch b of $\mathcal{T}$ such that $b \in \Delta^1_{2n+2}(\mathcal{T}, \mathcal{M}_k)$. Player II cannot accept the tree $\mathcal{T}$ is $\mathcal{T}$ has a last illfounded model because then he just loses $\mathcal{G}(\mathcal{M}, n)$. Then $\mathcal{M}_{\kappa+1} = \mathcal{M}^{\mathcal{T}}_b$ is the last model of $\mathcal{T}$. The players then go to round $k+1$. The first one to break the rules loses and if no one breaks the rules then player II wins.

Definition 4.42. We say that $\mathcal{M}$ is Π^1_{2n+2} iterable if player II has a winning strategy in the game $\mathcal{G}(\mathcal{M}, n)$.

Using the Spector-gandy theorem, it is then immediate that the set

$$\{\mathcal{M} : \mathcal{M} \text{ is } \Pi^1_{2n+2} \text{ iterable}\}$$

is a Π^1_{2n+2} set. Steel then shows in [24] that Π^1_{2n+1} iterability is sufficient for comparison of mice with $2n+1$ Woodin cardinals which are realizable into the background. We will assume this now until the end of the paper. The reader can consult [24] for a full proof of this fact. We now state and prove the main theorem of this section.

Theorem 4.43 (A., Sargsyan). *Assume $AD^{L(\mathbb{R})}$. Let T_{2n+2} be the canonical tree which projects to a universal Π^1_{2n+2} set. Then*

$$L[T_{2n+2}] = L[\mathcal{M}^{\#}_{2n+1,\infty}]$$

Proof. Define Steel's tree S_{2n+2} for Π^1_{2n+2}. This will be a tree on $\omega \times \omega \times \omega \times \kappa^1_{2n+3}$. Let $\mathcal{L}$ be the language of premice and let $\mathcal{L}^* = \mathcal{L} \cup \{\dot{a}_i : i < \omega\}$ where the a_i are constants. Let $\langle \varphi_n : n < \omega \rangle$ be a recursive enumeration of the sentence of $\mathcal{L}^*$. We say $x \in \mathbb{R}$ codes a premouse if

$$T_x = \{\phi_n : x(n) = 0\}$$

is a complete Henkinized theory of a premouse. If x codes a premouse, we let

$$\mathcal{R}_x = \{\dot{a}^x_i : i < \omega\}$$

be the premouse whose theory is T_x. Define G^- to be the set of triples such that:

(1) y codes a C_{2n+2} guided tree $\mathcal{T}_y$ on $\mathcal{M}^{\#}_{2n+1}$

(2) z codes a premouse $\mathcal{R}_z$ such that $\mathcal{M}(\mathcal{T}_y) \trianglelefteq \mathcal{R}_z \trianglelefteq L[\mathcal{M}(\mathcal{T}_y)]$ and $\mathcal{R}_z \vDash \mathrm{ZFC}^- +$"$\delta(\mathcal{T}_y)$ is the largest cardinal"

(3) w codes a branch b of $\mathcal{T}_y$ such that $\mathcal{R}_z \trianglelefteq \mathcal{M}_b$

The set G^- is a Δ^1_{2n+2} set. We let

$$G = \{(y, z, w) \in G^- : \text{ either } \mathcal{R}_z \vDash \delta(\mathcal{T}_y) \text{ is not Woodin or } \mathcal{M}(\mathcal{T}_y)^+ \trianglelefteq \mathcal{R}_z\},$$

where $\mathcal{M}(\mathcal{T}_y)^+ = C_{2n+2}(\mathcal{M}(\mathcal{T}_y))$ is the unique suitable premouse extending $\mathcal{M}(\mathcal{T}_y)$ such that $\delta(\mathcal{T}_y)$ is its largest Woodin cardinal. So in G we basically have two cases: the case where $\mathcal{T}_y$ is a short tree and the case where $\mathcal{T}_y$ is a maximal tree. Then the set G is a $\Pi^1_{2n+2}(x)$ set of reals where x codes $\mathcal{M}^\#_{2n+1}$. Define a scale on G as follows. Fix a Σ^1_{2n+2} scale $\vec{\varphi}$ on G^-. Extend $\mathcal{L}^*$ to $\mathcal{L}^{**}$ by introducing new constant symbols $\{\dot{\delta}\} \cup \{\dot{\tau}_i : i < \omega\}$. The intended meaning of the symbols is that if z codes a premouse $\mathcal{R}_z$ which is suitable then we interpret $\dot{\delta}_z$ as the Woodin cardinal of $\mathcal{R}_z$ and $\dot{\tau}_i^z$ as the theories $T_i^{\mathcal{R}_z}$, where i means we only look at the first i indiscernibles. Let $\mathcal{R}^+$ be the $\mathcal{L}^{**}$ structure obtained from $\mathcal{R}_z$. Let $\langle \theta_i : i < \omega \rangle$ be a recursive enumeration of the Σ_0 sentences of $\mathcal{L}^{**}$. Then let

$$T_z^+ = \{\theta_i : \mathcal{R}_z^+ \vDash \theta_i\}$$

Now let

$$\phi_i^0(y, z, w) = 0 \text{ if } \theta_i \in T_z^+ \text{ and } \phi_i^0(y, z, w) = 1 \text{ otherwise.}$$

If $\theta_n = \exists v < \dot{\delta}\psi(v)$ and $\theta_n \in T_z^+$, then we let

$$\phi_n^1(y, z, w) = \text{ least } k \text{ such that } \psi(\dot{a}_k) \in T_z^+$$

and otherwise we let $\phi_n^1(y, z, w) = 0$. Also if $(\dot{a}_k < \gamma_k^{\mathcal{R}_z}) \in T_z^+$ then let

$$\phi_{n,k}^2(y, z, w) = i_{\mathcal{R}_z,\infty}(\dot{a}_n^z)$$

so basically we code the embedding into the norms. Notice, just as in Steel, that the first-order theory of $\mathcal{R}^+$ is coded into the norms. The norms also code the elementary embedding $\pi_{\mathcal{R},\infty} \restriction \delta(\mathcal{T}_z)$. Now we code the whole thing as follows: let

$$\phi_{n,m}(y, z, w) = \langle \psi_n(y, z, w), \phi_n^0(y, z, w), \phi_n^1(y, z, w), \phi_{n,m}^2(y, z, w) \rangle$$

Using arguments from Steel one can show that this is a scale [3], see [25]. We actually go ahead and show the following claim:

Claim 4.44. *$\vec{\phi}_{n,m}$ is a scale on G.*

Proof. The lower semi-continuity property follows from the Dodd-Jensen property. We refer to Steel [25] for the details. Next we verify the convergence property. So let $(y_n, z_n, w_n) \to (y, z, w)$ with respect to $\vec{\phi_{n,m}}$. We then must see that $(y, z, w) \in G$. Since ψ_n is a scale, then $(y, z, w) \in G^-$. This then implies that $\mathcal{T}_z$ is C_{2n+2}-guided and that we have $\mathcal{R}_z \trianglelefteq \mathcal{M}(\mathcal{T}_z)^+$. Since $(y_n, z_n, w_n) \to (y, z, w)$ with respect to $\vec{\psi^0}$ then we can define $T^+_{z_n} \to T^+$, and T^+ is exists and codes the first-order theory of some unique $\mathcal{P}^+$. Since (y_n, z_n, w_n) converges to (y, z, w) with respect to $\vec{\phi^1}$, then $\mathcal{R}_z = \mathcal{P}$. Next we justify that $\mathcal{P}$ is wellfounded and suitable. For this we use the fact that $\vec{\phi^2}$ is a scale. Let

$$\gamma_n = \sup(\{\xi < \dot{\delta}^{\mathcal{P}^+} : (\xi \text{ is definable over } \mathcal{P} \text{ from } \dot{\tau}_n^{\mathcal{P}^+}\})$$

and let

$$\gamma = \sup_{n<\infty} \gamma_n.$$

Since $\gamma \leq \dot{\delta}^{\mathcal{P}^+} = \delta(\mathcal{T}_y)$ then γ is in the wellfounded part of $\mathcal{P}^+$. Let $\mathcal{P}_1 = \mathcal{H}_1^{\mathcal{P}}(\gamma \cup \{\dot{\tau}_n^{\mathcal{P}^+}\})$ be a Σ_1 Skolem hull which is collapsed on its wellfounded part. Let $\sigma : \mathcal{P}_1 \to \mathcal{P}$ be the canonical embedding Then we must have $crit(\sigma) = \gamma$ by elementarity, so that $\sigma \restriction \gamma = id$. Let $\pi_n : \mathcal{P}_{z_n} \to \mathcal{M}_{2n+1,\infty}$ and define $\pi : \mathcal{P}|\gamma \to \mathcal{M}_{2n+1,\infty}$ by $\pi(\dot{a}_j^z) =$ eventual value of $\pi_n(\dot{a}_j^{z_n})$ as $n \to \infty$. Notice that this eventual value must exist since if $\dot{a}_j^z < \gamma$, then there is $\varphi \in T_z^+$ such that $(\dot{a}_j^z < \gamma) \leftrightarrow \varphi$ and $\varphi \in T^+_{z_n}$ for all sufficiently large n. So there exists a $k < \infty$ such that $\dot{a}_j^{z_n} < \gamma_k^{\mathcal{P}_{z_n}}$. We now extend $\pi : \mathcal{P}|\gamma \to \mathcal{M}_{2n+1,\infty}$ to $\pi : \mathcal{P}_1 \to \mathcal{M}_{2n+1,\infty}$. Notice that this extension need not be an iteration embedding. We also let $\pi(\tau_n^{\dot{\mathcal{P}}^+}) = \tau_n^{\dot{\infty}}$.

Let $c \in \mathcal{P}_1$. Then there exists a $k < \infty$ and a Σ_0 formula φ of the language of premice, and parameters $\dot{a}_{i_0}^z, ..., \dot{a}_{i_n}^z < \gamma_k$ such that

$$c = \text{ the unique } v \text{ s.t } \mathcal{P}|\gamma \models \varphi[v, \dot{a}_{i_0}^z, ..., \dot{a}_{i_n}^z, \tau_n^{\dot{\mathcal{P}}^+}]$$

[3]The key is to show that we have fullness and to use the Dodd-Jensen property

We can do this since $\vec{\phi}^0$ is a scale and since the $T^+_{z_n}$ converge to T^+_z. Then we set

$$\pi(c) = \text{ the unique } v \text{ s.t } \mathcal{M}_{2n+1,\infty}|\gamma^\infty_n \vDash \varphi[v, \pi(\dot{a}^z_{i_0}), ..., \pi(\dot{a}^z_{i_n}), \dot{\tau}^\infty_n]$$

As usual the map $\pi : \mathcal{P}_1 \to \mathcal{M}_{2n+1,\infty}$ is Σ_1 elementary and welldefined. Now, since by a result of Woodin there exists suitable mice and by [25] we can apply the condensation lemma, then $\gamma = \delta(\mathcal{T}_y)$ as T_y is C_{2n+2} guided. So $\mathcal{P}_1 = \mathcal{P}$ and $\sigma = id$. The other alternative is that $\mathcal{P} \vDash \delta(\mathcal{T}_y)$ is not Woodin because the truth of this statement is kept by all theories $T^+_{z_n}$ then we have that either $\mathcal{R}_z = \mathcal{M}(\mathcal{T}_y)$ or $\mathcal{R}_z \vDash \delta(\mathcal{T}_y)$ is not Woodin so that $G(y, z, w)$ holds.

$\square$

As in Steel, one can show that the norms of the above scale are all in $\mathcal{M}^\#_{2n+1,\infty}$. In different work with Sargsyan and Woodin, we show that one can actually obtain parameter-free scales using a similar set up. The norms of the above scale ϕ_i can be computed to be in for every i in $\Game^{2n+1}\omega(i+1) - \Pi^1_1$ where we use only the first i indiscernibles, since the theories in i indiscernibles have same complexity $\Game^{2n+1}\omega(i+1) - \Pi^1_1$,i.e the types of the first i indiscernibles are exactly $\Game^{2n+1}\omega(i+1) - \Pi^1_1$. Thus each ϕ_n is $\Delta^1_{2n+1}(x)$. Let S_{2n+2} be the tree from this scale. By the proof of the uniqueness of the $L[T_{2n+2}]$ models we have that $L[T_{2n+2}] = L[S_{2n+2}]$. We'll be done if can show that $L[\mathcal{M}^\#_{2n+1,\infty}] = L[S_{2n+2}]$.

First because $\mathcal{M}_{2n+1,\infty}$ is $\Sigma^1_{2n+3}(\mathcal{M}^\#_{2n+1})$, then we have that $\mathcal{M}_{2n+1,\infty} \in L[S_{2n+2}] = L[T_{2n+2}]$, since by $\mathcal{Q}$-theory, $\mathcal{M}^\#_{2n+1} \in L[T_{2n+2}]$. Letting $i = i_{\mathcal{M}_{2n+1,\infty}} \restriction \delta^{\mathcal{M}_{2n+1}}$ then $i \in L[S_{2n+2}]$ because the iteration embedding i is also $\Sigma^1_{2n+3}(\mathcal{M}^\#_{2n+1})$. Thus we have $\mathcal{M}_{2n+1,\infty}, i \in L[S_{2n+2}]$. Hence $\mathcal{M}^\#_{2n+1,\infty} \in L[S_{2n+2}]$.

We next show that we have that $L[S_{2n+2}] \subseteq L[\mathcal{M}^\#_{2n+1,\infty}]$. Following an idea of Steel (as in [31] or [29] for instance), we build the direct limit tree S. It will be the case that $S \in L[\mathcal{M}^\#_{2n+1,\infty}]$ and that Steel's tree S_{2n+2} (and also T_{2n+2}, whichever way we decide to define it) belongs to $L[S]$ by the uniqueness of the $L[T_{2n+2}]$ models. We then define S to be the tree on $\omega \times \omega \times \omega \times \mathcal{M}_{2n+1,\infty}$ of all attempts to build $(x, \pi) \in (\mathbb{R}^3 \times \mathcal{M}^\omega_{2n+1,\infty})$ such that

(1) x codes the complete theory with parameters of a structure $\mathcal{P}_x$ for the language of premice with universe $\omega \setminus \{0\}$,

(2) $\pi(0)$ is a successor cardinal Woodin cutpoint of $\mathcal{P}_x$, and,

(3) $\pi \restriction (\omega \setminus \{0\})$ is an elementary embedding from $\mathcal{P}_x$ into $\mathcal{M}_{2n+1,\infty}|\pi(0)$.

Notice that $S_{2n+2} \subseteq S$. It then follows that $S_{2n+2} \in L[S]$, by Hjorth and since $S \in L[\mathcal{M}^{\#}_{2n+1,\infty}]$, we are done.

$\square$

We record the following which now follows from the generalization of the Kechris-Martin theorem, the uniqueness of the $L[T_{2n}]$ models and the above characterization of the $L[T_{2n}]$ in terms of HOD limits of directed systems of mice.

THEOREM 4.45 (Inner model characterization of Π^1_{2n+3}). *Assume $AD^{L(\mathbb{R})}$ and let κ be the least admissible above $\kappa^1_{2n+3} = \delta_{0,\infty}$. Then a set $A \subseteq \mathbb{R}$ is Π^1_{2n+3} if and only if*

$$A(x) \leftrightarrow L_\kappa[\mathcal{M}^{\#}_{2n+1,\infty}, x] \vDash \varphi(x),$$

where $\varphi \in \Sigma_1$.

4.4. $L[T_{2n}]$, CH and GCH: A Proof of a Conjecture of Woodin

In this section we give a positive solution to the following problem posed by Woodin:

CONJECTURE 4.46 (Woodin). $L[T_{2n+2}]$ satisfies the GCH for every $n \in \omega$.

The problem of showing that HOD $\vDash$ GCH is a central problem in inner model theory. A solution to this problem would increase our understanding of HOD. Recall that the models $L[T_{2n}]$ are analogs of HOD which lie somewhere between first order logic and second order logic, that is they are the equivalents of HOD at lower levels of definability. Therefore our task here is to show that the GCH holds for the HOD up to $\underset{\sim}{\delta}^2_1$. In previous work, Steel has shown that assuming AD and Γ-mouse capturing holds, $L[T_\Gamma]$ is an extender model and satisfies the GCH, where Γ is a scaled inductive like pointclass. Howver recall that in our case Γ is now a non scaled pointclass (i.e Π^1_{2n} in the case of the projective hierarchy). We would like to thank Sargsyan and Woodin for introducing us to the above conjecture and for discussions on the problem.

We first recall some background of $\mathcal{Q}$-theory. Recall that Q_{2n+3} is a subset of C_{2n+3}, where C_{2n+3} is the largest thin Π^1_{2n+3} set of reals. Also there is a Δ^1_{2n+3}-good wellorder on C_{2n+3} of length $\aleph_1$.

As a warm up and context, we reproduce the proofs of the following two theorem of [15]. Both proofs here are just as in [15]. The proof below should be compared to the proof of the same fact but using inner model theoretic methods, see [24].

THEOREM 4.47 (Martin). *There is a real w such that if $w \in L[T_{2n+1}, x]$ then $\mathbb{R} \cap HOD^{L[T_{2n+1},x]} = Q_{2n+3}$.*

PROOF. Let $x_1 \in Q_{2n+3}$ and let $\varphi : C_{2n+3} \to \rho_{2n+3}$ be the norm associated with a Δ^1_{2n+3}-good wellordering $<$ on C_{2n+3} and where ρ_{2n+3} is the order type of the increasing enumeration of the Δ^1_{2n+3} degree in C_{2n+3}. Then if $\varphi(x_1) = \alpha$ then for all $z \in \mathrm{WO}$, $|z| = \alpha$ we have that

$$x_1(n) = m \leftrightarrow \forall \varepsilon \in Q_{2n+3}(\varphi(\varepsilon) = |z| \to \varepsilon(n) = m) \leftrightarrow \exists y P(n, m, y, z),$$

where $P \subseteq \omega \times \omega \times \mathbb{R}^2$ is a Π^1_{2n+2} relation. Fix a $z_0 \in \mathrm{WO}$ such that $|z_0| = \alpha$ and for each $n, m \in \omega$ with $x_1(n) = m$ pick a witness $y_{m,n}$ such that $P(n, m, y_{n,m}, z)$ holds. Let $w = \langle w_0, m, n, y_{n,m} \rangle$. Then if $w \in L[T_{2n+1}, x]$, we have

$$x_1(n) = m \leftrightarrow L[T_{2n+1}, x] \models \exists z \exists y(z \in \mathrm{WO} \wedge |z| = \alpha \wedge P(n, m, y, z))$$

so that $x_1 \in \mathrm{HOD}^{L[T_{2n+1},x]}$. Since Q_{2n+3} is countable, then there is a z_0 such that

$$z_0 \in L[T_{2n+1}, x] \to Q_{2n+3} \subseteq \mathrm{HOD}^{L[T_{2n+1},x]}$$

For each $x \in \mathbb{R}$ and for each $\omega < \alpha < \omega_1$ let $<_{\alpha,x}$ be a canonical wellordering of $\mathbb{R}$ which are $\mathrm{OD}^{L_\alpha[T_{2n+1},x]}$. Let H_x be the set of all reals which are $\mathrm{OD}^{L_\alpha[T_{2n+1},x]}$ for some α and define $<_x$ a canonical well ordering on H_x, for $\omega < \alpha < \omega_1$ by if $\varepsilon_0, \varepsilon_1 \in H_x$ then $\varepsilon_0 <_x \varepsilon_1 \leftrightarrow$ (the least α s.t ε_0 is $\mathrm{OD}^{L[T_{2n+1},x]} <$ the least α s.t ε_1 is $\mathrm{OD}^{L[T_{2n+1},x]}$) $\vee$ ($\varepsilon_0, \varepsilon_1$ are constructed at the same level α_0 and $\varepsilon_0 <_{\alpha_0,x} \varepsilon_1$). Let $\Theta(x)$ be the order type of $<_x$.Then we have that $\Theta(x) \leq \omega_1^{L[T_{2n+1},x]} < \omega_1$. Also $\mathbb{R} \cap L[T_{2n+1}, x] \subseteq H_x$ and $<_x$ depends only on the Turing degree of x. For $\alpha < \Theta(x)$, let ε^x_α be the α^{th} real in $<_x$. So ε^x_α only depends on the Turing degree of x. Now if $\alpha < \omega_1$, then the set

$$P_\alpha(x) \leftrightarrow \alpha < \Theta(x)$$

is $\underset{\sim}{\Sigma}^1_{2n+2}$. So by $\mathrm{Det}(\underset{\sim}{\Sigma}^1_{2n+2})$, for each α, either P_α or its complement contains a cone of Turing degrees. Let

$$A = \{\alpha : \exists x_0 \forall x \geq_T x_0, P_\alpha(x)\} = \{\alpha : \exists x_0 \forall x \geq_T x_0(\alpha < \Theta(x))\}$$

Then $A \subseteq \omega_1$. If $\alpha \in A$ we claim that for all x in a Turing cone we have that

$$\varepsilon^x_\alpha = \varepsilon_\alpha \text{ is fixed },$$

where

$$\varepsilon_\alpha(n) = m \leftrightarrow \exists x_0 \forall x \geq_T x_0(\varepsilon^x_\alpha(n) = m)$$

To see this, notice that for each α the relation

$$R_\alpha(x, n, m) \leftrightarrow \alpha < \Theta(x) \wedge \varepsilon^x_\alpha(n) = m$$

is Σ^1_{2n+2} and so for each fixed α, n, m either $\{x : R_\alpha(x, n, m)\}$ or its complement contains a Turing cone of degrees, and thus for some x_0 for sufficiently high Turing degree and for all $n, m \in \omega$ if $x_0 \leq_T x$ we have

$$\varepsilon^x_\alpha(n) = m \leftrightarrow \varepsilon^{x_0}_\alpha(n) = m$$

and we are done.

Since the relation

$$w \in \mathrm{WO} \wedge \varepsilon^x_{|w|}(n) = m$$

is $\underset{\sim}{\Sigma}^1_{2n+2}$,it follows from

$$\varepsilon_\alpha(n) = m \leftrightarrow \exists x_0 \forall x_0 \leq_T x(\varepsilon^x_\alpha(n) = m \leftrightarrow \forall y \exists x \geq_T y(\varepsilon^x_\alpha(n) = m$$

that each ε_α is Δ^1_{2n+3} in a countable ordinal, thus

$$\{\varepsilon_\alpha : \alpha \in A\} \subseteq Q_{2n+3}.$$

But the map $\alpha \to \varepsilon_\alpha$ defined on A is $1-1$, since if $\alpha \neq \beta$ and x_0 is of enough large Turing degree so that $\alpha, \beta < \Theta(x_0)$ and $x \geq_T x_0 \to \varepsilon^x_\alpha = \varepsilon_\alpha, \varepsilon^x_\beta = \varepsilon_\beta$ we clearly have $\varepsilon^x_\alpha \neq \varepsilon^x_\beta$. So A is countable. Let $\alpha_0 = \sup\{\alpha : \alpha \in A\}$. Since $\alpha_0 \notin A$ we have that $\forall x \exists y \geq_T x(\Theta(y) \leq \alpha_0)$, thus $\exists x_0 \forall x \geq_T x_0(\Theta(x) \leq \alpha_0)$. So pick a $z \in \mathbb{R}$ such that $\forall x \geq_T z, \Theta(x) \leq \alpha_0$ and so for $\alpha < \Theta(x)$ we have $\varepsilon^x_\alpha = \varepsilon_\alpha$. Then for all $x \geq_T z$,

$$\mathrm{HOD}^{L[T_{2n+1},x]} \cap \mathbb{R} \subseteq H_x = \{\varepsilon^x_\alpha : \alpha < \Theta(x)\} \subseteq \{\varepsilon_\alpha : \alpha \in A\} \subseteq Q_{2n+1}$$

and we are done.

□

The next theorem of Woodin shows that relativizing to a real is the same as adjoining a real to HOD.

THEOREM 4.48 (Woodin). *For every real w there is a real z such that if $w, z \in L[T_{2n+1}, x]$ then $\mathbb{R} \cap HOD^{L[T_{2n+1},x]}_{T_{2n+1}}[w] = \mathbb{R} \cap HOD^{L[x]}_{T_{2n+1},w} = Q_{2n+3}$*

PROOF. The proof is in [15] in the case of the $\mathrm{HOD}^{L[x]}$ and can be generalized. It uses the Vopenka algebra. We omit it since we already included the proof of theorem 4.38.

□

The above two theorem first led us to incorrectly think that it may be possible that $\mathrm{HOD}^{L[T_{2n+1},x]}$ is $L[T_{2n+2}]$, but Woodin noticed that this cannot be true. What will help in correctly identifying $L[T_{2n+2}]$ from the point of view of inner model theory is a characterization of the reals of $L[T_{2n+2}]$. We show the following theorem:

THEOREM 4.49. *(The reals of $L[T_{2n+2}]$)*

Let Q_{2n+3} be the largest bounded Π^1_{2n+3} set of reals and let y_{2n+3} be the least nontrivial Π^1_{2n+3} singleton and let $y_{2n+3}(x)$ be the least nontrivial $\Pi^1_{2n+3}(x)$ singleton. Let $\mathcal{Y}_{2n+3} = Q_{2n+3} \cup \{y_{2n+3}\} \cup \{y_{2n+3}(x) : x \in Q_{2n+3}\}$. Therefore $L[T_{2n+2}]$ is y_{2n+1}-closed and $\mathbb{R} \cap L[T_{2n+2}] = \mathcal{Y}_{2n+3}$.

Notice that we can't have that the set of reals of $L[T_{2n+2}]$ be C_{2n+3}, where C_{2n+3} is the largest thin Π^1_{2n+3} set of reals, since this would imply that the set of reals of $L[T_{2n+2}]$

is C_{2n+4}, since again by $\mathcal{Q}$-theory, $L(C_{2n+3}) = L(C_{2n+4})$, but this would contradict the fact that $L[T_{2n+2}] = L[\mathcal{M}^{\#}_{2n+1,\infty}]$, as $\mathbb{R} \cap \mathcal{M}^{\#}_{2n+2} = C_{2n+4}$.

Proof. $L[T_{2n+2}]$ can compute left most branch of a Δ^1_{2n+3} scale on a Δ^1_{2n+3} set of reals and it is a result of Harrington that the real from the left most branch of the tree from this scale, provided the set $A \in \Delta^1_{2n+3}$ on which we put the scale, does not contain any Δ^1_{2n+3} real, is $\Delta^1_{2n+3}(\mathcal{M}^{\#}_{2n+1})$ and vice-versa. So the least non trivial Π^1_{2n+3} singleton is in $L[T_{2n+2}]$. Next, notice that by section 3, $Q_{2n+3} \subseteq L[T_{2n+2}]$, so $L[T_{2n+2}]$ can also compute the left most real of the tree of a $\Delta^1_{2n+3}(x)$ scale on a $\Delta^1_{2n+3}(x)$ set of reals, for every $x \in Q_{2n+3}$. So $y_{2n+3}(x) \in L[T_{2n+2}]$ for every $x \in Q_{2n+3}$.

$\square$

As mentioned above, recall that for $\alpha = \underset{\sim}{\delta}^1_{2n+1}$ then we have that $L[T_{2n+1}] \cap V_{\underset{\sim}{\delta}^1_{2n+1}}$ is an iterate of a $\mathcal{M}_{2n}$ cut a the least strong cardinal to its least Woodin cardinal and the height of that iterate is exactly $\underset{\sim}{\delta}^1_{2n+1}$, since δ^1_{2n+1} is the least strong to the bottom Woodin δ_∞ in the direct limit of all iterates of $\mathcal{M}_{2n}$. We recall how this computation takes place. The set up below is from [31]. Let Γ be a pointclass closed under $\forall^{\mathbb{R}}$ and which has the scale property. Let $U \subseteq \omega \times \mathbb{R}$ be a good universal for Γ sets and fix φ a Γ-norm on U onto some ordinal δ. Define the set $P_{\rho,G} \subseteq \omega \times \delta$ by

$$P_{\rho,\delta}(n, \alpha) \leftrightarrow \exists x(x \in U \wedge \varphi(x) = \alpha \wedge U(n, \alpha))$$

Then if AD holds we let $H_\Gamma = L[P_{\rho,U}]$.

Definition 4.50. A premouse $\mathcal{P}$ is Γ-properly small iff $\mathcal{P}$ is countable, has a largest cardinal which is a cutpoint of $\mathcal{P}$ and for every $\eta < o(\mathcal{P})$,

(1) $Lp^\Gamma(\mathcal{P}|\eta) \trianglelefteq \mathcal{P}$,
(2) $Lp^\Gamma(\mathcal{P}|\eta) \vDash \eta$ is not a Woodin cardinal,
(3) If η is a cutpoint of $\mathcal{P}$, then $Lp^\Gamma(\mathcal{P}|\eta) = \mathcal{P}|(\eta^+)^{\mathcal{P}}$.

Next we define a notion of iterability for Γ-properly small mice.

Definition 4.51. Let $\mathcal{P}$ be a Γ-properly small mouse. We say $\mathcal{P}$ is Γ-correctly iterable if whenever $\vec{\mathcal{T}}$ is a countable stack of Lp^{Γ} guided normal trees of successor lengths on $\mathcal{P}$ with last model $\mathcal{Q}$, then

(1) $\mathcal{Q}$ is wellfounded and if the branch from $\mathcal{P}$ to $\mathcal{Q}$ of $\vec{\mathcal{T}}$ does not drop, then $\mathcal{Q}$ if Γ-properly small and

(2) If $\mathcal{U}$ is an Lp^{Γ} guided normal tree on $\mathcal{Q}$ then

 (a) $\mathcal{U}$ is a short tree and

 (b) If $\mathcal{U}$ has a last model then it can be freely extended by one more ultrapower that is every putative normal iteration tree $\mathcal{T}$ extending $\mathcal{U}$ and having length $lh(\mathcal{U})+1$ has a wellfounded last model and moreover this last model is Γ-properly small if the leading branch does not drop, and

 (c) If $\mathcal{U}$ has limit length then $\mathcal{U}$ has a cofinal wellfounded branch b such that $\mathcal{Q}(b, U) = \mathcal{Q}(\mathcal{U})$ and $\mathcal{M}_b^{\mathcal{U}}$ is Γ properly small if the branch from $\mathcal{P}$ to $\mathcal{Q}$ to $\mathcal{M}_b^{\mathcal{U}}$ does not drop.

If Σ is the $(\omega, \omega_1, \omega_1)$ strategy of $\mathcal{P}$ given by the above then we say that it is Lp^{Γ} guided and the non-dropping iterates of $\mathcal{P}$ via Σ are Γ properly small. Σ is unique and has by the Dodd-Jensen property. This allows defining the direct limit of all non-dropping Lp^{Γ} guided iterates of $\mathcal{P}$. So let $\mathcal{I} = \{\mathcal{P} : \mathcal{P}$ is Γ-properly small and Γ-correctly iterable$\}$. For $\mathcal{P}, \mathcal{Q} \in \mathcal{I}$, we let

$$\mathcal{P} \prec \mathcal{Q} \leftrightarrow \exists \eta \text{ s.t } \eta \text{ is a strong cutpoint of } \mathcal{Q}, \mathcal{Q}|\eta \text{ is a } \Gamma\text{-correct iterate of } \mathcal{P}$$

It is then shown in [31] using a comparison argument that the system $(\mathcal{I}, \preceq)$ is a directed system of mice, and thus by the Dodd-Jensen property, the direct limit of this system, $\mathcal{M}_\infty$ is well-defined, wellfounded and that $\mathcal{M}_\infty = L[T_\Gamma]$. One first shows that $\mathcal{M}_\infty \subseteq H_\Gamma$ by providing a Suslin representation for Γ sets from $\mathcal{M}_\infty$ and then the Becker-Kechris theorem implies that $H_\Gamma \subseteq L[\mathcal{M}_\infty]$. Since $\underset{\sim}{\delta}_\Gamma = o(\mathcal{M}_\infty)$, then $\underset{\sim}{\delta}_\Gamma$ is the least $< \delta_\infty$-strong cardinal in HOD, where δ_∞ is the least Woodin cardinal of $\mathcal{M}_\infty$. To take a concrete example, suppose Γ is a Π^1_1-like pointclass, say $\underset{\sim}{\Pi}^1_3$. Then the model $H_3 = L[T_3]$ is the direct limit of all iterates of $\mathcal{M}_2$ cut off at the least cardinal strong to the least Woodin cardinal.

We now turn to the proof of the GCH in the models $L[T_{2n}]$. We are grateful to Hugh Woodin for guiding us to show the main theorem of this section. Following an idea of Hugh Woodin, we first show that the GCH holds in $L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}}$. Then the GCH will hold in $L[T_{2n+2}]$ using a usual Godel/Silver condensation argument for relative constructibility. The goal is then to show that $L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}}$ is a direct limit of fully sound structures. As in the theorem in the previous section, we will then show that $L[T_{2n+2}] = L[\mathcal{M}^{\#}]$ for some $\mathcal{M}$ which is a direct limit of fully sound structures and such that $L[\mathcal{M}^{\#}] \cap V_{\kappa^1_{2n+3}} = \mathcal{M}$. So we will require that $o(\mathcal{M}) = \kappa^1_{2n+3}$. We start with the following definition:

Definition 4.52 ($\mathcal{M}^{\#}_{2n+1}$-closed mouse). Let $\mathcal{M}$ be a premouse. Then we say that $\mathcal{M}$ is a $\mathcal{M}^{\#}_{2n+1}$-closed premouse if for every $A \in \mathcal{M}$, we have $\mathcal{M}^{\#}_{2n+1}(A) \in \mathcal{M}$. Also, $\mathcal{M}$ is a $\mathcal{M}^{\#}_{2n+1}$-closed mouse if it is a $\mathcal{M}$ is a $\mathcal{M}^{\#}_{2n+1}$-closed premouse and has an $(\omega, \omega_1, \omega_1)$-iteration strategy Σ.

Next we need to define the Woodin mice which will constitute our directed system below.

Definition 4.53. We say $\mathcal{N}$ is a n-Woodin mouse if the following conditions are satisfied:

(1) $\mathcal{N} = L(\mathcal{N})^{\#} \cap V_\delta$, where $\delta = o(\mathcal{N})$,
(2) $L(\mathcal{N}) \vDash \delta$ is a Woodin cardinal .
(3) $\mathcal{N}$ has n Woodin cardinals.

We next define the iteration strategy of an n-Woodin mouse in the case n is odd.

Definition 4.54 (Iterability for n-Woodin mice). Let $\mathcal{N}$ be an n-Woodin mouse. We say $\mathcal{N}$ is correctly iterable if whenever $\vec{\mathcal{T}}$ is a countable stack of C_{2n+2} guided normal trees of successor lengths on $\mathcal{N}$ with last model $\mathcal{Q}$, then

(1) $\mathcal{Q}$ is wellfounded and if the branch from $\mathcal{N}$ to $\mathcal{Q}$ of $\vec{\mathcal{T}}$ does not drop, then $\mathcal{Q}$ is an n-Woodin mouse and
(2) If $\mathcal{U}$ is a C_{2n+2} guided normal tree on $\mathcal{Q}$ then

(a) $\mathcal{U}$ is a short tree and

(b) If $\mathcal{U}$ has a last model then it can be freely extended by one more ultrapower that is every putative normal iteration tree $\mathcal{T}$ extending $\mathcal{U}$ and having length $lh(\mathcal{U})+1$ has a wellfounded last model and moreover this last model is an n-Woodin mouse if the leading branch does not drop, and

(c) If $\mathcal{U}$ has limit length then $\mathcal{U}$ has a cofinal wellfounded branch b such that $\mathcal{Q}(b,U)=\mathcal{Q}(\mathcal{U})$ and $\mathcal{M}_b^{\mathcal{U}}$ is an n-Woodin mouse if the branch from $\mathcal{N}$ to $\mathcal{Q}$ to $\mathcal{M}_b^{\mathcal{U}}$ does not drop.

By Steel, see [24], the above notion of iterability for n-Woodin mice is equivalent to Π^1_{2n+3} iterability. Let $\mathcal{N}$ be the least $2n+1$-Woodin mouse, that is if $\mathcal{S} \lhd \mathcal{N}$ then $\mathcal{S}$ fails one of the conditions above. Let $\Sigma_{\mathcal{N}}$ be the iteration strategy of $\mathcal{N}$. Define

$$\mathcal{I} = \{\mathcal{P} : \mathcal{P} \text{ is a } \Sigma\text{-iterate of } \mathcal{N}\}$$

and for $\mathcal{P}, \mathcal{Q} \in \mathcal{I}$, we let

$$\mathcal{P} \prec^* \mathcal{Q} \leftrightarrow \exists \eta (\eta \text{ is a Woodin cardinal cutpoint of } \mathcal{Q} \text{ and } \mathcal{Q}|\eta \text{ is a countable } \Sigma\text{-iterate of } \mathcal{P})$$

Then notice that $(\mathcal{I}, \prec^*)$ is a partial order.

Lemma 4.55. *$(\mathcal{I}, \prec^*)$ is countably directed.*

The proof of the above is as usual and we chose to omit it. One can read the proof in [30].

Let now $\mathcal{N}_\infty$ be the direct limit of the system $(\mathcal{I}, \prec^*)$. Then since $(\mathcal{I}, \prec^*)$ is countably directed, $\mathcal{N}_\infty$ is wellfounded. $\mathcal{N}_\infty$ is the direct limit of all countable iterates of the least $\mathcal{N}$ satisfying the above two conditions, and we can define this direct limit by the Dodd-Jensen property of the $\Sigma_{\mathcal{N}}$. Notice that $\mathcal{N}_\infty$ is itself a countable iterate of $\mathcal{N}$ via $\Sigma_{\mathcal{N}}$. It then follows by the proof in the above section that

$$L[T_{2n+2}] = L[\mathcal{N}_\infty^{\#}],$$

since the iteration strategy Σ_∞ of $\mathcal{N}_\infty$ is Π^1_{2n+3}. Notice that

$$\mathcal{N}_\infty = L[\mathcal{N}_\infty^\#] \cap V_{\delta_\infty} = L[\mathcal{N}_\infty^\#] \cap V_{\kappa^1_{2n+3}} = L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}}.$$

Therefore $L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}}$ is a direct limit of all Σ iterates of $\mathcal{N}$. Since $\mathcal{N}_\infty$ is fully sound then $L[T_{2n+2}] \cap V_{\kappa^1_{2n+3}} \vDash \mathrm{GCH}$. Then by a condensation argument as in Godel/Silver, $L[T_{2n+2}] \vDash$ GCH.

It then remains to show that $\mathcal{N}_\infty$ is $\mathcal{M}_1^\#$-closed and we finish by showing the following lemma. So $\mathcal{N}_\infty$ is the least active mouse closed under $\mathcal{M}_1^\#$ which projects to ω. It is sometimes referred to in the litterature as $\mathcal{M}_1^{\#^\#}$.

LEMMA 4.56. *$\mathcal{N}_\infty$ is $\mathcal{M}_1^\#$-closed. Therefore $\mathcal{N}_\infty$ does not project at or below δ_∞, $\mathcal{N}_\infty$ is fully sound and*

$$\rho_\omega(\mathcal{N}_\infty) > o(\mathcal{N}_\infty) = \delta_\infty.$$

PROOF. Suppose not and let $A \in \mathcal{N}_\infty$ such that $\mathcal{M}_1^\#(A) \notin \mathcal{N}_\infty$. Let $\mathcal{P} \in \mathcal{I}$ be a countable iterate of $\mathcal{N}$ such that $\pi_{\mathcal{P},\infty} : \mathcal{P} \to \mathcal{N}_\infty$ is the iteration embedding. Let $\pi : L(\mathcal{P}) \to L(\mathcal{N}_\infty)$ be elementary such that $\pi|\mathcal{P} = \pi_{\mathcal{P},\infty}$ and such that $\delta_\infty, \mathcal{N}_\infty, \mathcal{P}$ and $A \in ran(\pi)$. Let $\bar{A} \in \mathcal{P}$ such that $\pi(\bar{A}) = A$. Notice that $\mathcal{M}_1^\#(\bar{A})$ has same size as $\bar{A}$. It then follows it is a bounded subset of $\delta^{\mathcal{P}}$. Since the $\mathcal{M}_1^\#$ operator condenses well then we have that $\mathcal{M}_1^\#(\pi^{-1}(A)) = \pi^{-1}(\mathcal{M}_1^\#(A))$. So $\mathcal{M}_1^\#(\pi^{-1}(A)) \notin \mathcal{P}$. But then $L(\mathcal{P}) \nvDash \delta^{\mathcal{P}}$ is Woodin . Contradiction.

$\square$

The above can be generalized in the obvious way to all $\mathcal{M}_{2n+1}^\#$. It then follows that $L[T_{2n}] \vDash$ GCH. From the above it should now be possible to adapt the standard proofs that $\square_\kappa$ for $\kappa > \aleph_1$ a cardinal to show that if $V = L[T_{2n}]$ then for any cardinal $\kappa > \aleph_1$, $\square_\kappa$ holds. Then using failure of $\square_\kappa$, one could possibly derive how much boldface determinacy the $L[T_{2n}]$ satisfy. Using purely inner model theoretic tools, the analysis could possibly be pushed to pointclasses higher than those of the projective hierarchy. Or it may as well be possible that the very fine analysis of $L(\mathbb{R})$ is necessary to carry this analysis further.

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