DEFINING NON-EMPTY SMALL SETS FROM FAMILIES OF INFINITE SETS

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Abstract. We describe circumstances under which non-empty small sets can be defined from families of infinite sets. In the process, we establish generalizations of Mansfield's perfect set theorem for κ-Souslin sets and the Lusin-Novikov uniformization theorem.

An *intersecting family* is a collection of sets, any two of which have non-empty intersection. These have been the subject of much study in combinatorics, and have also recently come up in descriptive set theory.

In particular, while [CCM07] focused on σ-ideals associated with countable Borel equivalence relations, the main result there depended on the simple but surprising observation that for all sets X and all non-empty intersecting families $\mathscr{A}$ of finite subsets of X, a non-empty finite subset of X is definable from $\mathscr{A}$. This observation was generalized and strengthened in [CCCM09], where quantitative analogs were obtained for non-empty families of non-empty sets that do not contain infinite pairwise disjoint subfamilies.

To be precise, let $[X]^{\leq\kappa}_{+}$ denote the family of all non-empty subsets of X whose cardinality is at most κ, and let L denote the signature consisting of a unary relation symbol $\dot{\mathscr{A}}$ and a binary relation symbol $\dot{\in}$. Associated with each cardinal κ, set X, and family $\mathscr{A} \subseteq [X]^{\leq\kappa}_{+}$ is the L-structure

$$\mathscr{M}_{\mathscr{A}} = (X \cup [X]^{\leq\kappa}_{+}, \mathscr{A}, \in \cap (X \times [X]^{\leq\kappa}_{+})).$$

We do not specify κ in our notation as it will be clear from context. Observe that both X and $[X]^{\leq\kappa}_{+}$ are definable in $\mathscr{M}_{\mathscr{A}}$.

Theorem 1. *There is a disjunction of first-order L-formulae $\theta(x)$ with the property that if $k \in \omega$, X is a set, and $\mathscr{A} \subseteq [X]^{\leq k}_{+}$ is a non-empty family that does not have an infinite pairwise disjoint subfamily, then $\{x \mid \mathscr{M}_{\mathscr{A}} \models \theta(x)\}$ is a non-empty finite subset of X.*

2010 *Mathematics Subject Classification.* Primary 03E15; Secondary 05B30.
Key words and phrases. Definability, dichotomy theorems, intersecting families.
†The first author was supported in part by NSF Grant DMS-0801189.
‡The fourth author was supported in part by NSF VIGRE Grant DMS-0502315.

Proof. See [CCCM09]. ⊠

Our goal here is to investigate analogs of Theorem 1 in which k is replaced with an infinite cardinal. We begin with the problem of defining non-empty intersecting families from non-empty families that do not contain large pairwise disjoint subfamilies. Although it is possible to give a direct proof of the fact we have in mind, we first mention a useful auxiliary result. We say that a set is a *core* for a family of sets if it intersects every set in the family.

Proposition 2 (AC)**.** *Suppose that κ is an infinite cardinal, X is a set, and $\mathscr{A} \subseteq [X]^{\leq\kappa}_{+}$. Then the following are equivalent:*

1. *No pairwise disjoint subfamily of $\mathscr{A}$ has cardinality κ^+.*
2. *There is a core for $\mathscr{A}$ of cardinality at most κ.*
3. *The family $\mathscr{A}$ is the union of κ-many intersecting subfamilies.*

Proof. To see (1) $\implies$ (2), observe that if $\mathscr{B}$ is a maximal pairwise disjoint subfamily of $\mathscr{A}$, then $\bigcup \mathscr{B}$ is a core for $\mathscr{A}$ of cardinality at most κ. To see (2) $\implies$ (3), observe that if C is a core for $\mathscr{A}$, then $\mathscr{A}$ is the union of the intersecting families $\mathscr{A}_x = \{A \in \mathscr{A} \mid x \in A\}$ for $x \in C$. To see (3) $\implies$ (1), observe that if $\mathscr{A}$ is the union of intersecting families $\mathscr{A}_\alpha$ for $\alpha \in \kappa$ and $\mathscr{B}$ is a pairwise disjoint subfamily of $\mathscr{A}$, then no two sets in $\mathscr{B}$ are in the same $\mathscr{A}_\alpha$, thus $\mathscr{B}$ has cardinality at most κ. ⊠

We now establish the fact to which we alluded earlier:

Proposition 3 (AC)**.** *There is a first-order L-formula $\theta(x)$ with the property that if κ is an infinite cardinal, X is a set, and $\mathscr{A} \subseteq [X]^{\leq\kappa}_{+}$ is a non-empty family that does not have a pairwise disjoint subfamily of cardinality κ^+, then $\{x \mid \mathscr{M}_{\mathscr{A}} \models \theta(x)\}$ is a non-empty intersecting subfamily of $[X]^{\leq\kappa}_{+}$.*

Proof. Fix a first-order L-formula $\theta(x)$ with the property that if κ is an infinite cardinal, X is a set, $\mathscr{A} \subseteq [X]^{\leq\kappa}_{+}$, and $x \in X \cup [X]^{\leq\kappa}_{+}$, then $\mathscr{M}_{\mathscr{A}} \models \theta(x)$ if and only if x is a core for $\mathscr{A}$ containing a set in $\mathscr{A}$. Clearly $\{x \mid \mathscr{M}_{\mathscr{A}} \models \theta(x)\}$ is an intersecting family, and Proposition 2 ensures that if $\mathscr{A}$ is a non-empty family that does not have a pairwise disjoint subfamily of cardinality κ^+, then the family $\{x \mid \mathscr{M}_{\mathscr{A}} \models \theta(x)\}$ is non-empty, thus the formula $\theta(x)$ is as desired. ⊠

It only remains to give a way of defining non-empty small sets from non-empty intersecting families of infinite sets. However, a moment's reflection reveals that this is not always possible:

Example 4. Suppose that κ is an aleph, X is a set, and $Y \subseteq X$ is a set of cardinality κ, and define $\mathscr{A} = \{Z \in [X]^{\leq\kappa}_{+} \mid Y \triangle Z \text{ is finite}\}$.

Clearly $\mathscr{A}$ is a non-empty intersecting family, and it is not difficult to see that the automorphism group of $\mathscr{M}_{\mathscr{A}}$ is simply the group of permutations of $X \cup [X]^{\leq\kappa}_{+}$ induced by permutations τ of X with the property that $\tau[Y] \in \mathscr{A}$. It is clear that the latter group acts transitively on X, and it follows that there is no L-formula $\theta(x)$ for which $\{x \in X \mid \mathscr{M}_{\mathscr{A}} \models \theta(x)\}$ is a non-empty proper subset of X.

Remark 5. It is not difficult to see that if the complement of Y is Dedekind infinite, then the automorphism group of $\mathscr{M}_{\mathscr{A}}$ also acts transitively on $\mathscr{A}$, in which case it follows that there is no L-formula $\theta(x)$ for which $\{A \in \mathscr{A} \mid \mathscr{M}_{\mathscr{A}} \models \theta(A)\}$ is a non-empty proper subset of $\mathscr{A}$.

In light of Example 4, we will shift our focus to circumstances under which non-empty small sets can be defined. Given that we initially encountered intersecting families in the descriptive set-theoretic context, it is natural to first place definability constraints on $\mathscr{A}$. Towards this end, it will be convenient to deal with sequences rather than sets. Let ${}^{\kappa}X$ denote the family of κ-length sequences of elements of X. Set ${}^{<\kappa}X = \bigcup_{\alpha\in\kappa} {}^{\alpha}X$ and ${}^{\leq\kappa}X = {}^{<\kappa}X \cup {}^{\kappa}X$. We associate each sequence with its image, so that we can talk about pairs of sequences being comparable (under containment) or disjoint, and sets of sequences being chains (under containment), intersecting, or pairwise disjoint.

Suppose that κ is an aleph and X is a Hausdorff space. A set $A \subseteq X$ is *κ-Souslin* if it is a continuous image of a closed subset of ${}^{\omega}\kappa$, where κ is endowed with the discrete topology. It is easy to see that non-empty κ-Souslin sets are continuous images of ${}^{\omega}\kappa$ itself. A set is *bi-κ-Souslin* if it is κ-Souslin and its complement is κ-Souslin. A set is *analytic* if it is ω-Souslin. A set $B \subseteq X$ is *κ-Borel* if it is in the closure of the topology of X under complements and intersections (and therefore unions) of length strictly less than κ. A set is *Borel* if it is ω_1-Borel. A set $C \subseteq X$ is *ω-universally Baire* if $\varphi^{-1}(C)$ has the Baire property for every continuous function $\varphi\colon {}^{\omega}\omega \to X$.

In what follows, we will consider families $\mathscr{A} \subseteq [X]^{\leq\omega}_{+}$ for which the corresponding sets $A \subseteq {}^{\omega}X$ are κ-Souslin. Of course, Example 4 can be used to see that even with this additional constraint, we still cannot in general hope to define non-empty small sets from non-empty intersecting families. One way of getting around this is to rule out large pairwise incomparable subfamilies instead of large pairwise disjoint subfamilies. For each $n \in \omega$, define $\mathrm{proj}_n\colon {}^{\omega}X \to X$ by $\mathrm{proj}_n(x) = x(n)$.

Theorem 6 (AC)**.** *Suppose that κ is an infinite cardinal, X is a Hausdorff space, and $A \subseteq {}^{\omega}X$ is a κ-Souslin set that does not have a pairwise incomparable perfect subset. Then $|\bigcup_{n\in\omega} \mathrm{proj}_n(A)| \leq \kappa$.*

Proof. Clearly we can assume that A is non-empty. Fix a continuous surjection $\varphi \colon {}^{\omega}\kappa \to A$ and let R denote the pullback of the containment relation on A through φ. Then R is Borel, and therefore ω-universally Baire and bi-κ-Souslin. Miller's generalization [Mil09] of Theorem 5.1 of [HMS88] to such quasi-orders therefore implies that there are R-chains $B_\alpha \subseteq {}^{\omega}\kappa$ such that ${}^{\omega}\kappa = \bigcup_{\alpha \in \kappa} B_\alpha$. It only remains to check that $\bigcup_{n \in \omega} |\mathrm{proj}_n \circ \varphi[B_\alpha]| \leq \kappa$ for all $\alpha \in \kappa$. Towards this end, simply observe that otherwise, a straightforward transfinite induction yields a strictly increasing R-chain of length κ^+, which contradicts Miller's generalization [Mil09] of Theorem 3.1 of [HMS88] to ω-universally Baire bi-κ-Souslin quasi-orders. $\boxtimes$

The following example suggests that the conclusion of Theorem 6 cannot be substantially improved:

Example 7. Set $X = \mathbb{Q}$ and $\mathscr{A} = \{\{q \in \mathbb{Q} \mid q < r\} \mid r \in \mathbb{R} \setminus \mathbb{Q}\}$. Clearly $\mathscr{A}$ is a non-empty chain. It is not difficult to see that the automorphism group of $\mathscr{M}_{\mathscr{A}}$ is simply the group of permutations of $X \cup [X]^{\leq \omega}_{+}$ induced by order-preserving permutations of $\mathbb{Q}$. As all countable dense linear orders without endpoints are isomorphic, the latter group acts transitively on X and $\mathscr{A}$, thus there is no L-formula $\theta(x)$ for which $\{x \in X \mid \mathscr{M}_{\mathscr{A}} \models \theta(x)\}$ is a non-empty proper subset of X or $\{A \in \mathscr{A} \mid \mathscr{M}_{\mathscr{A}} \models \theta(A)\}$ is a non-empty proper subset of $\mathscr{A}$.

A somewhat different way around Example 4 is to work with a more expressive language. Let L^+ denote the signature consisting of a sequence of unary function symbols $(\dot{\varphi}_n)_{n \in \omega}$ and a binary relation symbol $\dot{\sqsubseteq}$. Associated with each cardinal κ, set X, and function $\varphi \colon {}^{\omega}\kappa \to {}^{\omega}X$ is the L^+-structure

$$\mathscr{M}_{\varphi} = (X \cup {}^{\leq \omega}\kappa, (\mathrm{proj}_n \circ \varphi)_{n \in \omega}, \sqsubseteq \cap ({}^{<\omega}\kappa \times {}^{\leq \omega}\kappa)),$$

where we adopt the convention that $(\mathrm{proj}_n \circ \varphi)(x) = \emptyset$ for $x \notin {}^{\omega}\kappa$. Again, we do not specify κ in our notation as it will be clear from context. Observe that X, ${}^{<\omega}\kappa$, and ${}^{\omega}\kappa$ are definable in $\mathscr{M}_{\varphi}$.

Theorem 8. *There is a disjunction of first-order L^+-formulae $\theta(x)$ with the property that if κ is an aleph, X is a Hausdorff space, $A \subseteq {}^{\omega}X$ is a κ-Souslin set that does not have a pairwise disjoint perfect subset, and $\varphi \colon {}^{\omega}\kappa \to {}^{\omega}X$ is a continuous function with $A = \varphi[{}^{\omega}\kappa]$, then $\{x \mid \mathscr{M}_{\varphi} \models \theta(x)\}$ is a non-empty subset of X of cardinality at most κ.*

Proof. It is clearly sufficient to show that if κ is an aleph, X is a Hausdorff space, $A \subseteq {}^{\omega}X$ is a κ-Souslin set that does not have a pairwise disjoint perfect subset, and $\varphi \colon {}^{\omega}\kappa \to {}^{\omega}X$ is a continuous function with $A = \varphi[{}^{\omega}\kappa]$, then there exist $s \in {}^{<\omega}\kappa$ and a first-order L^+-formula $\theta(s, x)$

such that $\{x \mid \mathscr{M}_\varphi \models \theta(s,x)\}$ is a non-empty finite subset of X. By Theorem 1, it is enough to show that there exist $n \in \omega$ and $s \in {}^{<\omega}\kappa$ for which the set $\{\varphi(x) \restriction n \mid x \in \mathcal{N}_s\}$ has no infinite pairwise disjoint subsets. Suppose, towards a contradiction, that this is not the case.

For each $n \in \omega$, we say that a function $u \colon {}^n2 \to {}^{<\omega}\kappa$ is *extended* by a function $v \colon {}^n2 \to {}^{<\omega}\kappa$ if $u(s) \sqsubseteq v(s)$ for all $s \in {}^n2$. An *n-approximation* is a function $v \colon {}^n2 \to {}^{<\omega}\kappa$ with the property that for all distinct $s,t \in {}^n2$ and all $x,y \in {}^\omega\kappa$ with $v(s) \sqsubseteq x$ and $v(t) \sqsubseteq y$, the sequences $\varphi(x) \restriction n$ and $\varphi(y) \restriction n$ are disjoint.

Lemma 9. *Suppose that $n \in \omega$ and $u \colon {}^n2 \to {}^{<\omega}\kappa$. Then there is an n-approximation which extends u.*

Proof of lemma. Fix an enumeration $(s_k)_{k \in 2^n}$ of ${}^n2$. Our assumption that for all $n \in \omega$ and $s \in {}^{<\omega}\kappa$ the set $\{\varphi(x) \restriction n \mid x \in \mathcal{N}_s\}$ has an infinite pairwise disjoint subset ensures that we can recursively choose $x_k \in \mathcal{N}_{u(s_k)}$ such that for all $j \in k$ the sequences $\varphi(x_j) \restriction n$ and $\varphi(x_k) \restriction n$ are disjoint. Fix a natural number $l \geq \max_{s \in {}^n2} |u(s)|$ sufficiently large that for all $j \in k \in 2^n$ and all $y_j, y_k \in {}^\omega\kappa$ with $x_j \restriction l = y_j \restriction l$ and $x_k \restriction l = y_k \restriction l$, the sequences $\varphi(y_j) \restriction n$ and $\varphi(y_k) \restriction n$ are disjoint. Clearly the function $v \colon {}^n2 \to {}^{<\omega}\kappa$ given by $v(s_k) = x_k \restriction l$ for $k \in 2^n$ is an n-approximation which extends u. ⊠

Let v_0 denote the 0-approximation given by $v_0(\emptyset) = \emptyset$. Given an n-approximation v_n, define $u_{n+1} \colon {}^{n+1}2 \to {}^{<\omega}\kappa$ by $u_{n+1}(s^\frown i) = u_n(s)^\frown i$ for $i \in 2$ and $s \in {}^n2$, and let v_{n+1} denote the $(n+1)$-approximation obtained from u_{n+1} by applying Lemma 9.

Define a continuous function $\psi \colon {}^\omega2 \to {}^\omega\kappa$ by setting

$$\psi(x) = \lim_{n \to \omega} v_n(x \restriction n),$$

and define $\pi \colon {}^\omega2 \to X$ by $\pi = \varphi \circ \psi$. We will obtain the desired contradiction by showing that $P = \pi[{}^\omega2]$ is a pairwise disjoint perfect set. It is clearly sufficient to show that if $x,y \in {}^\omega2$ are distinct, then $\pi(x)(i) \neq \pi(y)(j)$ for all $i,j \in \omega$. Towards this end, fix a natural number $n > i,j$ with $x \restriction n \neq y \restriction n$, and observe that $v_n(x \restriction n) \sqsubseteq \psi(x)$ and $v_n(y \restriction n) \sqsubseteq \psi(y)$, so the definition of n-approximation ensures that $\pi(x) \restriction n$ and $\pi(y) \restriction n$ are disjoint. ⊠

Remark 10. In the special case that $\mathsf{AD}_\mathbb{R}$ holds and X is an analytic Hausdorff space, Theorem 8 yields a definition of a non-empty countable subset of X for all subsets of ${}^\omega X$ which do not have pairwise disjoint perfect subsets.

Theorem 8 avoids the need for Proposition 3. Moreover, its hypothesis on the cardinality of pairwise disjoint subfamilies is weaker and its proof does not require the axiom of choice. It is therefore natural to ask whether the special case of Proposition 3 for suitably definable sets has an analogous improvement. In order to see that this is indeed the case, we will first establish a generalization of Mansfield's perfect set theorem for κ-Souslin sets (see Theorem 2C.2 of [Mos09]).

We say that ω-sequences x and y are *graph disjoint* if $x(i) \neq y(i)$ for all $i \in \omega$. We say that a set $A \subseteq {}^\omega X$ is *graph intersecting* if no two sequences in A are graph disjoint.

Theorem 11. *Suppose that κ is an aleph, X is a Hausdorff space, and $A \subseteq {}^\omega X$ is κ-Souslin. Then one of the following holds:*

1. *The set A is the union of κ-many graph intersecting sets which are κ^+-Borel when considered as subsets of A.*
2. *The set A has a pairwise disjoint perfect subset.*

Proof. A *graph* on X is an irreflexive set $G \subseteq X \times X$. A *κ-coloring* of G is a function $c \colon X \to \kappa$ such that $c(x) \neq c(y)$ for all $(x, y) \in G$.

Let G denote the graph on A given by

$$G = \{(x, y) \in A \times A \mid x \text{ and } y \text{ are graph disjoint}\}.$$

Lemma 12. *Suppose that there is a κ^+-Borel κ-coloring of G. Then A is the union of κ-many graph intersecting sets which are κ^+-Borel when considered as subsets of A.*

Proof of lemma. Simply observe that if $c \colon A \to \kappa$ is a κ^+-Borel κ-coloring of G, then the sets $A_\alpha = c^{-1}(\{\alpha\})$ for $\alpha \in \kappa$ are as desired. ⊠

Recall the graph G_0 from [KST99], which is obtained by fixing sequences $s_n \in {}^n 2$ such that $\forall s \in {}^{<\omega}2 \exists n \in \omega \ (s \sqsubseteq s_n)$, and setting

$$G_0 = \{(s_n \frown i \frown x, s_n \frown (1 - i) \frown x) \mid i \in 2, \ n \in \omega, \text{ and } x \in {}^\omega 2\}.$$

Lemma 13. *Suppose that there is a continuous homomorphism from G_0 to G. Then A has a pairwise disjoint perfect subset.*

Proof of lemma. Fix a continuous homomorphism $\varphi \colon {}^\omega 2 \to A$ from G_0 to G, and define $M = \{((i, x), y) \in (\omega \times X) \times {}^\omega 2 \mid x = \varphi(y)(i)\}$.

Sublemma 14. *Suppose that $(i, x) \in \omega \times X$. Then $M_{(i,x)}$ is meager.*

Proof of sublemma. By the proof of Proposition 6.2 of [KST99], if $M_{(i,x)}$ is not meager, then there exists $(y, z) \in G_0 \restriction M_{(i,x)}$, in which case $(\varphi(y), \varphi(z)) \in G$, contradicting the fact that $\varphi(y)(i) = x = \varphi(z)(i)$. ⊠

Define $R = \{(y, z) \in {}^\omega 2 \times {}^\omega 2 \mid \varphi(y) \text{ and } \varphi(z) \text{ are disjoint}\}$.

Sublemma 15. *The set R is comeager.*

Proof of sublemma. Sublemma 14 easily implies that every vertical section of R is comeager, so the desired result is a consequence of the Kuratowski-Ulam theorem (see Theorem 8.41 of [Kec95]). $\boxtimes$

Sublemma 15 and Mycielski's theorem (see Theorem 19.1 of [Kec95]) ensure that there is a perfect set $Q \subseteq {}^{\omega}2$ with $(x, y) \in R$ for all distinct $x, y \in Q$, and it follows that the set $P = \varphi[Q]$ is as desired. $\boxtimes$

As Kanovei's generalization [Kan97] of the Kechris-Solecki-Todorcevic dichotomy theorem [KST99] ensures that there is either a κ^+-Borel κ-coloring of G or a continuous homomorphism from G_0 to G, the desired result follows from Lemmas 12 and 13. $\boxtimes$

Remark 16. Mansfield's perfect set theorem for κ-Souslin sets is essentially the special case of Theorem 11 for constant sequences.

Remark 17. In the special case that $\mathtt{AD}^+$ holds, an analogous result for κ^+-Borel sets can be established using the Caicedo-Ketchersid version [CK09] of the Kechris-Solecki-Todorcevic theorem.

As a corollary, we obtain the desired version of Proposition 3:

Theorem 18 ($\mathtt{AC}$)**.** *There is a first-order L-formula $\theta(x)$ with the property that if κ is an infinite cardinal, X is a Hausdorff space, $A \subseteq {}^{\omega}X$ is a κ-Souslin set that does not have a pairwise disjoint perfect subset, and $\mathscr{A}$ is the corresponding family of countable sets in $[X]^{\leq\kappa}_{+}$, then $\{x \mid \mathscr{M}_{\mathscr{A}} \models \theta(x)\}$ is a non-empty intersecting subfamily of $[X]^{\leq\kappa}_{+}$.*

Proof. Theorem 11 implies that $\mathscr{A}$ is the union of κ-many intersecting subfamilies, so Proposition 2 ensures that $\mathscr{A}$ does not have a pairwise disjoint subfamily of cardinality κ^+, thus Proposition 3 yields the desired result. $\boxtimes$

Remark 19. The special case of Theorem 18 in which $\kappa = \omega$ can be pushed through without the axiom of choice. To see this, it is enough to show that $\mathscr{A}$ has a countable core without using the axiom of choice. Towards this end, simply follow the proof of Theorem 11 so as to obtain witnesses to the analyticity of countably many intersecting analytic sets $A_n \subseteq A$ whose union is A, use these witnesses to choose a single sequence out of each A_n, and observe that the set of points along the chosen sequences is the desired countable core for $\mathscr{A}$.

We close by noting that while there is a more direct proof of Theorem 11, our argument also yields the following generalization:

Theorem 20. *Suppose that κ is an aleph, X and Y are Hausdorff, and $R \subseteq X \times {}^{\omega}Y$ is κ-Souslin. Then one of the following holds:*

(1) *The set R is the union of κ-many sets whose vertical sections are graph intersecting and which are κ^+-Borel when considered as subsets of R.*
(2) *Some vertical section of R has a pairwise disjoint perfect subset.*

Proof. Let G denote the graph on R given by

$$G = \{((x, y), (x, z)) \in R \times R \mid y \text{ and } z \text{ are graph disjoint}\}.$$

Lemma 21. *Suppose that there is a κ^+-Borel κ-coloring of G. Then R is the union of κ-many sets whose vertical sections are graph intersecting and which are κ^+-Borel when considered as subsets of R.*

Proof of lemma. Simply observe that if $c \colon R \to \kappa$ is a κ^+-Borel κ-coloring of G, then the sets $R_\alpha = c^{-1}(\{\alpha\})$ for $\alpha \in \kappa$ are as desired. ⊠

Lemma 22. *Suppose that there is a continuous homomorphism from G_0 to G. Then there is a vertical section of R which has a pairwise disjoint perfect subset.*

Proof of lemma. Suppose that $\varphi \colon {}^\omega 2 \to R$ is a continuous homomorphism from G_0 to G, and let E_0 denote the equivalence relation on ${}^\omega 2$ given by

$$x E_0 y \iff \exists m \in \omega \forall n \in \omega \setminus m \; (x(n) = y(n)).$$

Then $\mathrm{proj}_X \circ \varphi$ is a continuous homomorphism from E_0 to $\Delta(X)$, and it is not difficult to see that every such function is constant. Let x denote its constant value, and let G_x denote the graph on R_x given by

$$G_x = \{(y, z) \in R_x \times R_x \mid y \text{ and } z \text{ are graph disjoint}\}.$$

Then $\mathrm{proj}_{{}^\omega Y} \circ \varphi$ is a continuous homomorphism from G_0 to G_x, so Lemma 13 yields a pairwise disjoint perfect subset of R_x. ⊠

As Kanovei's generalization of the Kechris-Solecki-Todorcevic dichotomy theorem ensures that there is either a κ^+-Borel κ-coloring of G or a continuous homomorphism from G_0 to G, the desired result follows from Lemmas 21 and 22. ⊠

Remark 23. The natural generalization of the Lusin-Novikov uniformization theorem (see Theorem 18.10 and Exercise 35.13 of [Kec95]) to κ-Souslin sets is essentially the special case of Theorem 11 for constant sequences.

Remark 24. In the special case that AD^+ holds, an analogous result for κ^+-Borel sets can be established using the Caicedo-Ketchersid version of the Kechris-Solecki-Todorcevic theorem.

Acknowledgments. We would like to thank Alexander S. Kechris for useful conversations concerning the material presented here.

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