OPEN FILTERS AND LARGE CARDINALS

SERHII BARDYLA, JAROSLAV ŠUPINA AND LYUBOMYR ZDOMSKYY

ABSTRACT. In this paper we investigate lattices which are order isomorphic to a poset $\mathbf{OF}(X)$ of all free open filters of a given Hausdorff space X. We characterize spaces whose posets of free open filters are lattices. For each $n \in \mathbb{N}$ we construct a scattered space X such that $\mathbf{OF}(X)$ is order isomorphic to the n-element chain. This result answers two questions of Mooney. We show that for each cardinal κ there exists a Hausdorff space X which possesses exactly κ-many free open filters. This result is a counterpart to the result of Pelant, Simon and Vaughan who showed that each Hausdorff space admits at least ω_1 free closed filters. Assuming the existence of n measurable cardinals, for every $m_0, \ldots, m_n \in \omega\backslash\{0\}$ we construct a space X such that $\mathbf{OF}(X)$ is isomorphic to the direct product $\prod_{i\in n+1} m_i$. We show that only complete distributive lattices can be isomorphic to $\mathbf{OF}(X)$ for some Hausdorff space X. This way we partially answered another question of Mooney. It is proved that the existence of a space X which possesses a free ω_1-complete open ultrafilter is equivalent to the existence of a measurable cardinal. This provides an answer to an old question of Liu.

1. Introduction and preliminaries

In this paper all spaces are Hausdorff. Each natural number n is identified with the set $\{0, \ldots, n-1\}$. We shall use a terminology from [13] and [19]. Let $\mathcal{F}$ be a filter on a set X and $Y \subset X$. If $Y \cap F \neq \emptyset$ for each $F \in \mathcal{F}$, then the filter $\{F \cap Y : F \in \mathcal{F}\}$ on Y is called a *trace of the filter* $\mathcal{F}$ on Y. The family $\mathcal{A}$ of subsets of X is said to *generate a filter* $\mathcal{F}$ if the set of all finite intersections of elements of the family $\mathcal{A}$ forms a base of $\mathcal{F}$. A filter $\mathcal{F}$ on a space X is called *open* if $\mathcal{F}$ possesses a base which consists of open sets. Moreover, if $\cap\{\overline{F} : F \in \mathcal{F}\} = \emptyset$, then the filter $\mathcal{F}$ is called *free*. For a subspace Y of X a filter $\mathcal{F}$ is called *open on* Y if the trace of $\mathcal{F}$ on Y is an open filter. For any element x of a space X by $\mathcal{N}(x)$ we denote the open filter on X whose base consists of all open neighborhoods of x. By $\mathbf{OF}(X)$ we denote the poset of all free open filters on a space X partially ordered by the inclusion, i.e., $\mathcal{F}_1 \leq \mathcal{F}_2$ iff $\mathcal{F}_1 \subset \mathcal{F}_2$ for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{OF}(X)$. A topological space X is called *H-closed* if $\mathbf{OF}(X) = \emptyset$. Let us note that a space X is H-closed iff X is a closed subset of any Hausdorff topological space Y which contains X as a subspace. Clearly, each compact space is H-closed, but the converse is not true.

Free open filters are useful in investigating H-closed extensions. A space Y is called an *extension* of a space X if Y contains X as a dense subspace. If, moreover, the space Y is H-closed, then it is called an *H-closed extension* of X. H-closed extensions is a classical topic in general topology. It was investigated by many authors in [15, 28, 29, 31, 32, 37, 39]. One can consider H-closed extensions as an analogue of compactifications in the category of Hausdorff spaces. By $\mathbf{H}(X)$ we denote the set of all H-closed extensions of a space X. The set $\mathbf{H}(X)$ carries a natural partial order $\leq$ defined as follows: for any $Y, Z \in \mathbf{H}(X)$, $Y \leq Z$ if there exists a continuous surjection $f : Z \to Y$ such that $f(x) = x$ for any $x \in X$. It can be checked [23] that $\mathbf{H}(X)$ is a complete upper semilattice with respect to the order defined above, i.e., each subset $A \subset \mathbf{H}(X)$ has supremum in $\mathbf{H}(X)$. The order structure of H-closed extensions and compactifications was investigated in [5].

2010 *Mathematics Subject Classification.* 54A20, 54D80, 03E55. ???

Key words and phrases. lattice of filters, H-closed extension, free open filter, open ultrafilter, κ-complete filter.

The first author was supported by the Austrian Science Fund FWF (Grant M 2967). The second author was supported by ???. The third author was supported by the Austrian Science Fund FWF (Grants I 3709-N35, I 2374-N35).

Free closed filters on topological spaces were investigated by Pelant, Simon and Vaughan [27]. In particular, they showed that every Hausdorff (Tychonoff, resp.) non-compact space admits at least ω_1 (ω_2, resp.) free closed filters. Also, they constructed a T_1 space which has exactly one free closed ultrafilter and made a call for investigating other types of free filters on topological spaces. In particular, they asked about the smallest possible non-zero number of free open filters on a given Hausdorff space. In [23] Mooney constructed a space X admitting a unique free open filter and this way answered the question of Pelant, Simon and Vaughan. Note that the Mooney space X admits a unique extension Y. Moreover, Y is H-closed and $Y \setminus X$ is singleton. Also, he showed that for each Bell number[1] $B(n)$ there exists a space X such that $|\mathbf{H}(X)| = B(n)$. This motivated Mooney to ask the following questions:

Question 1 ([23, Question 6.8]). *Is there a space with exactly three H-closed extensions? Four? Other non-Bell numbers?*

Question 2 ([23, Question 6.9]). *Is there a space with two one-point H-closed extensions and no other H-closed extensions?*

Question 3 ([23, Question 6.11]). *Characterize all complete upper semilattices S for which there is a space X such that $S = \mathbf{H}(X)$.*

An open filter $\mathcal{F}$ on a space X is called an *open ultrafilter* if for any open set $U \subset X$ such that $U \notin \mathcal{F}$ there exists $F \in \mathcal{F}$ such that $F \cap U = \emptyset$. Open ultrafilters were characterized by Mooney [24]. Namely, an open filter $\mathcal{F}$ an a space X is an open ultrafilter if and only if $\mathcal{F}$ satisfies the following two conditions:

- $\mathcal{F}$ contains all open dense subspaces of X;
- if $F = A \cup B$ is an element of $\mathcal{F}$, where the sets A, B are open and disjoint, then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

For a subspace Y of X a filter $\mathcal{F}$ on X is called an *open ultrafilter on* Y, if the trace of $\mathcal{F}$ on Y is an open ultrafilter. Note that if some open filter $\mathcal{F}$ contains an open ultrafilter $\mathcal{U}$, then $\mathcal{F} = \mathcal{U}$. The Axiom of Choice implies that every open filter is contained in some open ultrafilter. Free open ultrafilters were investigated in [4, 20, 24, 34, 35]. In particular, Carson and Porter [4] applied open ultrafilters to the study of maximal points and lower topologies in the poset of Hausdorff topologies on a given set. Open ultrafilters naturally appears in Čech-Stone compactifications of Tychonoff spaces. Namely, a known result of van Douwen [6] states that if X a non-pseudocompact Tychonoff space with a countable π-base, then $\beta(X) \setminus X$ contains a remote point, i.e., there exists $y \in \beta(X) \setminus X$ such that y is not in the closure of any nowhere dense subset of X. Using characterization of Mooney and the density of X in $\beta(X)$, one can check that a point $y \in \beta(X) \setminus X$ is remote if and only if the trace of $\mathcal{N}(y)$ on X is an open ultrafilter. See [2, 7, 9, 10, 11, 12, 38, 40] for more about remote points and their applications to non-homogeneity and butterfly points of Čech-Stone compactifications. The set of all free open ultrafilters on a space X is denoted by $\mathbf{U}(X)$. Recall that a filter $\mathcal{F}$ is called *κ-complete* if for any $\lambda \in \kappa$ and subfamily $\{F_\alpha : \alpha \in \lambda\} \subset \mathcal{F}$ the set $\cap_{\alpha\in\lambda} F_\alpha$ belongs to $\mathcal{F}$. The following old question belongs to Liu.

Question 4 ([20, Remark 3.12]). *Does there exist in ZFC a space X which possesses a free ω_1-complete open ultrafilter?*

In this paper we characterize spaces whose posets of free open filters are lattices. For each $n \in \mathbb{N}$ we construct a scattered space X such that $\mathbf{OF}(X)$ is order isomorphic to the n-element chain. This way we give an affirmative answer to questions 1 and 2. We show that for each cardinal κ there exists

[1]for a positive integer n the Bell number $B(n)$ is equal to the number of equivalence relations on an n-element set

a Hausdorff space X which possesses exactly κ-many free open filters. This result is a counterpart to the results of Pelant, Simon and Vaughan. Assuming the existence of n measurable cardinals, for every $m_0, \dots, m_n \in \omega \setminus \{0\}$ we construct a space X such that $\mathbf{OF}(X)$ is isomorphic to the direct product $\prod_{i \in n+1} m_i$. We show that only complete distributive lattices can be isomorphic to $\mathbf{OF}(X)$ for some Hausdorff space X. This sheds some light on Question 3. It is proved that the existence of a space X which possesses a free ω_1-complete open ultrafilter is equivalent to the existence of a measurable cardinal. This way we explicitly answer Question 4.

2. General facts about the poset of free open filters on a given space

In this section we investigate properties of a poset $\mathbf{OF}(X)$. A space X is called *locally H-closed* if for each $x \in X$ there exists an open neighborhood U of x such that $\overline{U}$ is H-closed. Next we recall two well-known H-closed extensions which will be useful later on.

By [28], for each locally H-closed non-H-closed space X the poset $\mathbf{H}(X)$ contains the infimum, which now is called the Obreanu-Porter extension $OP(X)$ [16]. The extension $OP(X)$ has a singleton remainder y and open neighborhood base $\mathcal{B}(y)$ at y consists of the sets $\{y\} \cup X \setminus K$, where K runs over H-closed subspaces of X. It is worth to note that the open filter $\mathcal{F}_{\text{inf}}$ on X generated by the family $\{X \setminus K : K \text{ is an H-closed subspace of } X\}$ is the infimum of the poset $\mathbf{OF}(X)$.

Following [29], the Katetov extension $K(X)$ of a space X is the set

$$X \cup \{\mathcal{F} : \mathcal{F} \text{ is a free open ultrafilter on } X\}$$

endowed with the topology τ which satisfies the following conditions:

- X is an open subspace of $(K(X), \tau)$;
- open neighborhood base at $\mathcal{F} \in K(X) \setminus X$ consists of the sets $\{\mathcal{F}\} \cup F$, where $F \in \mathcal{F}$ is open.

For any space X, the Katetov extension $K(X)$ is the supremum of $\mathbf{H}(X)$ and it can be considered as a generalization of the Čech-Stone compactification (see [29] or [31] for more details). A topological space X is called *almost H-closed* if it admits a unique free open ultrafilter.

The following lemma is a consequence of Theorem 4.1 from [15].

Lemma 2.1. *Every almost H-closed space is locally H-closed.*

Lemma 2.2. *Let X be an almost H-closed space and $\mathcal{T}$ be a nonempty subset of $\mathbf{OF}(X)$. Then $\inf \mathcal{T} = \cap\mathcal{T}$ and $\sup \mathcal{T}$ is generated by the family $\cup\mathcal{T}$.*

Proof. By the definition of the order on $\mathbf{OF}(X)$, to prove that $\inf \mathcal{T} = \cap\mathcal{T}$ it suffices to show that $\cap\mathcal{T}$ is a free open filter. Since the space X is almost H-closed, there exists a unique free open ultrafilter $\mathcal{F}_{\text{sup}}$ which contains every element of $\mathcal{T}$. By Lemma 2.1, X is locally H-closed. It is easy to check that for every free open filter $\mathcal{F}$ on X and H-closed subspace K of X, the set $X \setminus K \in \mathcal{F}$. Then, taking into account that X is locally H-closed, the open filter $\mathcal{F}_{\text{inf}}$ generated by the family $\{X \setminus K : K \text{ is a H-closed subspace of } X\}$ is free and coincides with the infimum of the poset $\mathbf{OF}(X)$. Hence $\mathcal{F}_{\text{inf}} \subset \mathcal{F} \subset \mathcal{F}_{\text{sup}}$ for each $\mathcal{F} \in \mathcal{T}$. At this point it is clear that $\cap\mathcal{T}$ is a free filter. To prove that the filter $\cap\mathcal{T}$ is open fix any $F \in \cap\mathcal{T}$. Since the set $\mathcal{T}$ consists of open filters, for each $\mathcal{F} \in \mathcal{T}$ there exists an open set $O_{\mathcal{F}} \in \mathcal{F}$ such that $O_{\mathcal{F}} \subset F$. Then $H = \cup_{\mathcal{F} \in \mathcal{T}} O_{\mathcal{F}}$ is open, $H \in \cap\mathcal{T}$ and $H \subset F$, witnessing that $\cap\mathcal{T}$ is a free open filter.

Since for each $\mathcal{F} \in \mathcal{T}$, $\mathcal{F} \subset \mathcal{F}_{\text{sup}}$, the family $\cup\mathcal{T}$ is centered. Let $\mathcal{H}$ be a filter generated by the family $\cup\mathcal{T}$. Since there exists $\mathcal{F} \in \mathcal{T}$ such that $\mathcal{F} \subset \mathcal{H}$ the filter $\mathcal{H}$ is free. The definition of the order on $\mathbf{OF}(X)$ implies that to complete the proof it remains to show that the filter $\mathcal{H}$ is open. For this fix any $H \in \mathcal{H}$. It follows that there exist $\mathcal{F}_0, \dots, \mathcal{F}_{n-1} \in \mathcal{T}$ and elements $F_i \in \mathcal{F}_i$, $i \in n$ such that

$\cap_{i\in n}F_i \subset H$. Since the filters $\mathcal{F}_i$, $i \in n$ are open, for every $i \in n$ there exists an open set $O_i \in \mathcal{F}_i$ such that $O_i \subset F_i$. Observe that the open set $\cap_{i\in n}O_i \subset H$ belongs to $\mathcal{H}$. □

A lattice $(L, \vee, \wedge)$ is called *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. It is well-known that the latter condition is equivalent to its dual: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Theorem 2.3. *For a topological space X the following conditions are equivalent:*

(1) *X is almost H-closed;*
(2) *the posets $\mathbf{H}(X)$ and $\mathbf{OF}(X)$ are order isomorphic;*
(3) *the poset $\mathbf{OF}(X)$ is a complete distributive lattice;*
(4) *the poset $\mathbf{OF}(X)$ is a lattice.*

Proof. (1)⇒(2). Let us show that any proper extension of X has a singleton reminder. To derive a contradiction, assume that Y is an extension of X such that $Y \setminus X$ contains two distinct points a, b. By Zorn's Lemma, the traces of the filters $\mathcal{N}(a)$ and $\mathcal{N}(b)$ on X can be enlarged to open ultrafilters $\mathcal{F}_a$ and $\mathcal{F}_b$, respectively. Since the space Y is Hausdorff, there exists $U_a \in \mathcal{N}(a)$ and $U_b \in \mathcal{N}(b)$ such that $U_a \cap U_b = \emptyset$. Therefore the filters $\mathcal{F}_a$ and $\mathcal{F}_b$ are distinct which contradicts to the almost H-closedness of X. Define $\phi : \mathbf{H}(X) \to \mathbf{OF}(X)$ by $\phi(Y) = \mathcal{F}_y$ where $Y \setminus X = \{y\}$ and $\mathcal{F}_y$ is the trace of the filter $\mathcal{N}(y)$ on X. Clearly, the filter $\mathcal{F}_y$ is open and free, witnessing that the map ϕ is well-defined. At this point it is straightforward to check that ϕ is an injective order homomorphism between posets $\mathbf{H}(X)$ and $\mathbf{OF}(X)$. Fix any free open filter $\mathcal{F}$ on X. Let τ be the topology on the set $X \cup \{\mathcal{F}\}$ which satisfies the following conditions:

- X is an open subspace of $(X \cup \{\mathcal{F}\}, \tau)$;
- the family $\{\{\mathcal{F}\} \cup F : F \in \mathcal{F}$ and F is open$\}$ forms a base at $\mathcal{F}$.

Then $Y = (X \cup \{\mathcal{F}\}, \tau) \in \mathbf{H}(X)$ and $\phi(Y) = \mathcal{F}$ which implies that the map ϕ is surjective.

(2)⇒ (1). Assume that the space X possesses two distinct free open ultrafilters $\mathcal{F}_1$, $\mathcal{F}_2$. Then there exist $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ such that $F_1 \cap F_2 = \emptyset$. It follows that there exists no open filter $\mathcal{H}$ on X such that $\mathcal{F}_1 \subset \mathcal{H}$ and $\mathcal{F}_2 \subset \mathcal{H}$. Consequently, the poset $\mathbf{OF}(X)$ does not have a supremum. On the other hand, the Katetov extension $K(X)$ is the supremum of $\mathbf{H}(X)$ (see [29]). Hence the posets $\mathbf{OF}(X)$ and $\mathbf{H}(X)$ cannot be order isomorphic.

(2)⇒(3). Since (1)⇔(2) the space X is almost H-closed. Lemma 2.1 implies that X is locally H-closed. By the previous arguments, we have that each H-closed extension of X has a singleton remainder. Proposition 1 from [16] provides that the set of all H-closed extensions with a singleton remainder of a space X forms a complete sublattice of $\mathbf{H}(X)$. Hence the lattice $\mathbf{OF}(X)$ is complete. Consider any free open filters $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ on X. Lemma 2.2 implies that the filter $\mathcal{F} \wedge (\mathcal{G} \vee \mathcal{H})$ is generated by the family $\mathcal{A} = \{F \cup (G \cap H) : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H}\}$, and the filter $(\mathcal{F} \wedge \mathcal{G}) \vee (\mathcal{F} \wedge \mathcal{H})$ is generated by the family $\mathcal{B} = \{(F \cup G) \cap (F \cup H) : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H}\}$. Obviously, $\mathcal{A} = \mathcal{B}$ witnessing that the lattice $\mathbf{OF}(X)$ is distributive.

The implication (3)⇒(4) is trivial.

(4)⇒(1). Assume that $\mathbf{OF}(X)$ is a nonempty lattice and nonetheless X admits two distinct free open ultrafilters $\mathcal{F}_1, \mathcal{F}_2$. Then there exists $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ such that $F_1 \cap F_2 = \emptyset$. Since $\mathbf{OF}(X)$ is a lattice there exists a filter $\mathcal{G} \in \mathrm{OF}(X)$ such that $\mathcal{G} = \max\{\mathcal{F}_1, \mathcal{F}_2\}$. It follows that $\{F_1, F_2\} \subset \mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{G}$, which implies a contradiction. □

Recall that a lattice is distributive if and only if it does not contain as a sublattice the following two lattices: here we need some pictures!

The latter fact and Theorem 2.3 imply the following corollary which sheds some light on Question 3.

Corollary 2.4. *There exist finite lattices which cannot be isomorphic to $\mathbf{OF}(X)$ for any space X.*

Corollary 2.4 implies the following natural problem, which will be studied in the next sections.

Problem 2.5. Characterise finite lattices which can be represented as the lattice $\mathbf{OF}(X)$ for some Hausdorff space X.

Lemma 2.6. *Let X be an almost H-closed space and $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{OF}(X)$. Then $\operatorname{Int}(\overline{F}) \in \mathcal{F}_2$ for each $F \in \mathcal{F}_1$.*

Proof. Let $\mathcal{F}_{\sup}$ and $\mathcal{F}_{\inf}$ be the supremum and infimum, respectively, of the lattice $\mathbf{OF}(X)$. Note that for any open filter $\mathcal{F}$ on X, $\overline{F} \in \mathcal{F}$ if and only if $\operatorname{Int}(\overline{F}) \in \mathcal{F}$. Taking into account the latter argument and the fact that $\mathcal{F}_{\inf}$ is the infimum of $\mathbf{OF}(X)$, it suffices to check that for each $F \in \mathcal{F}_{\sup}$, $\overline{F} \in \mathcal{F}_{\inf}$. To derive a contradiction, assume that there exists $F \in \mathcal{F}_{\sup}$ such that $\overline{F} \notin \mathcal{F}_{\inf}$. It follows that for each $H \in \mathcal{F}_{\inf}$ the set $H \setminus \overline{F}$ is nonempty. Then the family $\{H \setminus \overline{F} : H \in \mathcal{F}_{\inf}\}$ forms a base of some free open filter on X which is not contained in the filter $\mathcal{F}_{\sup}$. The obtained contradiction completes the proof of the lemma. □

Corollary 2.7. *If X is an almost H-closed space and $\mathcal{U}$ is a unique free open ultrafilter on X, then the infimum of $\mathbf{OF}(X)$ is generated by the family $\{\operatorname{Int}(\overline{U}) : U \in \mathcal{U}\}$.*

Proof. Clearly, the filter $\mathcal{W}$ generated by the family $\{\operatorname{Int}(\overline{U}) : U \in \mathcal{U}\}$ is an open filter. Note that for each $U \in \mathcal{U}$, $\overline{U} = \overline{\operatorname{Int}(\overline{U})}$. Since the filter $\mathcal{U}$ is free, the filter $\mathcal{W}$ is free as well. Lemma 2.6 provides that the filter $\mathcal{W}$ is contained in any other free open filter on X. Hence $\mathcal{W}$ is the infimum of $\mathbf{OF}(X)$. □

Lemma 2.8. *Let $\mathcal{F}, \mathcal{H}$ be two distinct free open filters on an almost H-closed space X such that $\mathcal{F} \subset \mathcal{H}$ and there exists no open filter $\mathcal{G}$ satisfying $\mathcal{F} \subsetneqq \mathcal{G} \subsetneqq \mathcal{H}$. Let P be any open set which belongs to $\mathcal{H} \setminus \mathcal{F}$. By $\mathcal{T}_P$ we denote the filter generated by the family $\{F \cap (\overline{P} \setminus P) : F \in \mathcal{F}\}$. Then the following statements hold:*

1. *the filter $\mathcal{H}$ is generated by the family $\mathcal{F} \cup \{P\}$;*
2. *the filter $\mathcal{F}$ is generated by the family $\{T \cup H : T \in \mathcal{T}_P$ and $H \in \mathcal{H}\}$;*
3. *the filter $\mathcal{T}_P$ is an open ultrafilter on $\overline{P} \setminus P$;*
4. *the filter $\mathcal{T}_P$ is generated by the family $\{F \cap (\overline{H} \setminus P) : F \in \mathcal{F}, H \in \mathcal{H}\}$;*
5. *the filter $\mathcal{T}_P$ is generated by the family $\{F \cap (\overline{H} \setminus H) : F \in \mathcal{F}, P \supset H \in \mathcal{H}\}$;*
6. *if for any open filter $\mathcal{G}$ on X the inclusion $\mathcal{F} \subsetneqq \mathcal{G}$ implies that $\mathcal{H} \subset \mathcal{G}$, then $\mathcal{F}$ is generated by the family $\{U \subset X : U$ is open and $U \cap (\overline{P} \setminus P) \in \mathcal{T}_P\}$.*

Proof. 1. Observe that the filter $\mathcal{G}$ generated by the family $\mathcal{F} \cup \{P\}$ is open and satisfies $\mathcal{F} \subset \mathcal{G} \subset \mathcal{H}$ which implies that either $\mathcal{G} = \mathcal{F}$ or $\mathcal{G} = \mathcal{H}$. Since $P \notin \mathcal{F}$ we deduce that $\mathcal{G} = \mathcal{H}$.

2. Let $\mathcal{Z}$ be the filter generated by the family $\{T \cup H : T \in \mathcal{T}_P, H \in \mathcal{H}\}$. Consider any $Z \in \mathcal{Z}$. By the definition of $\mathcal{Z}$, there exist $T \in \mathcal{T}_P$ and $H \in \mathcal{H}$ such that $T \cup H \subset Z$. By item 1, the filter $\mathcal{H}$ is generated by the family $\mathcal{F} \cup \{P\}$. It follows that there exists an element $F \in \mathcal{F}$ such that $F \cap P \subset H$. By Lemma 2.6 and the definition of $\mathcal{T}_P$, there exists $G \in \mathcal{F}$ such that $G \subset F$, $G \subset \overline{P}$ and $G \cap (\overline{P} \setminus P) \subset T$. The choice of F implies that $G \cap P \subset F \cap P \subset H$. Thus, $G \subset T \cup H \subset Z$ witnessing that $\mathcal{Z} \subset \mathcal{F}$.

To show the converse inclusion, fix any $F \in \mathcal{F}$. Put $H = F \cap P \in \mathcal{H}$ and $T = F \cap (\overline{P} \setminus P) \in \mathcal{T}_P$. Obviously, $T \cup H \in \mathcal{Z}$ and $T \cup H \subset F$ witnessing that $\mathcal{Z} = \mathcal{F}$.

3. Since the filter $\mathcal{F}$ is open, the filter $\mathcal{T}_P$ is open on $\overline{P} \setminus P$. To derive a contradiction, assume that $\mathcal{T}_P$ is not an open ultrafilter on $\overline{P} \setminus P$. Then the trace of $\mathcal{T}_P$ on $\overline{P} \setminus P$ is properly contained in an open filter $\mathcal{U}$ on $\overline{P} \setminus P$. It follows that there exists a set $U \in \mathcal{U} \setminus \mathcal{T}_P$ which is open in $\overline{P} \setminus P$. Then there exists an open set $V \subset X$ such that $V \cap (\overline{P} \setminus P) = U$. Let $W = V \cup P$. Consider the filter $\mathcal{Z}$ generated by the family $\mathcal{F} \cup \{W\}$. Since the set W is open we get that $\mathcal{Z}$ is an open filter. By the definition of $\mathcal{Z}$, $\mathcal{F} \subsetneqq \mathcal{Z} \subsetneqq \mathcal{H}$ which implies a contradiction.

4. Since $P \in \mathcal{H}$, the filter $\mathcal{Z}$ generated by the family $\{F \cap (\overline{H} \setminus P) : F \in \mathcal{F}, H \in \mathcal{H}\}$ contains the filter $\mathcal{T}_P$. By Statement 3, the filter $\mathcal{T}_P$ is an open ultrafilter on $\overline{P} \setminus P$. So, it suffices to check that the filter $\mathcal{Z}$ is open on $\overline{P} \setminus P$. For this fix any element $Z \in \mathcal{Z}$. With no loss of generality we can assume that $Z = F \cap (\overline{H} \setminus P)$ for some open sets $F \in \mathcal{F}$ and $P \supset H \in \mathcal{H}$. Let $V \in \mathcal{F}$ be any open set such that $V \subset F$ and $V \subset \overline{H}$, which exists by Lemma 2.6. Put $Z_1 = V \cap Z \subset Z$. The choice of V implies that

$$Z_1 = V \cap F \cap (\overline{H} \setminus P) = V \cap (\overline{H} \setminus P) = V \cap (\overline{P} \setminus P).$$

It follows that Z_1 is open in $\overline{P} \setminus P$ and $Z_1 \in \mathcal{Z}$. Thus, $\mathcal{T}_P$ is generated by the family $\{F \cap (\overline{H} \setminus P) : F \in \mathcal{F}, H \in \mathcal{H}\}$.

5. Let $\mathcal{Z}$ be the filter generated by the family $\{F \cap (\overline{H} \setminus H) : F \in \mathcal{F}, P \supset H \in \mathcal{H}\}$. Statement 4 implies that $\mathcal{Z} \subset \mathcal{T}_P$. Fix any $T \in \mathcal{T}_P$. Pick any $H \in \mathcal{H}$ such that $H \subset P$ and put $F = H \cup T$. By Statement 2, the set F belongs to $\mathcal{F}$. Then the set

$$Z = F \cap (\overline{H} \setminus H) = (H \cup T) \cap (\overline{H} \setminus H) = T \cap (\overline{H} \setminus H)$$

belongs to $\mathcal{Z}$ and $Z \subset T$. Hence $\mathcal{Z} = \mathcal{T}_P$.

6. Assume that for any open filter $\mathcal{G}$ on X the inclusion $\mathcal{F} \subsetneq \mathcal{G}$ implies that $\mathcal{H} \subset \mathcal{G}$. Consider the filter $\mathcal{Z}$ which is generated by the family $\{U \subset X : U \text{ is open and } U \cap (\overline{P} \setminus P) \in \mathcal{T}_P\}$. Obviously the filter $\mathcal{Z}$ is open and $\mathcal{F} \subset \mathcal{Z}$. If $\mathcal{Z} \neq \mathcal{F}$, the assumption of the statement implies that $\mathcal{H} \subset \mathcal{Z}$. But this is not possible, since $P \notin \mathcal{Z}$. The obtained contradiction provides that $\mathcal{Z} = \mathcal{F}$. □

3. Linearly ordered finite lattices of free open filters

In what follows ordinals are assumed to carry the discrete topology, if the converse not stated. The Čech-Stone compactification of a space X is denoted by $\beta(X)$. For a cardinal κ we shall identify a point $\alpha \in \kappa \subset \beta(\kappa)$ with the corresponding principal ultrafilter. A space X is called *scattered* if each subspace of X contains an isolated point. A *height* of a scattered space X is the minimal ordinal $ht(X)$ such that the $ht(X)$-th Cantor-Bendixson derivative of X is empty. Recall that Cantor-Bendixson derivatives of a scattered space X is defined by transfinite induction as follows, where X' is the set of all limit accumulation? points of X:

- $X^0 = X$;
- $X^{\alpha+1} = \big(X^\alpha\big)'$;
- $X^\alpha = \cap_{\beta<\alpha} X^\beta$, if α is limit ordinal.

The set $X^\alpha \setminus X^{\alpha+1}$ is called a *Cantor-Bendixson level* of X and is denoted by $X^{(\alpha)}$.

A subset X of a space Y is called *strongly discrete* if there exists a family $(U_x)_{x\in X}$ of open pairwise disjoint subsets of Y such that $U_x \cap X = \{x\}$ for every $x \in X$. The following lemma is a folklore. Nonetheless, we give an easy proof of it.

Lemma 3.1. *Let $X = \{x_\alpha\}_{\alpha\in\lambda}$ be a strongly discrete subset of $\beta(\kappa)$. Then the space $\overline{X}$ is homeomorphic to $\beta(\lambda)$.*

Proof. Since the set X is strongly discrete there exists a family $\{A_\alpha : \alpha \in \lambda\}$ of open pairwise disjoint subsets of $\beta(\kappa)$ such that $X \cap A_\alpha = \{x_\alpha\}$ for any $\alpha \in \lambda$. Since the space $\beta(\kappa)$ is zero-dimensional, with no loss of generality we can assume that A_α is closed for each $\alpha \in \lambda$. Fix any function $f : X \to [0, 1]$. Define the function $f_1 : \kappa \to [0, 1]$ as follows:

$$f_1(\xi) = \begin{cases} f(x_\alpha), & \text{if } \xi \in A_\alpha \cap \kappa; \\ 0, & \text{if } \xi \in \kappa \setminus (\cup_{\alpha\in\lambda} A_\alpha). \end{cases}$$

By the definition of $\beta(\kappa)$, there exists a continuous extension $\hat{f}_1 : \beta(\kappa) \to [0,1]$ of f_1. Let $g = \hat{f}_1|_{\overline{X}}$. Since $\hat{f}_1(x_\alpha) = f(x_\alpha)$, g is a continuous extension of f. Hence every function $f : X \to [0,1]$ can be extended to a continuous function $g : \overline{X} \to [0,1]$. Corollary 3.6.3 from [13] implies that $\overline{X}$ is homeomorphic to $\beta(X)$. Since X is a discrete set of cardinality λ, the space $\overline{X}$ is homeomorphic to $\beta(\lambda)$. □

Let ρ be an equivalence relation on a set X. Then for each $x \in X$ such that $x\rho y$ iff $x = y$ we agree to I think "agree to" is redundant. denote the equivalence class $\{x\}$ simply by x. Recall that for a point x of a space X by $\mathcal{N}(x)$ we denote the filter generated by open neighborhoods of x. The following scheme was invented by Mooney [23] and it will be crucial in constructing almost H-closed spaces with certain properties of their lattices of free open filters.

Construction 1. Let X be a non-H-closed topological space and x^* be any non-isolated point of X. By $\mathbf{U}$ we denote the set of all free open ultrafilters on X. Let $Y = X \cup \mathbf{U}$ endowed with the topology τ defined as follows:

- X is an open subspace of Y;
- open neighborhood base at a point $\mathcal{F} \in Y \setminus X$ consists of the sets $F \cup \{\mathcal{F}\}$, where $F \in \mathcal{F}$ and F is open in X.

One can see that Y is homeomorphic to the Katetov extension [29] of X. Therefore Y is H-closed.

Consider the Tychonoff product $Y \times \{0,1\}$ where the set $\{0,1\}$ carries the discrete topology. Let Z be the quotient space $Y \times \{0,1\}/_\rho$ where the equivalence relation ρ is defined as follows: $(a,i) \sim (b,j)$ iff $a = b$ and $i = j$, or $a = b \in \mathbf{U}$. Finally, put $\mathrm{M}(X, x^*) = Z \setminus \{(x^*, 0)\}$. One can easily check that the extension $\mathrm{M}(X, x^*)$ is scattered if and only if X is scattered.

Lemma 3.2. *Let $\mathcal{F}, \mathcal{G}$ be filters on a set X and $\mathcal{A}$ a finite family of subsets of X such that $\cup\mathcal{A} \in \mathcal{F} \cap \mathcal{G}$. If for each $A \in \mathcal{A}$ traces of the filters $\mathcal{F}$ and $\mathcal{G}$ on A coincide, then $\mathcal{F} = \mathcal{G}$.*

Proof. Fix any $F \in \mathcal{F}$ and for each $A \in \mathcal{A}$ let $F_A = F \cap A$. By the assumption, for every $A \in \mathcal{A}$ there exists $\cup\mathcal{A} \supset G_A \in \mathcal{G}$ such that $G_A \cap A \subset F_A$. Then $\cap_{A \in \mathcal{A}} G_A \in \mathcal{G}$ and $(\cap_{A \in \mathcal{A}} G_A) \subset \cup_{A \in \mathcal{A}} F_A \subset F$, witnessing that $\mathcal{F} \subset \mathcal{G}$. Similarly one can show the converse inclusion. □

The following theorem provides an affirmative answer to Questions 1, 2 and a partial answer to Problem 2.5.

Theorem 3.3. *For each $n \in \mathbb{N}$ there exists a scattered space M such that $\mathbf{OF}(M)$ is an n-element chain.*

Proof. Fix any $n \in \mathbb{N}$. We divide our proof into two steps. At the first step we construct a desired space M. At the second step we will show that $\mathbf{OF}(M)$ is an n-element chain.

Step 1. We start with the construction of a scattered subspace $X \subset \beta(\omega)$ such that $\omega \subset X$, $ht(X) = n+1$, $X^{(n)}$ consists of one point which we denote by x^* and X satisfies the following condition:

(∗) for each $m < n$ the family $\{U \cap X^{(m)} : U \in \mathcal{N}(x^*)\}$ is an ultrafilter on $X^{(m)}$.

Let $X_0 = \omega$. Assume that the strongly discrete subset $X_i \subset \beta(\omega)$ is already constructed for some $i \in n-1$. Then put X_{i+1} be any countable infinite discrete subset of $\overline{X_i} \setminus X_i$. The countability of X_{i+1} provides that the set X_{i+1} is strongly discrete in $\beta(\omega)$. This way we construct the sets X_i for $i \leq n-1$. Finally, fix any point $x^* \in \overline{X_{n-1}} \setminus X_{n-1}$ and let X be the set $\cup_{i \in n} X_i \cup \{x^*\} \subset \beta(\omega)$ endowed with the subspace topology. One can easily check that X is scattered, $X^{(i)} = X_i$ for $i < n$ and $X^{(n)} = \{x^*\}$. It follows that $ht(X) = n+1$. To check condition (∗) fix any $m < n$. We can assume that $m > 0$, because condition (∗) obviously holds in the case $m = 0$. Since X_m is a countable strongly discrete subset of

$\beta(\omega)$, Lemma 3.1 implies that the subspace $\overline{X^{(m)}}$ is homeomorphic to $\beta(\omega)$. Since $x^* \in \overline{X^{(m)}}$ we obtain that the family $\{U \cap X^{(m)} : U \in \mathcal{N}(x^*)\}$ is a base of some ultrafilter on $X^{(m)}$.

Finally, let $M = \mathrm{M}(X, x^*)$. Since the space X is scattered, Construction 1 implies the space M is scattered as well.

Step 2. Let us check that $\mathbf{OF}(M)$ is an n-element chain. For this fix any free open filter $\mathcal{F}$ on M. By Construction 1, the subspace $M \setminus ((X \setminus \{x^*\}) \times \{0\})$ is H-closed. Hence $(X \setminus \{x^*\}) \times \{0\} \in \mathcal{F}$. Note that $\omega \times \{0\}$ is a dense open discrete subspace of $(X \setminus \{x^*\}) \times \{0\}$. Thus $F \cap (\omega \times \{0\}) \neq \emptyset$ for each $F \in \mathcal{F}$. Observe that $\{H \times \{0\} : H \in x^*\}$ is the only free in M open filter on $\omega \times \{0\}$. Therefore, the filter $\mathcal{F}$ traces on $\omega \times \{0\}$ the mentioned above ultrafilter.

Let k be the maximum integer such that $F \cap (X^{(k)} \times \{0\}) \neq \emptyset$ for any $F \in \mathcal{F}$. For $m < k$ by Φ_m we denote the filter on $X^{(m)} \times \{0\}$ which is generated by the family $\{(U \cap X^{(m)}) \times \{0\} : U \in \mathcal{N}(x^*)\}$. We claim that for each $m \leq k$ the trace of the filter $\mathcal{F}$ on the set $X^{(m)} \times \{0\}$ coincides with Φ_m. By condition $(*)$, for any $m \leq k$ the filter Φ_m is an ultrafilter. Hence it is suffices to show that $\Phi_m \subset \{F \cap (X^{(m)} \times \{0\}) : F \in \mathcal{F}\}$ for any $m \leq k$. For this, fix any $m \leq k$ and $(U \cap X^{(m)}) \times \{0\} \in \Phi_m$, where $U \in \mathcal{N}(x^*)$. With no loss of generality we can assume that U is a basic open neighborhood of x^*, that is an ultrafilter $x \in X$ belongs to U iff $U \cap \omega \in x$. Since the trace of $\mathcal{F}$ on $\omega \times \{0\}$ coincides with $\{H \times \{0\} : H \in x^*\}$ there exists an open set $F \in \mathcal{F}$ such that $F \cap (\omega \times \{0\}) \subset (U \cap \omega) \times \{0\}$. Let us show that $F \cap (X^{(m)} \times \{0\}) \subset (U \cap X^{(m)}) \times \{0\}$. Indeed, take any point $(z, 0) \in (F \cap (X^{(m)} \times \{0\}))$. By the definition of the space X, z is an ultrafilter on ω. Since the set F is open and taking into account the definition of topology on X, we get that $U \cap \omega$ is an element of the ultrafilter z. Consequently, $(z, 0) \in U \times \{0\}$, by the choice of U. Hence $F \cap (X^{(m)} \times \{0\}) \subset (U \cap X^{(m)}) \times \{0\}$, witnessing that $\Phi_m = \{F \cap (X^{(m)} \times \{0\}) : F \in \mathcal{F}\}$ for each $m \leq k$.

By $\Phi^{\leq k}$ we denote the open filter on M generated by the family $\{(U \cap (\cup_{i \leq k} X^{(i)})) \times \{0\} : U \in \mathcal{N}(x^*)\}$. By the arguments above, for each $i \leq k$ traces of the filters $\mathcal{F}$ and $\Phi^{\leq k}$ on $X^{(i)} \times \{0\}$ coincide. Taking into account that $\cup_{i \leq k}(X^{(i)} \times \{0\}) \in \mathcal{F} \cap \Phi^{\leq k}$, Lemma 3.2 implies that $\mathcal{F} = \Phi^{\leq k}$.

To sum up, the space M admits exactly n distinct free open filters, namely, $\Phi^{\leq k}$, $k \in \{0, 1, \ldots, n-1\}$. Moreover, $\Phi^{\leq k} \subset \Phi^{\leq m}$ iff $m \leq k$. Hence $\mathbf{OF}(M)$ is an n-element chain. □

Note that Remark 1.7 from [27] implies that each non-compact regular space possesses at least ω_1 free open filters. So, after constructing for each positive integer n a space which admits exactly n free open filters, it is natural to ask whether there exists a space which possesses κ many free open filters for a given cardinal κ.

Assume that for each $\alpha \in \kappa$, $\mathcal{F}_\alpha$ is a filter on a set X_α. Then for a set $A \subset \kappa$ by $\wedge_{\alpha \in A} \mathcal{F}_\alpha$ we denote the filter on the set $\bigcup_{\alpha \in A} X_\alpha$ generated by the family $\{\bigcup_{\alpha \in A} F_\alpha : F_\alpha \in \mathcal{F}_\alpha\}$. The following proposition affirmatively answers the above question. Nothing about almost H-closed is possible to state?

Proposition 3.4. *For each cardinal κ there exists a space Y which possesses exactly κ-many free open filters.*

Proof. If $\kappa < \omega$, then the proof follows from Theorem 3.3. So, fix any infinite cardinal κ. For each $\alpha \in \kappa$ let X_α be a space admitting a unique free open filter $\mathcal{F}_\alpha$, which exists by Theorem 3.3. Let Y be the disjoint union $\sqcup_{\alpha \in \kappa} X_\alpha \sqcup \{z\}$ endowed with the topology τ satisfying the following conditions:

- X_α is an open subspace of (Y, τ) for each $\alpha \in \kappa$;
- open neighborhood base at z consists of the sets $Y \setminus (\cup_{\alpha \in A} X_\alpha)$, where $A \in [\kappa]^{<\omega}$, i.e., A is a finite subset of κ.

We claim that the set of all free open filters on the space Y coincides with the set $\{\wedge_{\alpha \in A} \mathcal{F}_\alpha : A \in [\kappa]^{<\omega}\}$ which clearly has cardinality κ. By the definition of the topology on Y, any filter of the form $\wedge_{\alpha \in A} \mathcal{F}_\alpha$, $A \in [\kappa]^{<\omega}$ is open and free. So, it remains to show the converse inclusion. For this fix any free open filter $\mathcal{F}$ on the space Y. Since $\mathcal{F}$ is free there exists $F \subset \mathcal{F}$ such that $z \notin \overline{F}$. It follows that there exists a finite subset $A \subset \kappa$ such that $F \subset \cup_{\alpha\in A} X_\alpha$. Let

$$B = \{\alpha \in \kappa : F \cap X_\alpha \neq \emptyset \quad \text{for each } F \in \mathcal{F}\}.$$

The arguments above imply that $B \subset A$. The set B is nonempty, because otherwise there would exist elements $F_\alpha \in \mathcal{F}$, $\alpha \in A$ such that $F_\alpha \subset \bigcup_{\alpha\in A} X_\alpha$ and $F_\alpha \cap X_\alpha = \emptyset$ providing that $\emptyset = \cap_{\alpha\in A} F_\alpha \in \mathcal{F}$ which is impossible. Since for each $\alpha \in B$ the space X_α possesses the unique free open filter $\mathcal{F}_\alpha$ we obtain that for each $\alpha \in B$ the trace of the filter $\mathcal{F}$ on X_α coincides with $\mathcal{F}_\alpha$. Since the subspaces X_α, $\alpha \in \kappa$ are clopen and pairwise disjoint it is straightforward to check that $\mathcal{F} = \wedge_{\alpha\in B}\mathcal{F}_\alpha$. □

A *character* of a filter $\mathcal{F}$ is the smallest cardinal κ such that $\mathcal{F}$ possesses a base of size κ. A subset A of a poset P is called an *antichain* if $a \not\leq b$ for any two distinct elements $a, b \in A$. The next proposition shows that a lattice of free open filters can contain arbitrary large chains and antichains. For typographical reasons we denote 2^κ as $\exp(\kappa)$.

Proposition 3.5. *For every cardinal κ there exists a space X such that the lattice $\mathbf{OF}(X)$ contains a chain of length $\exp(\exp(2^\kappa))^+$ and an antichain of cardinality $\exp(\exp(\exp(2^\kappa)))$.*

Proof. Let κ be any infinite cardinal and $K(\kappa)$ the Katetov extension of the discrete space κ. Fix any ultrafilter $\mathcal{U}$ on κ such that every element of $\mathcal{U}$ has cardinality κ. For instance, one can consider any ultrafilter which contains the filter $\{S \subset \kappa : |\kappa \setminus S| < \kappa\}$. By X we denote the subspace $K(\kappa) \setminus \{\mathcal{U}\} \subset K(\kappa)$. Since κ is a dense discrete subspace of X, $\mathcal{U}$ is the unique free open ultrafilter on X. Corollary 2.7 implies that the filter $\mathcal{W}$ generated by the family $\{\mathrm{Int}(\overline{U}) : U \in \mathcal{U}\}$ is the infimum of $\mathbf{OF}(X)$. Set $X^* = X \setminus \kappa$. Let $\mathcal{G}$ is the trace of the filter $\mathcal{W}$ on the set X^*. Clearly, the character of $\mathcal{U}$ is at most 2^κ. It follows that the character of the filter $\mathcal{G}$ is $\leq 2^\kappa$ as well. It is easy to see that the closure of every subset $A \in [\kappa]^\kappa$ in $K(\kappa)$ is open and homeomorphic to $K(\kappa)$. It follows that every element $G \in \mathcal{G}$ has cardinality $\exp(2^\kappa)$. The latter two facts imply that $\mathcal{G}$ is not an ultrafilter on $X \setminus \kappa$ and the set $\mathbf{A}$ of all ultrafilters on $X \setminus \kappa$ which contain $\mathcal{G}$ has cardinality $\exp(\exp(\exp(2^\kappa)))$. Maybe we need here more explanations? It is straightforward to check that for each $\mathcal{F} \in \mathbf{A}$ the filter $\mathcal{T}_\mathcal{F}$ on X generated by the family $\{F \cup U : F \in \mathcal{F}, U \in \mathcal{U}\}$ is open and free. Then the family $\{\mathcal{T}_\mathcal{F} : \mathcal{F} \in \mathbf{A}\} \subset \mathbf{OF}(X)$ forms an antichain of cardinality $\exp(\exp(\exp(2^\kappa)))$. Pick any $\mathcal{F} \in \mathbf{A}$ such that $\{S \subset X^* : |X^* \setminus S| < \exp(2^\kappa)\}$. By the choice of $\mathcal{F}$, every element of $\mathcal{F}$ has cardinality $\geq \exp(2^\kappa)$. Let λ is a character of $\mathcal{F}$. Clearly, check?$\lambda \geq \exp(2^\kappa)^+$. Fix any base $\mathcal{B}_\mathcal{F} = \{B_\alpha : \alpha \in \lambda\}$ of the filter $\mathcal{F}$ such that $B_\alpha \notin \mathcal{G}$ for each $\alpha \in \lambda$. Fix any base $\mathcal{B}_\mathcal{G}$ of $\mathcal{G}$ of cardinality $\leq 2^\kappa$. For every $\alpha \in \lambda$ there exists $\theta(\alpha)$ such that $B_{\theta(\alpha)}$ does not belong to the filter generated by the family $\mathcal{B}_\mathcal{G} \cup \{B_\xi : \xi \in \alpha\}$. The ordinal $\theta(\alpha)$ is well-defined, because $|\mathcal{B}_\mathcal{G} \cup \{B_\xi : \xi \in \alpha\}| < \lambda$. Let $\delta(\alpha) = \sup\{\theta(\alpha), \theta(\theta(\alpha)), \ldots, \theta^n(\alpha), \ldots\}$. Clearly, $B_{\delta(\alpha)}$ does not belong to the filter generated by the family $\mathcal{B}_\mathcal{G} \cup \{B_\xi : \xi \in \delta(\alpha)\}$. For each $\alpha \in \lambda$ let $\mathcal{H}_\alpha$ be the filter on X^* generated by the set $\{B_\xi : \xi \in \delta(\alpha)\} \cup \mathcal{B}_\mathcal{G}$. Then for every $\alpha \in \lambda$, the filter $\mathcal{F}_\alpha$ on X generated by the family $\{H \cup U : H \in \mathcal{H}_\alpha, U \in \mathcal{U}\}$ is free and open. Moreover, $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ if and only if $\alpha \in \beta$. Hence the set $\{F_\alpha : \alpha \in \lambda\} \subset \mathbf{OF}(X)$ is a chain of cardinality $\lambda \geq \exp(2^\kappa)^+$. □

4. Finite nonlinear lattices of free open filters

Assume that for each $n \in \omega$, $\mathcal{F}_n$ is a filter on a set X_n. Then by $\prod_{i\in n} \mathcal{F}_i$ we denote the filter on the set $\prod_{i\in n} X_i$ generated by the family $\{\prod_{i\in n} F_i : F_i \in \mathcal{F}_i, i \in n\}$. If $n = 2$, then the product of filters $\mathcal{F}_0$ and $\mathcal{F}_1$ is denoted by $\mathcal{F}_0 \times \mathcal{F}_1$.

Recall that an ultrafilter $\mathcal{F}$ on a cardinal κ is called *κ-complete* if for any $\lambda \in \kappa$ and subfamily $\{F_\alpha : \alpha \in \lambda\} \subset \mathcal{F}$ the set $\cap_{\alpha\in\lambda} F_\alpha$ belongs to $\mathcal{F}$. An uncountable cardinal κ which possesses a free

κ-complete ultrafilter is called *measurable*. The existence of a measurable cardinal is consistent with ZFC. I think that the latter sentence should be restated.

The following result was proved by Blass in his PhD-thesis.

Lemma 4.1. *For an ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on sets X and Y, respectively, the following conditions are equivalent:*

- *$\mathcal{F}\times\mathcal{G}$ is an ultrafilter;*
- *for every function $f : X \to \mathcal{G}$ there exists $F \in \mathcal{F}$ such that $\cap_{x\in F} f(x) \in \mathcal{G}$.*

Let $\{\kappa_i : 1 \leq i \leq n\}$ be an increasing sequence of measurable cardinals. For any $i \geq 1$ fix any free κ_i-complete ultrafilter $\mathcal{F}_i$ on κ_i. Let $\kappa_0 = \omega$ and U_0 be any free ultrafilter on ω. For each $0 < m \leq n$ by $\mathcal{U}_m$ we denote the filter $\mathcal{U}_{m-1}\times\mathcal{F}_m$. Lemma 4.1 implies the following.

Corollary 4.2. *For any $m \leq n$, $\mathcal{U}_m$ is an ultrafilter.*

Proof. Assume that for some $m < n$, $\mathcal{U}_m$ is an ultrafilter. Observe that $|\prod_{i\leq m} \kappa_i| = \kappa_m < \kappa_{m+1}$ and the filter $\mathcal{F}_{m+1}$ is κ_{m+1}-complete. Therefore, Lemma 4.1 implies that $\mathcal{U}_{m+1} = \mathcal{U}_m\times\mathcal{F}_{m+1}$ is an ultrafilter. □

Lemma 4.3. *Let κ be a measurable cardinal, $\mathcal{D} = \{x_\alpha : \alpha \in \kappa\}$ a strongly discrete subset of $\beta(\kappa)$, $x \in \overline{\mathcal{D}} \setminus \mathcal{D}$ and $\phi : \overline{\mathcal{D}} \to \beta(\kappa)$ a homeomorphism. If the ultrafilters x_α, $\alpha \in \kappa$ are κ-complete and $\phi(x)$ is κ-complete, then x is κ-complete.*

Proof. Let us note that the homeomorphism ϕ exists by Lemma 3.1. Since ϕ is a homeomorphism we deduce that $\phi(\mathcal{D}) = \kappa$. To derive a contradiction, assume that x is not κ-complete. Then there exists a cardinal $\lambda < \kappa$ and a family $\{F_\xi : \xi \in \lambda\} \subset x$ such that $\cap_{\xi\in\lambda} F_\xi = \emptyset$. For each $\xi \in \kappa$ let $X_\xi = \{x_\alpha : F_\xi \in x_\alpha\}$. Since $x \in \overline{\{x_\alpha : \alpha \in \kappa\}}$ we deduce the the sets X_ξ are nonempty. Taking into account that ϕ is a homeomorphism, it is easy to see that $\phi(X_\xi) \in \phi(x)$ for each $\xi \in \lambda$. Since the filter $\phi(x)$ is κ-complete the set $\cap_{\xi\in\lambda}\phi(X_\xi)$ is nonempty and belongs to $\phi(x)$. Pick any

$$x_\gamma \in \phi^{-1}(\cap_{\xi\in\lambda}\phi(X_\xi)) = \cap_{\xi\in\lambda}\phi^{-1}(\phi(X_\xi)) = \cap_{\xi\in\lambda}X_\xi.$$

Note that $F_\xi \in x_\gamma$ for each $\xi \in \lambda$. Since the filter x_γ is κ-complete the set $\cap_{\xi\in\lambda}F_\xi$ belongs to x_γ and hence it is nonempty, which contradicts the assumption. □

The next lemma provides a similar construction to one of Step 1 in the proof of Theorem 3.3.

Lemma 4.4. *Let κ be a measurable cardinal. Then for each positive integer n there exists a scattered subspace $S \subset \beta(\kappa)$ which consists of κ-complete ultrafilters and satisfies the following conditions:*

- *$\kappa \subset S$, $ht(S) = n + 1$ and $S^{(n)} = \{x^*\}$;*
- ($*_\kappa$) *for each $m < n$ the family $\{U \cap S^{(m)} : U \in \mathcal{N}(x^*)\}$ is a base of some κ-complete ultrafilter on $S^{(m)}$.*

Proof. Let $S_0 = \kappa$. Assume that for some $i < n - 1$ we constructed a strongly discrete set $S_i = \{x_\alpha : \alpha \in \kappa\} \subset \beta(\kappa)$ of cardinality κ which consists of κ-complete ultrafilters. Next we shall construct a strongly discrete subset S_{i+1}. Decompose κ into pairwise disjoint subsets K_ξ, $\xi \in \kappa$ which satisfy the following conditions: $|K_\xi| = \kappa$ and $\cup_{\xi\in\kappa}K_\xi = \kappa$. For each $\xi \in \kappa$ let $Y_\xi = \{x_\alpha : \alpha \in K_\xi\}$. Lemma 3.1 implies that for each $\xi \in \kappa$ there exists a homeomorphism $\phi_\xi : \overline{Y_\xi} \to \beta(\kappa)$. Pick any free κ-complete ultrafilter $\mathcal{F}$ on κ. Let $S_{i+1} = \{\phi_\xi^{-1}(\mathcal{F}) : \xi \in \kappa\}$. Since S_i consists of κ-complete ultrafilters, Lemma 4.3 implies that S_{i+1} also consists of κ-complete ultrafilters. Let us show that S_{i+1} is strongly discrete. Since the set S_i is strongly discrete there exist pairwise disjoint sets $F_\alpha \in x_\alpha$, $\alpha \in \kappa$. Note that for each $\xi \in \kappa$ the set $H_\xi = \cup_{\alpha\in K_\xi}F_\alpha$ belongs to $\phi_\xi^{-1}(\mathcal{F})$. Since the sets K_ξ, $\xi \in \kappa$ are pairwise disjoint we get that the sets H_ξ are pairwise disjoint as well, witnessing that the set S_{i+1} is strongly discrete. This way we can construct the sets S_i for $i \leq n-1$. Similarly, let x^* be $\phi^{-1}(\mathcal{F})$, where $\phi : \overline{S_{n-1}} \to \beta(\kappa)$ is any homeomorphism (which exists by Lemma 3.1) and $\mathcal{F}$ be a free κ-complete ultrafilter on κ. Lemma 4.3 ensures that the ultrafilter x^* is κ-complete. Let S be the subspace $\cup_{i\in n} S_i \cup \{x^*\}$ of $\beta(\kappa)$. One can easily check that S is scattered, $S^{(i)} = S_i$ for $i < n$ and $S^{(n)} = \{x^*\}$. It follows that $ht(S) = n+1$. To check condition $(*_\kappa)$ fix any $m < n$. We can assume that $m > 0$, because the ultrafilter x^* is κ-complete and, therefore, condition $(*_\kappa)$ obviously holds in the case $m = 0$. Since S_m is strongly discrete in $\beta(\kappa)$, Lemma 3.1 implies that the subspace $\overline{S^{(m)}}$ is homeomorphic to $\beta(\kappa)$. Since $x^* \in \overline{S^{(m)}}$ we obtain that the family $\{U \cap S^{(m)} : U \in \mathcal{N}(x^*)\}$ is a base of some ultrafilter $\mathcal{U}$ on $S^{(m)}$. Finally, the κ-completeness of x^* yields that $\mathcal{U}$ is κ-complete as well. □

Recall that each natural number $n = \{0, \dots, n-1\}$ carries a natural partial order $\leq$ which turns n into the n-element chain.

Theorem 4.5. *If there exists an increasing sequence of measurable cardinals $\{\kappa_i : 1 \leq i \leq n-1\}$, then for each sequence $m_0, \dots, m_{n-1}$ of natural numbers there exists a Hausdorff space M such that the lattice $\mathbf{OF}(M)$ is order isomorphic to $\prod_{i\in n} m_i$.*

Proof. Fix any $n > 1$ and an increasing sequence of measurable cardinals $\{\kappa_i : 1 \leq i \leq n-1\}$. Additionally set $\kappa_0 = \omega$. Similarly as in the proof of Theorem 3.3 we shall divide our proof into two steps. At first we construct a space M and then describe the lattice $\mathbf{OF}(M)$.

Step 1. Let S_0 be a scattered subspace of $\beta(\omega)$ of height $m_0 + 1$ such that $S_0^{(m_0)} = \{x_0^*\}$, $\omega \subset S_0$ and S_0 satisfies condition $(*)$ (see Step 1 in the proof of Theorem 3.3). For each $1 \leq i < n$ let S_i be the scattered subspace of $\beta(\kappa_i)$ of height $m_i + 1$ which was constructed in Lemma 4.4. In particular, $\kappa_i \subset S_i$, $S_i^{(m_i)} = \{x_i^*\}$, S_i consists of κ_i-complete ultrafilters and satisfies condition $(*_{\kappa_i})$. For $i \in n$ put

$$E_i = \{x_0^*\} \times \ldots \times \{x_{i-1}^*\} \times S_i \times \{x_{i+1}^*\} \times \ldots \times \{x_{n-1}^*\}.$$

By X we denote the subspace $(\bigcup_{i\in n} E_i) \cup \prod_{i\in n} \kappa_i$ of the Tychonoff product $\prod_{i\in n} S_i$. Note that by Corollary 4.2, the Tychonoff product $\prod_{i\in n} S_i$ can be naturally identified with a subspace of $\beta(\prod_{i\in n} \kappa_i)$. Namely, a point $y = (y_0, \dots, y_{n-1})$ we can identify with the ultrafilter $\phi(y) = \prod_{i\in n} y_i$ on $\prod_{i\in n} \kappa_i$. Moreover, the definition of the topology on X provides that ϕ is an embedding of X into $\beta(\prod_{i\in n} \kappa_i)$. In particular, it follows that for each $x \in X$ the filter $\mathcal{N}(x)$ traces on $\prod_{i\in n} \kappa_i$ an ultrafilter. For convenience we shall denote the point $(x_0^*, \dots, x_{n-1}^*) \in X$ by x^*.

Finally, let $M = \mathrm{M}(X, x^*)$ (see Construction 1).

Step 2. Let us check that $\mathbf{OF}(M)$ is order isomorphic to $\prod_{i\in n} m_i$. For this fix any free open filter $\mathcal{F}$ on M. By Construction 1, the subspace $M \setminus ((X \setminus \{x^*\}) \times \{0\})$ is H-closed. Hence $(X \setminus \{x^*\}) \times \{0\} \in \mathcal{F}$. Note that $\prod_{i\in n} \kappa_i \times \{0\}$ is a dense open discrete subspace of $(X \setminus \{x^*\}) \times \{0\}$. Thus $F \cap (\prod_{i\in n} \kappa_i \times \{0\}) \neq \emptyset$ for each $F \in \mathcal{F}$. The definition of X implies that the trace of $\mathcal{N}(x^*)$ on $\prod_{i\in n} \kappa_i$ coincide with the filter $\prod_{i\in n} x_i^*$. Moreover, Corollary 4.2 implies that $\prod_{i\in n} x_i^*$ is an ultrafilter. By the definition of M, the ultrafilter $\mathcal{U}$ generated by the family $\{F \times \{0\} : F \in \prod_{i\in n} x_i^*\}$ is the unique free open filter on $\prod_{i\in n} \kappa_i \times \{0\}$. Therefore, the filter $\mathcal{F}$ traces on $\prod_{i\in n} \kappa_i \times \{0\}$ the ultrafilter $\mathcal{U}$.

For each $i \in n$ and $m \in m_i$ put

$$\hat{E}_i^{(m)} = \{x_0^*\} \times \ldots \times \{x_{i-1}^*\} \times S_i^{(m)} \times \ldots \times \{x_{n-1}^*\},$$

and by Φ_i^m we denote the filter generated by the family

$$\{(U \cap \hat{E}_i^{(m)}) \times \{0\} : U \in \mathcal{N}(x^*)\}.$$

For each $i \in n$ let k_i be the minimum integer such that there exists $F \in \mathcal{F}$, $F \cap (\hat{E}_i^{(m)} \times \{0\}) = \emptyset$. Clearly, $0 \leq k_i \leq m_i$, $i \in n$. Let $K = \{i \in n : k_i \neq 0\}$. Note that if $K = \emptyset$, then the filter $\mathcal{F}$ equals to the filter $\mathcal{U}$.

We claim that for each $i \in K$ and $m \in k_i$ the trace of the filter $\mathcal{F}$ on the set $\hat{E}_i^{(m)} \times \{0\}$ coincides with Φ_i^m. Fix any $i \in K$ and $m \in k_i$. By condition $(*_{\kappa_i})$, the filter Φ_m^i is an ultrafilter. Hence it is suffices to show that Φ_i^m is contained in the trace of $\mathcal{F}$ on the set $\hat{E}_i^{(m)} \times \{0\}$. For this, fix any $(U \cap \hat{E}_i^{(m)}) \times \{0\} \in \Phi_i^m$, where $U \in \mathcal{N}(x^*)$ is a basic open neighborhood of x^*. Since the trace of $\mathcal{F}$ on $\prod_{i \in n} \kappa_i \times \{0\}$ equals to the filter $\{H \times \{0\} : H \in \prod_{i \in n} x_i^*\}$ there exists an open set $F \in \mathcal{F}$ such that $F \cap (\prod_{i \in n} \kappa_i \times \{0\}) \subset (U \cap \prod_{i \in n} \kappa_i) \times \{0\}$. Let us show that $F \cap (\hat{E}_i^{(m)} \times \{0\}) \subset (U \cap \hat{E}_i^{(m)}) \times \{0\}$. Indeed, take any point

$$(x_0^*, \dots, x_{i-1}^*, z, x_{i+1}^*, \dots, x_{n-1}^*, 0) \in F \cap (\hat{E}_i^{(m)} \times \{0\}).$$

By the definition of the space S_i, z is an ultrafilter on κ_i. Since the set F is open and taking into account the definition of topology on X, we get that the projection on the i-th factor of the set $U \cap \prod_{i \in n} \kappa_i$ is an element of the ultrafilter z. Then, by the definition of the topology on X,

$$(x_0^*, \dots, x_{i-1}^*, z, x_{i+1}^*, \dots, x_{n-1}^*, 0) \in (U \cap \hat{E}_i^{(m)}) \times \{0\}.$$

Hence $F \cap (\hat{E}_i^{(m)} \times \{0\}) \subset (U \cap \hat{E}_i^{(m)}) \times \{0\}$, witnessing that the filter Φ_i^m coincides with the trace of the filter $\mathcal{F}$ on the set $\hat{E}_i^{(m)} \times \{0\}$ for each $i \in K$ and $m \in k_i$.

For $\vec{v} = (v_0, \dots, v_{n-1}) \in \prod_{i \in n} m_i$ let $K_{\vec{v}} = \{i \in n : v_i \neq 0\}$. By $\Phi_{\vec{v}}$ we denote the filter generated by the family

$$\{(U \cap (\bigcup_{i \in K_{\vec{v}}} \bigcup_{j \in v_i} \hat{E}_i^{(j)} \cup \prod_{i \in n} \kappa_i)) \times \{0\} : U \in \mathcal{N}(x^*)\}.$$

By the arguments above, for each $i \in K_{\vec{v}}$ and $j \in v_i$ traces of the filters $\mathcal{F}$ and $\Phi_{\vec{v}}$ on the set $\hat{E}_i^{(j)} \times \{0\}$ coincide. Also, recall that the trace of $\mathcal{F}$ on the set $\prod_{i \in n} \kappa_i \times \{0\}$ is the filter $\mathcal{U}$ which equals to the trace of the filter $\Phi_{\vec{v}}$ on $\prod_{i \in n} \kappa_i \times \{0\}$. Taking into account that $(\bigcup_{i \in K_{\vec{v}}} \bigcup_{j \in v_i} \hat{E}_i^{(j)} \cup \prod_{i \in n} \kappa_i) \times \{0\} \in \mathcal{F} \cap \Phi_{\vec{v}}$, Lemma 3.2 implies that $\mathcal{F} = \Phi_{\vec{v}}$.

Since for every $\vec{v} \in \prod_{i \in n} m_i$ the filter $\Phi_{\vec{v}}$ is open and free we get that $\mathbf{OF}(M) = \{\Phi_{\vec{v}} : \vec{v} \in \prod_{i \in n} m_i\}$. It remains to observe that for vectors $\vec{v} = (v_0, \dots, v_{n-1})$ and $\vec{k} = (k_0, \dots, k_{n-1})$ the inclusion $\Phi_{\vec{k}} \subset \Phi_{\vec{v}}$ holds iff $k_i \geq v_i$ for each $i \in n$. Since for any natural number n the chains $(n, \leq)$ and $(n, \geq)$ are order isomorphic we deduce that the lattice $\mathbf{OF}(M)$ is order isomorphic to $\prod_{i \in n} m_i$. □

Since every lattice of cardinality ≤ 3 is a chain, Theorem 3.3 implies that for every lattice L with $|L| \leq 3$ there exists a space X such that $\mathbf{OF}(X)$ is order isomorphic to L. Theorems 3.3 and 4.5 provide that the existing of a measurable cardinal implies that for each lattice L of cardinality 4, there exists a space X such that $\mathbf{OF}(X)$ is order isomorphic to L. Theorem 2.3 yields the existence of two (non-distributive) five-element lattice which are not isomorphic to $\mathbf{OF}(X)$ for any space X. The next example shows that assuming the existence of two measurable cardinals for any five-element distributive lattice L there exists a space X such that $\mathbf{OF}(X)$ is isomorphic to L. Let $L_0 = \{(0,0), (1,1), (1,2), (2,1), (2,2)\}$ and $L_1 = \{(0,0), (1,0), (0,1), (1,1), (2,2)\}$ are sublattices of $(\omega^2, \max, \min)$.

Example 4.6. If there exist measurable cardinals $\kappa_1 < \kappa_2$, then there exist spaces Y_0, Y_1 such that the lattice $\mathbf{OF}(Y_i)$ is isomorphic to L_i, $i \in 2$.

Proof. Set $\kappa_0 = \omega$ and for every $i \in 3$ fix any κ_i-complete ultrafilter $\mathcal{F}_i$ on κ_i. Let X_i be the subspace $\kappa_i \cup \{\mathcal{F}_i\}$ of $\beta(\kappa_i)$. For convenience we denote the point $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2) \in \prod_{i \in 3} X_i$ by x^*.

Construction of Y_0. Let X be the subspace $(X_0\times X_1\times\kappa_2)\cup\{x^*\}$ of the Tychonoff product $\prod_{i\in 3} X_i$. Put $Y_0 = \mathrm{M}(X,x^*)$ (see Construction 1). Similarly as in the proofs of Theorem 3.3 and Theorem 4.5 one can check that $\mathbf{OF}(Y_0)=\{\Phi_{(i,j)} : (i,j)\in L_0\}$, where the free open filters $\Phi_{(i,j)}$ are defined as follows:

- $\Phi_{(2,2)}$ is generated by the family $\{(\prod_{i\in 3} F_i)\times\{0\} : F_i\in\mathcal{F}_i\}$;
- $\Phi_{(1,2)}$ is generated by the family $\{(F_0\cup\{\mathcal{F}_0\})\times F_1\times F_2\times\{0\} : F_i\in\mathcal{F}_i\}$;
- $\Phi_{(2,1)}$ is generated by the family $\{F_0\times(F_1\cup\{\mathcal{F}_1\})\times F_2\times\{0\} : F_i\in\mathcal{F}_i\}$;
- $\Phi_{(1,1)}$ is generated by the family $\{U\cup V : U\in\Phi_{(1,2)}$ and $V\in\Phi_{(2,1)}\}$;
- $\Phi_{(0,0)}$ is generated by the family $\{(F_0\cup\{\mathcal{F}_0\})\times(F_1\cup\{\mathcal{F}_1\})\times F_2\times\{0\} : F_i\in\mathcal{F}_i\}$.

It is straightforward to check that the map $\phi:\mathbf{OF}(Y_0)\to L_0$, $\phi(\Phi_{(i,j)})=(i,j)$ is an order isomorphism.

Construction of Y_1. Let X be the subspace

$$(X_0\times\kappa_1\times\kappa_2)\cup(\{\mathcal{F}_0\}\times\{\mathcal{F}_1\}\times\kappa_2)\cup(\{\mathcal{F}_0\}\times\kappa_1\times\{\mathcal{F}_2\})\cup\{x^*\}$$

of the Tychonoff product $\prod_{i\in 3} X_i$. Put $Y_1=\mathrm{M}(X,x^*)$. Similarly as above one can check that $\mathbf{OF}(Y_1) = \{\Phi_{(i,j)} : (i,j)\in L_1\}$, where the free open filters $\Phi_{(i,j)}$ are defined as follows:

- $\Phi_{(2,2)}$ is generated by the family $\{(\prod_{i\in 3} F_i)\times\{0\} : F_i\in\mathcal{F}_i\}$;
- $\Phi_{(1,1)}$ is generated by the family $\{(F_0\cup\{\mathcal{F}_0\})\times F_1\times F_2\times\{0\} : F_i\in\mathcal{F}_i\}$;
- $\Phi_{(0,1)}$ is generated by the family $\{U\cup(\{\mathcal{F}_0\}\times\{\mathcal{F}_1\}\times F_2\times\{0\}) : U\in\Phi_{(1,1)}$ and $F_2\in\mathcal{F}_2\}$;
- $\Phi_{(1,0)}$ is generated by the family $\{U\cup(\{\mathcal{F}_0\}\times F_1\times\{\mathcal{F}_2\}\times\{0\}) : U\in\Phi_{(1,1)}$ and $F_1\in\mathcal{F}_1\}$;
- $\Phi_{(0,0)}$ is generated by the family $\{U\cup V : U\in\Phi_{(1,0)}$ and $V\in\Phi_{(0,1)}\}$.

It is straightforward to check that the map $\phi:\mathbf{OF}(Y_1)\to L_1$, $\phi(\Phi_{(i,j)})=(i,j)$ is an order isomorphism. □

5. Infinite linear orders

A subset A of a scattered space X is called *high* if $X^{(\alpha)}\cap A\neq\emptyset$ for every $\alpha\in ht(X)$. For a subset $A\subset\kappa$ put $\langle A\rangle=\{\mathcal{F}\in\beta(\kappa) : A\in\mathcal{F}\}$. Recall that the set $\langle A\rangle$ is clopen in $\beta(\kappa)$. Let $\mathcal{F}$ be an ultrafilter on κ. We write that $\mathfrak{h}(\mathcal{F})\geq\alpha$ (and say that the *height* of $\mathcal{F}$ is $\geq\alpha$) if there exists a scattered space $X\subset\beta(\kappa)$ satisfying the following conditions:

(1) $\mathcal{F}\in X^{(\alpha)}$ and for any $\xi\in\alpha$ the open neighborhood filter of $\mathcal{F}$ traces on each $X^{(\xi)}$, $\xi<\alpha$ an ultrafilter $\mathcal{F}_\xi$;
(2) for any $0<\gamma\leq\alpha$, for each selector $\langle F_\xi\in\mathcal{F}_\xi : \xi\in\gamma\rangle$ such that the set $U=\cup_{\xi\in\gamma}F_\xi$ is open in X there exists $F\in\mathcal{F}$ satisfying $\langle F\rangle\cap(\cup_{\xi\in\gamma}X^{(\xi)})\subset U$.

Recall that, by Lemma 3.1, if for some $\xi\in\alpha$ the set X^ξ is strongly discrete in $\beta(\kappa)$, then $\mathcal{F}_\xi$, is an ultrafilter.

Lemma 5.1. *Let $\mathcal{F}$ be an ultrafilter on κ. If $\mathfrak{h}(\mathcal{F})\geq\alpha$, then $\mathfrak{h}(\mathcal{F})\geq\xi$ for any $\xi\in\alpha$.*

Proof. Fix any scattered space $X\subset\beta(\kappa)$ which is a witness for $\mathfrak{h}(\mathcal{F})\geq\alpha$, i.e., X satisfies conditions (1) and (2). Fix any $\xi\in\alpha$. Consider the subspace $Y=\cup_{\gamma\in\xi}X^{(\gamma)}\cup\{\mathcal{F}\}$ of X. At this point it is straightforward to check that the space Y is a witness for $\mathfrak{h}(\mathcal{F})\geq\xi$. □

We shall write $\mathfrak{h}(\mathcal{F})=\alpha$ if $\mathfrak{h}(\mathcal{F})\geq\xi$ for any $\xi\in\alpha$, but $\neg(\mathfrak{h}(\mathcal{F})\geq\alpha)$. Clearly, $\mathfrak{h}(\mathcal{F})=1$ if and only if $\mathcal{F}$ is a principal ultrafilter. The next lemma shows that consistently there exist free ultrafilters of height 2.

Lemma 5.2. $\mathfrak{h}(\mathcal{F})=2$ *for each P-point* $\mathcal{F}\in\beta(\omega)\setminus\omega$.

The defined above stratifications of ultrafilters are motivated by the following result, which prove resembles step 2 of the proof of Theorem 3.3.

Proposition 5.3. *If there exists an ultrafilter $\mathcal{F}$ on κ such that $\mathfrak{h}(\mathcal{F})\geq\alpha$, then for every $\theta\leq\alpha$ there exists a scattered space M such that $\mathbf{OF}(M)$ is order isomorphic to $(\theta+1,\geq)$.*

Proof. Fix a scattered subspace $Y\subset\beta(\kappa)$ such that $\mathcal{F}\in Y^{(\alpha)}$, for any $\xi\in\alpha$ the filter $\mathcal{N}(\mathcal{F})$ traces on each $Y^{(\xi)}$, $\xi<\alpha$ an ultrafilter $\mathcal{F}_\xi$, and for any $\gamma\leq\alpha$, for each selector $\langle F_\xi\in\mathcal{F}_\xi:\xi\in\gamma\rangle$ such that the set $U=\cup_{\xi\in\gamma}F_\xi$ is open in Y there exists $F\in\mathcal{F}$ satisfying $\langle F\rangle\cap(\cup_{\xi\in\gamma}Y_\xi)\subset U$. With no loss of generality we can assume that $\kappa\subset Y$. Fix any $\theta\leq\alpha$. Set $X=\cup_{\xi\in\theta}X^{(\xi)}\cup\{\mathcal{F}\}$. Let $M=\mathrm{M}(X,\mathcal{F})$. Construction 1 implies that the space M is scattered.

Let us show that $\mathbf{OF}(M)$ is order isomorphic to $(\theta+1,\geq)$. For this fix any free open filter $\mathcal{H}$ on M. By Construction 1, the subspace $M\setminus((X\setminus\{\mathcal{F}\})\times\{0\})$ is H-closed. Hence $(X\setminus\{\mathcal{F}\})\times\{0\}\in\mathcal{H}$. Note that $\kappa\times\{0\}$ is a dense open discrete subspace of $(X\setminus\{\mathcal{F}\})\times\{0\}$. Thus $H\cap(\kappa\times\{0\})\neq\emptyset$ for each $H\in\mathcal{H}$. Observe that $\{F\times\{0\}:F\in\mathcal{F}\}$ is the only free in M open filter on $\kappa\times\{0\}$. Therefore, the filter $\mathcal{H}$ traces on $\kappa\times\{0\}$ the mentioned above ultrafilter.

Let γ be the smallest ordinal such that exists $H\in\mathcal{H}$ which satisfies $H\cap(X^{(\gamma)}\times\{0\})=\emptyset$. For $\beta\in\gamma$ by Φ_β we denote the filter on $X^{(\beta)}\times\{0\}$ which is generated by the family $\{U\times\{0\}:U\in\mathcal{F}_\beta\}$. We claim that for each $\beta\in\gamma$ the trace of the filter $\mathcal{H}$ on the set $X^{(\beta)}\times\{0\}$ coincides with Φ_β. By the choice of X and $\mathcal{F}$, for any $\beta\in\gamma$ the filter Φ_β is an ultrafilter. Hence it is suffices to show that $\Phi_\beta\subset\{H\cap(X^{(\beta)}\times\{0\}):H\in\mathcal{H}\}$ for any $\beta\in\gamma$. For this, fix any $\beta\in\gamma$ and $U\times\{0\}\in\Phi_\beta$, where $U\in\mathcal{F}_\beta$. With no loss of generality we can assume that $U=\langle V\rangle\cap X^{(\beta)}$ where $V\in\mathcal{F}$. Since the trace of $\mathcal{H}$ on $\kappa\times\{0\}$ coincides with $\{F\times\{0\}:F\in\mathcal{F}\}$ there exists an open set $H\in\mathcal{H}$ such that $H\cap(\kappa\times\{0\})\subset(V\cap\kappa)\times\{0\}$. Let us show that

$$H\cap(X^{(\beta)}\times\{0\})\subset(\langle V\rangle\cap X^{(\beta)})\times\{0\}=U\times\{0\}.$$

Pick any point $(z,0)\in(H\cap(X^{(\beta)}\times\{0\}))$. By the definition of the space X, $z\in X^{(\beta)}$ is an ultrafilter on κ. Since the set H is open and taking into account the choice of V, we get that V is an element of the ultrafilter z. Consequently, $(z,0)\in(\langle V\rangle\cap X^{(\beta)})\times\{0\}=U\times\{0\}$. Hence $H\cap(X^{(\beta)}\times\{0\})\subset U\times\{0\}$, witnessing that $\Phi_\beta=\{H\cap(X^{(\beta)}\times\{0\}):H\in\mathcal{H}\}$ for each $\beta\in\gamma$.

By $\Phi^{<\gamma}$ we denote the open filter on M generated by the family $\{(\langle F\rangle\cap(\cup_{\beta\in\gamma}X^{(\beta)}))\times\{0\}:F\in\mathcal{F}\}$. By the arguments above, for each $\beta\in\gamma$ traces of the filters $\mathcal{H}$ and $\Phi^{<\gamma}$ on $X^{(\beta)}\times\{0\}$ coincide. Let us show that $\mathcal{H}=\Phi^{<\gamma}$. Pick any element $(\langle F\rangle\cap(\cup_{\beta\in\gamma}X^{(\beta)}))\times\{0\}\in\Phi^{<\gamma}$. There exists an open set $H_1\in\mathcal{H}$ such that $H_1\cap(\kappa\times\{0\})\subset F\times\{0\}$. By the definition of γ, there exists an open set $H_2\in\mathcal{H}$ such that $H_2\cap(X^{(\gamma)}\times\{0\})=\emptyset$. Put $H=H_1\cap H_2$. Taking into the account that the set H is open, it is easy to verify that $H\subset(\langle F\rangle\cap(\cup_{\beta\in\gamma}X^{(\beta)}))\times\{0\}$. Hence $\Phi^{<\gamma}\subset\mathcal{H}$. To show the converse inclusion, fix any open set $H\in\mathcal{H}$. For each $\beta\in\gamma$, the set $F_\beta=\{z\in X^{(\beta)}:(z,0)\in H\}$ belongs to the filter $\mathcal{F}_\beta$. Since the set H is open in M, the set $\cup_{\beta\in\gamma}F_\beta$ is open in X. By the definition of $\mathcal{F}$, there exists $F\in\mathcal{F}$ such that $\langle F\rangle\cap(\cup_{\beta\in\gamma}X^{(\beta)})\subset\cup_{\beta\in\gamma}F_\beta$. Consequently, $(\langle F\rangle\cap(\cup_{\beta\in\gamma}X^{(\beta)}))\times\{0\}\subset H$, witnessing that $\Phi^{<\gamma}=\mathcal{H}$.

To sum up, $\mathbf{OF}(M)=\{\Phi^{<\gamma},\gamma\in\theta+1\}$, and $\Phi^{<\gamma_1}\subset\Phi^{<\gamma_2}$ iff $\gamma_2\leq\gamma_1$. Hence $\mathbf{OF}(M)$ is order isomorphic to $(\theta+1,\geq)$. ☐

Theorem 5.4. *(CH) There exists an ultrafilter $\mathcal{F}$ on ω such that $\mathfrak{h}(\mathcal{F})\geq\omega$.*

Proof. Observe that during step 1 of the proof of Theorem 3.3 for every $n\in\mathbb{N}$ we constructed a countable scattered subspace $X_n\subset\beta(\omega)$ which satisfies the following conditions:

- $ht(X_n)=n+1$ and $X_n^{(n)}=\{\mathcal{F}_n\}$ (we denoted the filter $\mathcal{F}_n$ by x^*);
- For each $m\in n$ the Cantor-Bendixson level $X_n^{(m)}$ is strongly discrete, implying that the filter $\mathcal{N}(\mathcal{F}_n)$ traces on $X_n^{(m)}$ an ultrafilter.

Consider any partition $\{C_n : n \in \omega\}$ of ω into disjoint infinite subsets. For each $n \in \omega$, X_n is homeomorphic to a subspace of $\langle C_n \rangle \cong \beta(\kappa)$. Let Y be a topological sum of $\{X_n : n \in \omega\}$. Identify Y with a subspace of $\beta(\omega)$ such that $X_n \subset \langle C_n \rangle$. Enumerate the set of all high open subsets of Y as $\{U_\alpha : \alpha \in \omega_1\}$. Moreover, we assume that each high open subset of Y_θ repeats cofinally many times in the enumeration. Also, enumerate all subsets of ω as $\{B_\alpha : \alpha \in \omega_1\}$. Note that the latter two enumerations require CH. The desired ultrafilter $\mathcal{F}$ on ω will be constructed by induction of length ω_1. Fix any $\beta \in \omega_1$ and assume that we already constructed a family $\mathcal{V}_\beta = \{V_\alpha : \alpha \in \beta\}$ of subsets of ω which satisfies the following conditions:

(a) the family $\mathcal{V}_\beta$ is centered and, thus, generates a filter which we denote by $\mathcal{W}_\beta$;
(b) for any $\alpha \in \beta$ either $B_\alpha \in \mathcal{W}_\beta$ or $\omega \setminus B_\alpha \in \mathcal{W}_\beta$;
(c) for any $W \in \mathcal{W}_\beta$ the set $\langle W \rangle$ is high in Y, i.e., $\langle W \rangle \cap Y^{(m)} \neq \emptyset$ for any $m \in \omega$.

The filter $\mathcal{W}_\beta$ is an approximation to the desired filter $\mathcal{F}$. At stage β two cases are possible:

(i) for any $n \in \omega$ there exists $W_n \in \mathcal{W}_\beta$ such that $\langle W_\delta \rangle \cap Y^{(n)} \subset U_\beta$;
(ii) there exists $n \in \omega$ such that $\langle W \rangle \cap (Y^{(n)} \setminus U_\beta) \neq \emptyset$ for any $W \in \mathcal{W}_\beta$.

If case (ii) holds, then put $W_\beta = \omega$. Informally speaking the set U_β is at this stage "irrelevant" for our filter $\mathcal{W}_\beta$.

Assume that case (i) holds. Since the ordinal β is countable the filter $\mathcal{W}_\beta$ admits a countable nested base $\mathcal{B} = \{C_n : n \in \omega\}$, i.e., $C_n \subset C_m$ whenever $m \leq n$. Moreover, with no loss of generality we can assume that $C_n \subset \cap_{i \leq n} W_i$ for any $n \in \omega$.

First we inductively construct an auxiliary strongly discrete set $E_\beta = \{e_n : n \in \omega\} \subset Y$. Assume that some $n \in \omega$ we already constructed a set $\{e_k : k \in n\}$ such that $e_k \in \langle C_k \rangle \cap X^{(k)}_{\phi(k)}$ and $\phi(k_1) \neq \phi(k_2)$ for any distinct $k_1, k_2 \in n$. Taking into account that the set $T = \cup_{i \in n} X_{\phi(i)}$ is not high and the set C_n is high, we get that there exists a point $e_n \in \langle C_n \rangle \cap (Y^{(n)} \setminus T)$. Recall that we identify Y with the subspace of $\beta(\omega)$, so we consider points e_n, $n \in \omega$ as ultrafilters on ω. For every $n \in \omega$ fix any element $S_n \in e_n$ such that $S_n \subset X_{\phi(n)} \cap C_n$ and $\langle S_n \rangle \cap Y^{(n)} = \{e_n\}$, which exists since the set $\langle C_n \rangle \cap X_{\phi(n)} \ni e_n$ is open in Y. Finally set $V'_\beta = \cup_{n \in \omega} S_n$. The choice of S_n, $n \in \omega$ together with the injectivity of the function ϕ ensure that for any $y \in Y$ there exists at most one $n \in \omega$ such that $S_n \in y$. Hence

$$\langle V'_\beta \rangle \cap Y = \bigcup_{n \in \omega} (\langle S_n \rangle \cap Y) \subset \bigcup_{n \in \omega} (\langle C_n \rangle \cap (\cup_{i \leq n} Y^{(i)})) \subset \bigcup_{n \in \omega} (\langle \cap_{i \leq n} W_i \rangle \cap (\cup_{i \leq n} Y^{(i)})) \subset U_\beta.$$

Claim. At least one of the following assertions holds:

(†) for any $W \in \mathcal{W}_\beta$ the set $Z = \langle W \cap V'_\beta \cap B_\beta \rangle$ is high;
(††) for any $W \in \mathcal{W}_\beta$ the set $Z = \langle W \cap V'_\beta \cap (\kappa \setminus B_\beta) \rangle$ is high.

Proof. To derive a contradiction assume that both assertions fail. Then there exist $W_1, W_2 \in \mathcal{W}_\beta$ and $n_1, n_2 \in \omega$ such that for every $m \geq n = \max\{n_1, n_2\}$ the following equalities hold:

$$\langle W_1 \cap V'_\beta \cap B_\beta \rangle \cap Y^{(n_1)} = \emptyset \quad \text{and} \quad \langle W_2 \cap V'_\beta \cap (\kappa \setminus B_\beta) \rangle \cap Y^{(n_2)} = \emptyset.$$

Put $W = W_1 \cap W_2$. It is straightforward to check that for any $m \geq n$

$$\langle V'_\beta \cap W \rangle \cap Y^{(m)} \subset \langle W_1 \cap V'_\beta \cap B_\beta \rangle \cap Y^{(m)} \cup \langle W_2 \cap V'_\beta \cap (\kappa \setminus B_\beta) \rangle \cap Y^{(m)} = \emptyset.$$

On the other hand, there exists $m \geq n$ such that $C_m \subset W$ and

$$e_m \in C_m \cap \langle V'_\beta \rangle \cap Y^{(m)} \subset \langle W \cap V'_\beta \rangle \cap Y^{(m)},$$

which implies a contradiction. ☐

If assertion (†) of Claim 5 holds, then put $V_\beta = V'_\beta \cap B_\beta$. Otherwise, set $V_\beta = V'_\beta \cap (\kappa \setminus B_\beta)$. Clearly, the family $\mathcal{V}_{\beta+1} = \{V_\alpha : \alpha \in \beta+1\}$ satisfies the inductive hypothesis. So, after completing the induction we obtain a centered family $\mathcal{V}_{\omega_1} = \{V_\alpha : \alpha \in \omega_1\}$. Let $\mathcal{F}$ be the filter generated by the family $\mathcal{V}_{\omega_1}$. Fix any subset $B \subset \kappa$. There exists $\xi \in \omega_1$ such that $B = B_\xi$. By the construction of the family $\mathcal{V}_{\omega_1}$, either $V_\xi \subset B_\xi$ or $V_\xi \subset \kappa \setminus B_\xi$. Thus, either $B \in \mathcal{F}$ or $\kappa \setminus B \in \mathcal{F}$, witnessing that $\mathcal{F}$ is an ultrafilter. Let $X = Y \cup \{\mathcal{F}\}$ be the subspace of $\beta(\kappa)$. The definition of $\mathcal{F}$ (see condition (c)) implies that $X^{(\omega)} = \{\mathcal{F}\}$. By $\mathcal{F}_n$ we denote the trace of $\mathcal{N}(\mathcal{F})$ on $X^{(n)}$. Observe that the definition of Y implies that for each $n \in \omega$ the set $X^{(n)}$ is strongly discrete. Hence $\mathcal{F}_n$ is an ultrafilter for every $n \in \omega$.

So, to prove that $\mathfrak{h}(\mathcal{F}) \geq \omega$ it remains to show that for any $0 < \gamma \leq \omega$ for each selector $\langle F_\xi \in \mathcal{F}_\xi : \xi \in \gamma\rangle$ such that the set $U = \cup_{\xi\in\gamma} F_\xi$ is open in X there exists $F \in \mathcal{F}$ satisfying $\langle F\rangle \cap (\cup_{\xi\in\gamma} X^{(\xi)}) \subset U$. It is easy to see that for $\gamma < \omega$ the latter condition is automatically fulfilled, as the filter $\mathcal{F}$ is closed under finite intersections. Consider any selector $\langle F_n \in \mathcal{F}_n : n \in \omega\rangle$ such that the set $U = \cup_{n\in\omega} F_n$ is open in X. Since the set U is open and high in Y there exists a cofinal subset $\Xi \subset \omega_1$ such that $U = U_\xi$ for any $\xi \in \Xi$. For each $n \in \omega$ there exists a basic open neighborhood $\langle O_n\rangle$ of $\mathcal{F}$ which witnesses that $F_n \in \mathcal{F}_n$, that is, $\langle O_n\rangle \cap X^{(n)} = \langle O_n\rangle \cap Y^{(n)} \subset F_n$. By the construction of $\mathcal{F}$ for each $n \in \omega$ there exists $m(n) \in \omega$ and a family $\{V_{\xi_0}, \dots, V_{\xi_{m(n)}}\} \subset \mathcal{V}_{\omega_1}$ such that $\cap_{i\leq m(n)} V_{\xi_i} \subset O_n$. Set $\delta_n = \max\{\xi_0, \dots, \xi_{m(n)}\}$. Since the set Ξ is unbounded in ω_1 there exists $\mu \in \Xi$ such that $\mu > \sup\{\delta_n : n \in \omega\}$. Then at stage μ of our induction the set $U = U_\mu$ will be already "relevant" for the approximating filter $\mathcal{W}_\mu$, that is case (i) holds. Then, taking into account that $Y = \cup_{n\in\omega} X^{(n)}$, the inequality $\langle V_\mu\rangle \cap Y \subset \langle V'_\mu\rangle \cap Y \subset U$ implies that $\mathfrak{h}(\mathcal{F}) \geq \omega$. □

Theorem 5.5. *If there exists an ultrafilter $\mathcal{F}$ such that $\mathfrak{h}(\mathcal{F}) \geq \alpha > \omega$, then there exists a measurable cardinal.*

Proof. Assume that there exists a cardinal κ and an ultrafilter $\mathcal{F} \in \beta(\kappa)$ such that $\mathfrak{h}(\mathcal{F}) \geq \alpha > \omega$. Then there exists a scattered subspace $X \subset \beta(\kappa)$ satisfying the following two conditions:

(1) $\mathcal{F} \in X^{(\alpha)}$ and for any $\xi \in \alpha$ the open neighborhood filter of $\mathcal{F}$ traces on each $X^{(\xi)}$, $\xi < \alpha$ an ultrafilter $\mathcal{F}_\xi$;
(2) for any $0 < \gamma \leq \alpha$, for each selector $\langle F_\xi \in \mathcal{F}_\xi : \xi \in \gamma\rangle$ such that the set $U = \bigcup_{\xi\in\gamma} F_\xi$ is open in X there exists $F \in \mathcal{F}$ satisfying $\langle F\rangle \cap (\bigcup_{\xi\in\gamma} X^{(\xi)}) \subset U$.

Lemma 5.1 implies that with no loss of generality we can assume that $\alpha = \omega + 1$ and $X^{(\omega+1)} = \{\mathcal{F}\}$. To derive a contradiction, assume that there exist no measurable cardinals. Then the ultrafilters $\mathcal{F}$ and $\mathcal{F}_\omega$ are not ω_1-complete. Then there exist families $\{H_n : n \in \omega\} \subset \mathcal{F}$ and $\{G_n : n \in \omega\} \subset \mathcal{F}_\omega$ such that $\bigcap_{n\in\omega} H_n = \emptyset = \bigcap_{n\in\omega} G_n$. By the definition of $\mathcal{F}_\omega$, for every $n \in \omega$ there exists $T_n \in \mathcal{F}$ such that $\langle T_n\rangle \cap X^{(\omega)} \subset G_n$. For every $n \in \omega$ put $S_n = \bigcap_{i\leq n} H_n \cap (\bigcap_{i\leq n} T_n)$. Then $\{S_n : n \in \omega\}$ is a decreasing sequence of elements of $\mathcal{F}$ such that $\bigcap_{n\in\omega} S_n = \emptyset$ and $\bigcap_{n\in\omega}(\langle S_n\rangle \cap X^{(\omega)}) \subset \bigcap_{n\in\omega} G_n = \emptyset$. For any $n \in \omega$ put $F_n = \langle S_n\rangle \cap X^{(n)} \in \mathcal{F}_n$. Since $S_n \subset S_m$ whenever $m \leq n$, the set $\bigcup_{i\in\omega} F_i$ is open in X. Condition (2) implies the existence of a set $F \in \mathcal{F}$ such that $\langle F\rangle \cap (\bigcup_{i\in\omega} X^{(i)}) \subset \bigcup_{i\in\omega} F_i$. Denote $G = \langle F\rangle \cap X^{(\omega)} \in \mathcal{F}_\omega$. Since the set $\Phi = \langle F\rangle \cap (\bigcup_{i\in\omega} X^{(i)})$ is open, $\Phi \subset \bigcup_{i\in\omega} F_i$ and the set $\bigcup_{i\in\omega} F_i$ is open, we get that the set $(\bigcup_{i\in\omega} F_i) \cup G$ is open as well. Since $\bigcap_{n\in\omega}(\langle S_n\rangle \cap X^{(\omega)}) = \emptyset$ there exists $n \in \omega$ such that $W = G \setminus (\langle S_n\rangle \cap X^{(\omega)}) \neq \emptyset$. Consider any ultrafilter $\mathcal{U} \in W$. It follows that $S_n \notin \mathcal{U}$, witnessing that $\kappa \setminus S_n \in \mathcal{U}$. Since the set $(\bigcup_{i\in\omega} F_i) \cup G$ is open in X the exists $U \in \mathcal{U}$ such that $\langle U\rangle \cap X \subset (\bigcup_{i\in\omega} F_i) \cup G$. Then $V = U \cap (\kappa \setminus S_n) \in \mathcal{U}$ and $\langle V\rangle \cap X \subset (\bigcup_{i\in\omega} F_i) \cup G$. Taking into account that $\mathcal{U} \in X^{(\omega)}$, $\emptyset \neq \langle V\rangle \cap X^{(n)} \subset F_n$. But since $V \cap S_n = \emptyset$, the definition of F_n implies that $\langle V\rangle \cap F_n = \emptyset$. The obtained contradiction finalizes the proof. □

Proposition 5.6. *If there exists a measurable cardinal μ and an ultrafilter $\mathcal{F}$ such that $\mathfrak{h}(\mathcal{F}) \geq \alpha < \mu$, then for every $n \in \omega$ there exists an ultrafilter $\mathcal{F}_n$ such that $\mathfrak{h}(F_n) \geq \alpha + n$.*

6. Complete open ultrafilters

In this section we give an affirmative answer to Question 4. Let us note that in the following theorem there are no restrictions on separation axioms of the space X.

Theorem 6.1. *There exists a space X possessing a free ω_1-complete open ultrafilter if and only if there exists a measurable cardinal.*

Proof. If λ is a measurable cardinal, then obviously the discrete space λ possesses a λ-complete open ultrafilter.

Assume that a space X possesses a free ω_1-complete open ultrafilter $\mathcal{F}$. Let

$$\lambda = \min\{\kappa : \exists \mathcal{F}' \in [\mathcal{F}]^\kappa : \mathrm{Int}(\cap_{G \in \mathcal{F}'} \overline{G}) \notin \mathcal{F}\}.$$

Since the filter $\mathcal{F}$ is free and ω_1-complete we get that $\omega_1 \leq \lambda \leq |X|$. Let

$$\lambda' = \min\{\kappa : \exists \mathcal{F}' \in [\mathcal{F}]^\kappa : \mathrm{Int}(\cap_{G \in \mathcal{F}'} \overline{G}) = \emptyset\}.$$

Claim 1. $\lambda = \lambda'$.

Proof. Since $\emptyset \notin \mathcal{F}$ we get that $\lambda \leq \lambda'$. By the definition of λ, there exists a family $\mathcal{G} \subset \mathcal{F}$ of cardinality λ such that $A = \mathrm{Int}(\cap_{G \in \mathcal{G}} \overline{G}) \notin \mathcal{F}$. Since $\mathcal{F}$ is an open ultrafilter, there exists an open set $B \in \mathcal{F}$ such that $A \cap B = A \cap \overline{B} = \emptyset$. Then the family $\mathcal{H} = \mathcal{G} \cup \{B\}$ has cardinality λ and $\mathrm{Int}(\cap_{H \in \mathcal{H}} \overline{H}) = \emptyset$. Thus, $\lambda = \lambda'$. □

Next we shall inductively construct a subset $\{F_\alpha : \alpha \in \lambda\} \subset \mathcal{F}$ which satisfies the following conditions:

(1) $F_0 = X$;
(2) $F_{\alpha+1} \subset F_\alpha = \mathrm{Int}(\overline{F_\alpha})$;
(3) $F_\alpha \setminus \overline{F_{\alpha+1}} \neq \emptyset$;
(4) $F_\xi = \mathrm{Int}(\cap_{\alpha \in \xi} \overline{F_\alpha})$ if the ordinal ξ is limit;
(5) $\mathrm{Int}(\cap_{\alpha \in \lambda} \overline{F_\alpha}) = \emptyset$.

Fix any family $\mathcal{G} = \{G_\alpha : \alpha \in \lambda\} \subset \mathcal{F}$ such that $\cap_{\alpha \in \lambda} \overline{G_\alpha} = \emptyset$, which exists by Claim 1. For each $\alpha \in \lambda$ let $H_\alpha = \mathrm{Int}(\overline{G_\alpha})$. Since $\overline{H_\alpha} = \overline{G_\alpha}$ we deduce that $\mathrm{Int}(\cap_{\alpha \in \lambda} H_\alpha) = \mathrm{Int}(\cap_{\alpha \in \lambda} G_\alpha) = \emptyset$. Put $F_0 = X$. Assume that we already constructed F_α for each $\alpha < \xi$. Depending on ξ we do the following:

- assuming that $\xi = \theta + 1$, fix any $x \in F_\theta$ and find $P \in \mathcal{F}$ such that $x \notin \overline{P}$. Observe that the set P exists, because the filter $\mathcal{F}$ is free. Then put $F_\xi = \mathrm{Int}(\overline{F_\theta} \cap \overline{H_\theta} \cap \overline{P})$.
- if ξ is limit, then put $F_\xi = \mathrm{Int}(\cap_{\alpha < \xi} \overline{F_\alpha})$.

One can easily check that the constructed above family $\{F_\alpha, \alpha \in \lambda\}$ satisfies conditions (1)–(5).

For each $\alpha \in \lambda$ let $D_\alpha = F_\alpha \setminus \overline{F_{\alpha+1}}$.

Claim 2. *The set $D = \cup_{\alpha \in \lambda} D_\alpha$ is open and dense in X.*

Proof. Being a union of open sets, the set D is open. To derive a contradiction, assume that the set D is not dense. Then there exists a nonempty open set $V \subset X \setminus D$. Since $F_0 = X$, $V \subset F_0$. Assume that for any $\alpha \in \xi$ we have that $V \subset \overline{F_\alpha}$. If $\xi = \eta + 1$, then $V \subset \overline{F_\eta}$ and $V \cap (F_\eta \setminus \overline{F_\xi}) = \emptyset$, because $F_\eta \setminus \overline{F_\xi} \subset D$ and $V \cap D = \emptyset$. Condition (2) implies that $V \subset \overline{F_\xi}$. If the ordinal ξ is limit, we get that $V \subset \overline{F_\alpha}$ for every $\alpha \in \lambda$. Then $V \subset \cap_{\alpha \in \lambda} \overline{F_\alpha}$. Since V is open we get that $V \subset \mathrm{Int}(\cap_{\alpha \in \lambda} \overline{F_\alpha}) = F_\xi$.

Hence the above arguments provide that $V \subset \cap_{\alpha\in\lambda}\overline{F_\alpha} = \emptyset$ which contradicts the choice of V. Hence the set D is dense. □

By $\hat{\mathcal{F}}$ we denote the filter generated by the base $\mathcal{B} = \{\operatorname{Int}(\overline{F}) : F \in \mathcal{F}\}$. Let us check that the filter $\hat{\mathcal{F}}$ is λ-complete. For this fix any family $\{\operatorname{Int}(\overline{F_\alpha}) : \alpha < \beta < \lambda\} \subset \mathcal{B}$. Since the filter $\mathcal{F}$ is open, $\operatorname{Int}(\overline{F_\alpha}) \in \mathcal{F}$ for each $\alpha \in \beta$. By the λ-completemess of the filter $\mathcal{F}$, the set $Z = \cap_{\alpha\in\beta}\operatorname{Int}(\overline{F_\alpha})$ belongs to $\mathcal{F}$. Since $Z \subset \operatorname{Int}(\overline{F_\alpha})$ we get that $\operatorname{Int}(\overline{Z}) \subset \operatorname{Int}(\overline{\operatorname{Int}(\overline{F_\alpha})}) = \operatorname{Int}(\overline{F_\alpha})$ for every $\alpha \in \beta$. It follows that $\operatorname{Int}(\overline{Z}) \subset Z$, witnessing that $Z \in \hat{\mathcal{F}}$. Hence the filter $\hat{\mathcal{F}}$ is λ-complete.

Consider the filter $\mathcal{W}$ on the cardinal λ generated by the family $\{W_F : F \in \hat{\mathcal{F}}\}$, where $W_F = \{\xi \in \lambda : D_\xi \cap F \neq \emptyset\}$. The λ-completeness of $\hat{\mathcal{F}}$ implies that the filter $\mathcal{W}$ is λ-complete as well. To show that $\mathcal{W}$ is an ultrafilter, consider any partition $\lambda = M \cup N$. Let $I = \cup_{\alpha\in M} D_\alpha$ and $J = \cup_{\alpha\in N} D_\alpha$. Clearly, the disjoint sets I, J are open and $D = I \cup J$. Since D is an open dense subset of X and the filter $\mathcal{F}$ is an open ultrafilter, we get that $D \in \mathcal{F}$. The characterization of Mooney [24] of open ultrafilters (see the first section) implies that either $I \in \mathcal{F}$ or $J \in \mathcal{F}$. It is straightforward to check that $\operatorname{Int}(\overline{I}) \cap \operatorname{Int}(\overline{J}) = \emptyset$. Assuming that $I \in \mathcal{F}$ we get that $\operatorname{Int}(\overline{I}) \in \hat{\mathcal{F}}$ and $M = W_{\operatorname{Int}(\overline{I})} \in \mathcal{W}$. Otherwise, $\operatorname{Int}(\overline{J}) \in \hat{\mathcal{F}}$ and $N = W_{\operatorname{Int}(\overline{I})} \in \mathcal{W}$. Hence $\mathcal{W}$ is a λ-complete ultrafilter, witnessing that the cardinal λ is measurable. □

A filter $\mathcal{F}$ on a space X is called *closed* if $\mathcal{F}$ possesses a base consisting of closed sets. A closed filter $\mathcal{F}$ is a *closed ultrafilter* if for any closed subset $A \notin \mathcal{F}$ there exists $F \in \mathcal{F}$ such that $F \cap A = \emptyset$.

Lemma 6.2. *There exists a space X possessing a free ω_1-complete closed ultrafilter $\mathcal{F}$ such that* $\operatorname{Int}(F) \neq \emptyset$ *for any $F \in \mathcal{F}$ if and only if there exists a measurable cardinal.*

Proof. Since any λ-complete ultrafilter on a measurable cardinal λ is closed, the "if" part of the proof is trivial.

Assume that a space X possesses an ω_1-complete closed ultrafilter $\mathcal{F}$ such that $\operatorname{Int}(F) \neq \emptyset$ for any $F \in \mathcal{F}$ By $\hat{\mathcal{F}}$ we denote the open filter generated by the base $\mathcal{B} = \{\operatorname{Int}(F) : F \in \mathcal{F}\}$. Since for any closed subsets $A, B \subset X$, $\operatorname{Int}(A) \cap \operatorname{Int}(B) = \operatorname{Int}(A \cap B)$ we obtain that the filter $\hat{\mathcal{F}}$ is well-defined. Fix any family $\{F_n : n \in \omega\} \subset \mathcal{F}$. Taking into account that $\cap_{i\in\omega} F_i \in \mathcal{F}$ and $\operatorname{Int}(\cap_{i\in\omega} F_i) \subset \cap_{i\in\omega}\operatorname{Int}(F_i)$ we get that the open filter $\hat{\mathcal{F}}$ is ω_1-complete. Consider any open set $U \notin \hat{\mathcal{F}}$. To derive a contradiction assume that $\overline{U} \in \mathcal{F}$. Observe that the closed nowhere dense subset $\overline{U} \setminus U \notin \mathcal{F}$. Then there exists $\overline{U} \supset F \in \mathcal{F}$ such that $F \cap \overline{U} \setminus U = \emptyset$. It follows that $F \subset U \in \mathcal{F}$ which contradicts our assumption. Hence $\overline{U} \notin \mathcal{F}$ and, as a consequence, $X \setminus U \in \mathcal{F}$. Then $V = X \setminus \overline{U} \in \hat{\mathcal{F}}$ and $V \cap U = \emptyset$. Thus, $\hat{\mathcal{F}}$ is an open ultrafilter. Theorem 6.1 implies the existence of a measurable cardinal. □

Lemma 6.3. *A space X is Lindelöf if and only if X possesses no free ω_1-complete closed filters.*

Proof. Fix any free closed filter $\mathcal{F}$ on a Lindelöf space X. Let $\mathcal{B}$ is a base of $\mathcal{F}$ consisting of closed subsets. Clearly the family $\mathcal{U} = \{X \setminus B : B \in \mathcal{B}\}$ is an open cover of X. By the Lindelöfness of X, there exists a countable subcover $\mathcal{V} = \{X \setminus B_i : i \in \omega\} \subset \mathcal{U}$. Then $\cap_{i\in\omega} B_i = \emptyset$, witnessing that $\mathcal{F}$ is not ω_1-complete.

To derive a contradiction, assume that a space X is not Lindelöf, but possesses no free ω_1-complete closed filters. Consider any open cover $\mathcal{U}$ of X which contains no countable subcover. Consequently, $X \setminus (\cup\mathcal{V}) \neq \emptyset$ for any countable subset $\mathcal{V}$ of $\mathcal{U}$. Then the family $\mathcal{B} = \{X \setminus (\cup\mathcal{V}) : \mathcal{V} \in [\mathcal{U}]^{\leq\omega}\}$ is a base of some free ω_1-complete closed filter $\mathcal{F}$ on X. But this contradicts the choice of X.

In fact, the latter is the indirect proof: *Consider any open cover $\mathcal{U}$ of X which contains no countable subcover. Consequently, $X \setminus (\cup\mathcal{V}) \neq \emptyset$ for any countable subset $\mathcal{V}$ of $\mathcal{U}$. Then the family $\mathcal{B} = \{X \setminus (\cup\mathcal{V}) : \mathcal{V} \in [\mathcal{U}]^{\leq\omega}\}$ is a base of some free ω_1-complete closed filter $\mathcal{F}$ on X.* □

In what follows we assume that ordinals carry the order topology. A filter $\mathcal{F}$ on an ordinal λ of uncountable cofinality is called a *club filter* is $\mathcal{F}$ is generated by closed unbounded subsets of λ. For more about the club filter see Chapter II.6 of [19]. By $cf(\lambda)$ we denote the cofinality of λ.

Lemma 6.4. *For each ordinal λ of uncountable cofinality the club filter is a free $cf(\lambda)$-complete closed ultrafilter.*

Proof. Fix any closed subset $A \subset \lambda$. If A is unbounded, then $A \in \mathcal{F}$. Otherwise, there exists $\xi = \sup A$. Then the set $\{\alpha \in \lambda : \alpha \geq \xi + 1\}$ is an element of the club filter disjoint with A. Hence the club filter is a closed ultrafilter. Since each element of λ has a bounded open neighborhood, the club filter is free. Finally, the $cf(\lambda)$-completeness of the club filter follows from [19, Lemma 6.8]. □

By $\mathfrak{t}$ we denote the minimal cardinality of a maximal tower on ω. See ? for more details. It is known that $\omega_1 \leq \mathfrak{t} \leq \mathfrak{c}$.

Proposition 6.5. *There exists a separable first-countable normal locally compact space X of cardinality ω_1 which possesses a free ω_1-complete closed ultrafilter. Moreover, if $\mathfrak{t} = \omega_1$, then X is sequentially compact.*

Proof. The space X will be a subspace of a space constructed by Franklin and Rajagopalan in [14] (see also Example 7.1 in [8]). Fix an increasing maximal tower $\mathcal{T} = \{T_\alpha \mid \alpha \in \kappa\}$ on ω. That is $T \subset [\omega]^\omega$, $T_\alpha \subset^* T_\beta$, $|T_\beta \setminus T_\alpha| = \omega$ for any $\alpha \in \beta$, and there exists no subset $P \subset \omega$ such that $|\omega \setminus P| = \omega$ and $T_\alpha \subset^* P$ for all $\alpha \in \kappa$. The maximality of the tower $\mathcal{T}$ implies that the ordinal κ has an uncountable cofinality. Consider the space $Y(\mathcal{T}) = \mathcal{T} \cup \omega$ which is topologized as follows. Points of ω are isolated and a basic open neighborhood of $T \in \mathcal{T}$ has the form

$$B(S, T, F) = \{P \in \mathcal{T} \mid S \subset^* P \subseteq^* T\} \cup ((T \setminus S) \setminus F),$$

where $S \in \mathcal{T} \cup \{\emptyset\}$ satisfies $S \subset^* T$ and F is a finite subset of ω. By Example 7.1 from [8], $Y(\mathcal{T})$ is separable normal locally compact and sequentially compact. Note that the subspace $\mathcal{T}$ of $Y(\mathcal{T})$ is homeomorphic to the ordinal κ. Let $X = \omega \cup \{T_\alpha : \alpha \in \omega_1\}$ be a subspace of $Y(\mathcal{T})$. It is easy to see that the subspace $Z = \{T_\alpha : \alpha \in \omega_1\}$ of X is homeomorphic to the cardinal ω_1. Lemma 6.4 implies that the subspace Z possesses a free ω_1-complete closed ultrafilter $\mathcal{F}$. Since the set Z is closed in X, the filter $\mathcal{F}$ is a free ω_1-complete closed ultrafilter on X. Since ω is a dense subset of X, the space X is separable. Being an open subspace of a locally compact space Y, the space X remains locally compact. First-countablity of X follows from the definition of topology on Y. Consider any two closed disjoint subsets A, B of X. If $|A| = |B| = \omega_1$, then $A \cap Z$ and $B \cap Z$ are closed unbounded. It follows that $A \cap B \neq \emptyset$ which contradicts our assumption. Hence without loss of generality we can assume that $|A| = \omega$. is the space X normal?

If $\mathfrak{t} = \omega_1$, then we can assume that $\kappa = \omega_1$. In this case put $X = Y(\mathcal{T})$. □

Recall that a space X is called *countably compact* if X possesses no free closed filters with countable base.

It follows that every closed filter on a countably compact space can be enlarged to a ω_1-complete closed filter. Indeed, if $\mathcal{F}$ is a closed filter on a countably compact space, then the filter $\mathcal{H}$ generated by the base $\{\cap H : H \in [\mathcal{F}]^{\leq\omega}\}$ is closed and ω_1-complete. Hence we get the following.

Lemma 6.6. *Any free closed filter on a countably compact space can be enlarged to a free ω_1-complete closed ultrafilter.*

Proof. Let X be a countably compact non-compact space possessing a closed free filter $\mathcal{F}'$. Enlarge $\mathcal{F}'$ to a free closed ultrafilter $\mathcal{F}$. Consider the filter $\mathcal{H}$ on X generated by the base $\{\cap H : H \in [\mathcal{F}]^{\leq\omega}\}$.

Taking into account the countable compactness of X and the inclusion $\mathcal{F} \subset \mathcal{H}$, we obtain that $\mathcal{F} = \mathcal{H}$ is a ω_1-complete free closed ultrafilter. □

Lemma 6.6 does not hold for pseudocompact spaces. Consider Mrowka space.

A space X is called *perfectly normal* if it is normal and every closed subset in X is G_δ. In [26] Ostaszewski under $\diamondsuit$ constructed a famous example of a hereditary separable perfectly normal first-countable locally compact countably compact non-compact space of cardinality ω_1. For more information about $\diamondsuit$ see Chapter II.7 of [19]. Since the Ostaszewski space is not compact it admits a free closed filter. Then Lemma 6.6 implies the following.

Proposition 6.7. *($\diamondsuit$) There exists a hereditary separable perfectly normal first-countable locally compact countably compact space of cardinality ω_1 which possesses a free ω_1-complete closed ultrafilter.*

Weiss showed that MA implies that every perfectly normal countably compact space is compact. Thus Proposition 6.7 cannot be proved within ZFC.

According to [3] a space X is called *screened* if every open cover of X admits a σ-disjoint open refinement.

Proposition 6.8. *There exists a screened space X possessing a free ω_1-complete closed ultrafilter if and only if there exists a measurable cardinal.*

Proof. The "if" part of the proof is similar to those of Lemma 6.2.

Assume that a paracompact space X possesses a free ω_1-complete closed ultrafilter $\mathcal{F}$. Consider the open cover $\mathcal{V} = \{X \setminus F : F \text{ is a closed element of } \mathcal{F}\}$. Since the space X is screened, the cover $\mathcal{V}$ admits a refinement $\mathcal{W} = \bigcup_{n\in\omega} \mathcal{W}_n$, where for each $n \in \omega$ the family $\mathcal{W}_n$ consists of pairwise disjoint open sets. Let $\mathbf{W_n} = \bigcup \mathcal{W}_n$, $n \in \omega$ and $M = \{n \in \omega : \mathbf{W_n} \in \mathcal{F}\}$. The following argument ensures that $M \neq \emptyset$. If for any $n \in \omega$, $\mathbf{W_n} \notin \mathcal{F}$ then, by ω_1-completeness of $\mathcal{F}$, $\emptyset = \bigcap_{n\in\omega}(X \setminus \mathbf{W_n}) \in \mathcal{F}$ which implies a contradiction. Fix any $n \in M$. Enumerate $\mathcal{W}_n$ as $\{W_\alpha : \alpha \in \kappa\}$. For any $F \in \mathcal{F}$ define $H_F = \{\alpha \in \kappa : F \cap W_\alpha \neq \emptyset\}$. Consider the filter $\mathcal{H}$ on κ generated by the base $\{H_F : F \in \mathcal{F}\}$. By the definition of the cover $\mathcal{V}$, $W_\alpha \notin \mathcal{F}$ for any $\alpha \in \kappa$. Therefore, the filter $\mathcal{H}$ is free. Since the filter $\mathcal{F}$ is ω_1-complete, then so is $\mathcal{H}$. Pick any subset $A \subset \kappa$. Assume that $A \notin \mathcal{H}$. Then for every $F \in \mathcal{F}$ the set $F \cap (\bigcup_{\alpha\in\kappa\setminus A} W_\alpha) \neq \emptyset$. It follows that $F \cap (X \setminus (\bigcup_{\alpha\in A})W_\alpha) \neq \emptyset$ for all $F \in \mathcal{F}$. Since the set $G = X \setminus (\bigcup_{\alpha\in A})W_\alpha$ is closed and $\mathcal{F}$ is a closed ultrafilter, we get that $G \in \mathcal{F}$. Fix any $F \in \mathcal{F}$ such that $F \subset \mathbf{W_n}$. Then $\mathcal{H} \ni H_{F\cap G} \subset \kappa \setminus A$, witnessing that $\kappa \setminus A \in \mathcal{H}$. Hence $\mathcal{H}$ is an ultrafilter and the cardinal κ is measurable. □

By Corollary 2.4 from [3], every paracompact space is screened. Since metrizable spaces are paracompact we obtain the following.

Corollary 6.9. *There exists a metrizable space X possessing a free ω_1-complete closed ultrafilter if and only if there exists a measurable cardinal.*

References

[1] L. Acosta, I. Rubio, *On star compactifications*, Applied General Topology **14**:1 (2013) 17–32.

[2] J. Brown, A. Dow, *Remote points under the continuum hypothesis*, Algebra Universalis **60**:3 (2009), 345–364.

[3] D. Burke, *Covering properties*, in: K. Kunen, J.E. Vaughan (Eds.), *Handbook of set-theoretic topology*, North-Holland, Amsterdam (1984), 349–416.

[4] N. Carlson, J. Porter, *On open ultrafilters and maximal points*, Topol. Appl. **156** (2009) 2317–2325.

[5] R. Chandler, G. Faulkner, *Singular compactifications: the order structure*, Proc. Amer. Math. Soc. **100**:2 (1987) 377–382.

[6] E. van Douwen, *Existence and applications of remote points*, Bull. Amer. Math. Soc. **84**:1 (1978), 161–163.

[7] E. van Douwen, *Remote points*, Dissertationes Math. **188** (1981), 1–45.
[8] E. van Douwen, *The integers and topology*, in: K. Kunen, J.E. Vaughan (Eds.), *Handbook of set-theoretic topology*, North-Holland, Amsterdam (1984), 111–167.
[9] A. Dow, *Remote points in spaces with π-weight ω_1*, Fund. Math. **124** (1984), 197–205.
[10] A. Dow, *A separable space with no remote points*, Trans. Amer. Math. Soc. **312**:1 (1989), 335–353.
[11] A. Dow, T. Peters, *Products and remote points: examples and counterexamples*, Proc. Amer. Math. Soc. **104**:4 (1988), 1296–1304.
[12] A. Dow, T. Peters, *Game strategies yield remote points*, Topology Appl. **27**:3 (1987), 245–256.
[13] R. Engelking, *General topology*, Revised and completed ed., Berlin, Heldermann, 1989.
[14] S. Franklin, M. Rajagopalan, *Some examples in topology*, Trans. Amer. Math. Soc., **155**:2 (1971), 305-–314.
[15] M. Girou, *Properties of locally H-closed spaces*, Proc. Amer. Math. Soc. **113**:1 (1991) 287–295.
[16] H. Herrlich, *Essential extensions of Hausdorff spaces*, Applied Categorical Structures **2** (1994) 101–105.
[17] Jech, T. *Set theory.* The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
[18] M. Koushesh, *Compactification-like extensions*, Dissertationes Math. **476** (2011), 1–88.
[19] K. Kunen, *Set Theory, Studies in Logic and Foundations of Mathematics*, Vol. 102, Amsterdam: North-Holland, 1980.
[20] C. T. Liu, *The α-closure αX of a topological space X*, Proc. Amer. Math. Soc. **22**:3 (1969), 620–624.
[21] J. Mack, M. Rayburn, R. Woods, *Lattices of topological extensions*, Trans. Amer. Math. Soc. **189** (1974), 163–174.
[22] F. Mendivil, *Function algebras and the lattice of compactifications*, Proc. Amer. Math. Soc. **127**:6 (1999), 1863–1871.
[23] D. Mooney, *Spaces with unique Hausdorff extensions*, Topology Appl. **61** (1995), 241–256.
[24] D. Mooney, *H-bounded sets*, Topology Proceedings **18** (1993), 195–207.
[25] D. Mooney, T. Richmond, *Cardinality and structure of semilattices of ordered compactifications*, Annals of the New York Academy of Sciences **767**:1 (1995), 188–193.
[26] A. Ostaszewski, *On countably compact, perfectly normal spaces*, J. Lond. Math. Soc., **s2-14** no. 3, (1976) 505—516.
[27] J. Pelant, P. Simon, J. Vaughan, *The smallest number of free closed filters*, Fund. Math. **131** (1988), 215–221.
[28] J. Porter, *On locally H-closed spaces*, Proc. Lond. Math. Soc. **s3-20**:2 (1970), 193–204.
[29] J. Porter, C. Votaw, *H-closed extensions I*, Topology Appl. **3**:3 (1973), 211–224.
[30] J. Porter, C. Votaw, *H-closed extensions II*, Trans. Amer. Math. Soc. **202** (1975), 193–209.
[31] J. Porter, J. Vermeer, R.G. Woods, *H-closed extensions of absolutes*, Houston J. Math. **11**:1 (1985), 109–120.
[32] J. Porter, R.G. Woods, *Extensions of Hausdorff spaces*, Pacific J. Math. **103**:1 (1982), 111–134.
[33] J. Porter, R.G. Woods, *Extensions and Absolutes of Hausdorff spaces*, New York, Springer, 1988.
[34] E. Pytkeev, A. Chentsov, *Open ultrafilters and separability with the use of the operation of closure*, Proc. Stekl. Inst. Math. **299** (2017), 177–190.
[35] E. Pytkeev, A. Chentsov, *Some properties of open ultrafilters*, Izv. Inst. Mat. Inform. Udmurt. Univ. **46**:2 (2015), 140–148.
[36] T. Richmond, *Posets of ordered compactifications*, Bull. Austral. Math. Soc. **47** (1993), 59–72.
[37] K. Srivastava, *On singular H-closed extensions*, Proc. Amer. Math. Soc. **120**:1 (1994), 295–300.
[38] J. Terasava, *On the non-normality of $\beta X \setminus \{p\}$ for non-discrete spaces*, Topology Proc. **27**:1 (2003), 335–344.
[39] M. Tikoo, *Remainders of H-closed extensions*, Topology Appl. **23**:2 (1986), 117–128.
[40] A. Veksler, *Maximal nowhere dense sets and their applications to problems of existence of remote points and of weak P-points*, Mathematische Nachrichten **150** (1991), 263–275.
[41] J. Vermeer, *Closed subspaces of H-closed spaces*, Pacific J. Math. **118**:1 (1985), 229–247.
[42] W. Weiss, *Countably Compact Spaces and Martin's Axiom*, Can. J. Math. **30**:2 (1978), 243–249.

Universität Wien, Institut für Mathematik, Kurt Gödel Research Center, Wien, Austria.
Email address: `sbardyla@yahoo.com`
URL: `http://www.logic.univie.ac.at/~bardylas55/`

Institute of Mathematics, P.J. Šafárik University in Košice, Slovakia
Email address: `jaroslav.supina@upjs.sk`
URL: `https://umv.science.upjs.sk/supina/index.html`

Universität Wien, Institut für Mathematik, Kurt Gödel Research Center, Wien, Austria.
Email address: lzdomsky@gmail.com
URL: http://www.logic.univie.ac.at/~lzdomsky/