Local Banach-space dichotomies

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Banach space webinar

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We will work in real spaces, but everything works as well in complex spaces.
Gowers’ classification program

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In his famous paper *An infinite Ramsey theorem and some Banach-space dichotomies* (Ann. Math. ’02), Gowers suggests a weak classification program for separable Banach spaces, up to subspaces. The goal is to build a list of classes of separable Banach spaces (usually called a Gowers list), as fine as possible, and satisfying the following conditions:

(1) the classes are hereditary: if X belongs to a class C then all subspaces of X also belong to C (or, in the case of classes defined by properties of bases, all block-subspaces of X belong to C);
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3. knowing that a space belongs to a class gives much information about the structure of this space;

4. every Banach space contains a subspace belonging to one of the classes.
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**Theorem ("The first dichotomy", Gowers, '96)**

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- *either $Y$ has an unconditional basis;*
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**Theorem ("The first dichotomy", Gowers, '96)**

*Every Banach space X contains a subspace Y such that:*

- either Y has an unconditional basis;
- or Y is hereditarily indecomposable (HI), that is, it contains no topological direct sum of two subspaces.
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Gowers’ first dichotomy provides a Gowers list with two classes: the class of spaces having an unconditional basis, and the class of HI spaces.
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Gowers’ first dichotomy provides a Gowers list with two classes: the class of spaces having an unconditional basis, and the class of HI spaces. The fact that HI spaces have very few operators justifies property (3) for this class.
Ferenczi–Rosendal’s third dichotomy

Definition
A Banach space is **minimal** if it embeds into all of its subspaces.
Ferenczi–Rosendal’s third dichotomy

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If \((x_i)_{i \in I}\) is a family of vectors in a Banach space, we denote by \([x_i \mid i \in I]\) the closed subspace it spans.

Definition

A Banach space \(Y\) is tight in a basis \((e_i)\) if there exists a sequence of successive intervals of integers \(I_0 < I_1 < \ldots\) such that for every infinite \(A \subseteq \mathbb{N}\), \(Y\) cannot be embedded into \([e_i \mid i \notin \bigcup_{n \in A} I_n]\).

A basis \((e_i)\) is tight if every Banach space is tight in it. A Banach space \(X\) is tight if it admits a tight basis. For instance, Tsirelson’s space is tight.
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This dichotomy, combined with the first one, provides a Gowers list with three classes:

- minimal spaces with an unconditional basis;
- spaces having an unconditional and tight basis;
- tight HI spaces.
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- spaces having an unconditional and tight basis;
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Other dichotomies by Gowers and Ferenczi–Rosendal extend this list to 6 classes (all of whose are known to be nonempty) and 19 possible subclasses.
A non-example: James’ space

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Hence the idea to prove local dichotomies, i.e., dichotomies that are similar to the latter ones but where the outcome space can always be found “locally”, that is, in a fixed family of subspaces. The word “local” used here was stolen to Ramsey-theorists.
Consider a local property \((P)\) of Banach spaces, that is, a property such that:

- a Banach space \(X\) has \((P)\) with constant \(C\) if and only if all of its finite-dimensional subspaces have \((P)\) with constant \(C\);
- if a space \(X\) has \((P)\) with constant \(C\), and if \(Y\) is \(K\)-isomorphic to \(X\), then \(Y\) has \((P)\) with constant \(KC\).

This includes, for example, being Hilbertian, having type \(p\), cotype \(q\)...
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**Informal fact**

*Usual Banach-space dichotomies can be localized to spaces failing property \((P)\).*

We can actually generalize this to certain properties that are not local; such properties can be defined via the notion of degree.
A pair \((X, F)\) where \(X\) is a (finite- or infinite-dimensional) Banach space, and \(F\) is a finite-dimensional subspace of \(X\) will be called an approximation pair.

Denote by \(\AP\) the class of all approximation pairs.

Definition

A degree is a mapping \(d: \AP \to \mathbb{R}^+\) such that for every operators \(S, T\) making the following diagram commute:

\[
\begin{array}{ccc}
F & \xrightarrow{\iota} & G \\
\downarrow & & \downarrow \\
X & \xrightarrow{\iota} & Y
\end{array}
\]

we have

\[d(Y, G) \leq \|S\| \cdot \|T\| \cdot d(X, F)\].

The degree \(d\) is said to be local if \(d(X, F)\) only depends on \(F\) (in which case it is simply denoted by \(d(F)\)).
A pair \((X, F)\) where \(X\) is a (finite- or infinite-dimensional) Banach space, and \(F\) is a finite-dimensional subspace of \(X\) will be called an **approximation pair**. Denote by \(AP\) the class of all approximation pairs.

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The degree \(d\) is said to be *local* if \(d(X, F)\) only depends on \(F\) (in which case it is simply denoted by \(d(F)\)).
Degrees

Definition

Let $d$ be a degree. A Banach space is said to be $d$-small if $\sup_{F \subseteq X} d(X, F) < \infty$, and $d$-large otherwise.
Degrees

**Definition**

Let $d$ be a degree. A Banach space is said to be $d$-small if $\sup_{F \subseteq X} d(X, F) < \infty$, and $d$-large otherwise.

A local property and a local degree are the same thing. If $(P)$ is a local property, let $d(F)$ be the infimum of constants $C$ such that $F$ has property $(P)$ with constant $C$. Then $d$ is a local degree, and being $d$-small is equivalent to having property $(P)$.
(1) \( d(F) = \dim(F) \) is a local degree. \( d \)-small spaces are exactly finite-dimensional spaces.
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(2) $d_2(F) = d_{BM}(F, \ell_2^{\dim(F)})$ is a local degree, the Hilbertian degree. $d_2$-small spaces are exactly Hilbertian spaces.
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(3) Fix $1 \leq p \leq 2$. Then $d(F) = T_p(F)$, the type constant, is a degree. $d$-small spaces are exactly those having type $p$. 

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(3) Fix $1 \leq p \leq 2$. Then $d(F) = T_p(F)$, the type constant, is a degree. $d$-small spaces are exactly those having type $p$. The same works for cotype.
(4) Fix $1 \leq p \leq \infty$. Define $d(X, F)$ as the infimum of the $M$’s for which the canonical inclusion of $F$ into $X$ $M$-factorizes through some $\ell^n_p$, meaning that there exists $n \in \mathbb{N}$ and operators $U : F \to \ell^n_p$ and $V : \ell^n_p \to X$ with $\|U\| \cdot \|V\| = M$, making the following diagram commute:

\[ \ell^n_p \quad \downarrow U \quad \quad \quad \quad \quad \downarrow V \quad \ell^n_p \]

\[ F \quad \downarrow \iota \quad \quad \quad \quad \quad \downarrow \iota \quad X \]

Then $d$ is a non-local degree.
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![Diagram](https://via.placeholder.com/150)

Then $d$ is a non-local degree. By a result by Lindenstrauss and Rosenthal, we have that:

- if $1 < p < \infty$, a space is $d$-small if and only if it is either a $\mathcal{L}_p$-space, or a Hilbertian space;
- if $p = 1$ or $p = \infty$, a space is $d$-small if and only if it is a $\mathcal{L}_p$-space.
(5) Let $d(X, F)$ be the infimum of the $M$'s for which the canonical inclusion of $F$ into $X$ $M$-factorizes through some space with a 1-unconditional basis. Then $d$ is a non-local degree.
(5) Let \( d(X, F) \) be the infimum of the \( M \)'s for which the canonical inclusion of \( F \) into \( X \) \( M \)-factorizes through some space with a 1-unconditional basis. Then \( d \) is a non-local degree. A space is \( d \)-small iff it has Gordon-Lewis local unconditional structure.
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(6) More generally, recall that an normed operator ideal $\mathcal{U}$ is given by, for all Banach spaces $X$ and $Y$, a vector subspace $\mathcal{U}(X, Y) \subseteq \mathcal{L}(X, Y)$ and a complete norm $N$ on $\mathcal{U}(X, Y)$, such that, whenever $S \in \mathcal{L}(X, Y)$, $T \in \mathcal{U}(Y, Z)$ and $U \in \mathcal{L}(Z, W)$, we have $UTS \in \mathcal{U}(X, W)$ and $N(UTS) \leq \|U\| \cdot N(T) \cdot \|S\|$, and such that moreover $Id_{\mathbb{R}} \in \mathcal{U}(\mathbb{R}, \mathbb{R})$ with $N(Id_{\mathbb{R}}) = 1$. 
(5) Let $d(X, F)$ be the infimum of the $M$’s for which the canonical inclusion of $F$ into $X$ $M$-factorizes through some space with a 1-unconditional basis. Then $d$ is a non-local degree. A space is $d$-small iff it has Gordon-Lewis local unconditional structure.

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(7) Let $d(X, F)$ be the infimum of the $K$’s such that $F$ is $K$-complemented in $X$. Then $d$ is not a degree. This is somewhat surprising, since if it were one, $d$-small spaces would exactly be Hilbertian spaces.
Proposition

Let $d$ be a degree.

- Finite-dimensional spaces are $d$-small.
Properties of degrees

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- Finite-dimensional spaces are \( d \)-small.
- The properties of being \( d \)-small and \( d \)-large are preserved by isomorphism.
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- The properties of being $d$-small and $d$-large are preserved by isomorphism.
- A complemented subspace of a $d$-small space is $d$-small.
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**Proposition**

*Let $d$ be a degree.*

- *Finite-dimensional spaces are $d$-small.*
- *The properties of being $d$-small and $d$-large are preserved by isomorphism.*
- *A complemented subspace of a $d$-small space is $d$-small.*
- *If $d$ is local, every subspace of a $d$-small space is $d$-small.*
Proposition

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- Finite-dimensional spaces are $d$-small.
- The properties of being $d$-small and $d$-large are preserved by isomorphism.
- A complemented subspace of a $d$-small space is $d$-small.
- If $d$ is local, every subspace of a $d$-small space is $d$-small.
- If $(X_n)$ is a decreasing sequence of $d$-large spaces, then there is a $d$-large space $X_\infty$ such that $X_\infty \subseteq^* X_n$ for every $n$ (meaning that $X_n$ contains a finite-codimensional subspace of $X_\infty$).
Better FDD’s

Given a degree $d$, it is not clear whether every $d$-large Banach space should contain a $d$-large subspace with a basis (this is open for $d_2(F) = d_{BM}(F, \ell_2^{\dim(F)})$).

Definition

An FDD $(F_n)$ of a Banach space $X$ is said to be $d$-better if $d(X, F_n) \to \infty$.

If a space has a $d$-better FDD, then it is $d$-large. Conversely:

Lemma

Let $X$ be a $d$-large Banach space. Then $X$ has a subspace spanned by a $d$-better FDD.

Standard results about bases, such as Bessaga–Pełczyński selection principle, still hold for better FDD’s.
Better FDD’s

Given a degree $d$, it is not clear whether every $d$-large Banach space should contain a $d$-large subspace with a basis (this is open for $d_2(F) = d_{BM}(F, \ell_2^{\dim(F)})$). Hence we can’t prove local dichotomies between classes defined by properties of bases. We use finite-dimensional decompositions (FDD’s) instead.
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Let $X$ be a $d$-large Banach space. Then $X$ has a subspace spanned by a $d$-better FDD.
Better FDD’s

Given a degree $d$, it is not clear whether every $d$-large Banach space should contain a $d$-large subspace with a basis (this is open for $d_2(F) = d_{BM}(F, \ell_2^{\dim(F)})$). Hence we can’t prove local dichotomies between classes defined by properties of bases. We use finite-dimensional decompositions (FDD’s) instead.

**Definition**

An FDD $(F_n)$ of a Banach space $X$ is said to be $d$-better if $d(X, F_n) \to \infty$.

If a space has a $d$-better FDD, then it is $d$-large. Conversely:

**Lemma**

Let $X$ be a $d$-large Banach space. Then $X$ has a subspace spanned by a $d$-better FDD.

Standard results about bases, such as Bessaga–Pełczyński selection principle, still hold for better FDD’s.
We fix a degree $d$. We will localize Gowers’ first dichotomy to the class of $d$-large spaces, that is, we will ensure that the outcome space is $d$-large.
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A space is said to be $d$-HI if it is $d$-large and contains no topological direct sum of two $d$-large subspaces.
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**Theorem (The first dichotomy, local version)**

Let $X$ be a $d$-large Banach space. Then $X$ has a $d$-large subspace $Y$ such that:

- $Y$ is spanned by a $d$-better UFDD;
- $Y$ is $d$-HI.

When $d$ is the dimension, we recover the original dichotomy.
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A local version of Ferenczi–Rosendal’s third dichotomy

**Definition**

A Banach space is said to be *d-minimal* if it is *d*-large and embeds into all of its *d*-large subspaces.
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A Banach space is said to be \textit{\textit{d-minimal}} if it is \textit{d-large} and embeds into all of its \textit{d-large} subspaces.

Definition

- A Banach space \(Y\) is \textit{tight in a FDD} \((F_i)\) if there exists a sequence of successive intervals of integers \(I_0 < I_1 < \ldots\) such that for every infinite \(A \subseteq \mathbb{N}\), \(Y\) cannot be embedded into \([F_i \mid i \notin \bigcup_{n \in A} I_n]\).

Theorem (The third dichotomy, local version)

Every \(d\)-large Banach space either has a \(d\)-minimal subspace, or has a \(d\)-tight subspace.
A local version of Ferenczi–Rosendal’s third dichotomy

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- A Banach space *Y* is **tight** in a FDD \((F_i)\) if there exists a sequence of successive intervals of integers \(l_0 < l_1 < \ldots\) such that for every infinite \(A \subseteq \mathbb{N}\), *Y* cannot be embedded into \([F_i \mid i \notin \bigcup_{n \in A} I_n]\).
- A FDD \((F_i)\) is **d-tight** if every *d*-large Banach space is tight in it.

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- A FDD \((F_i)\) is \textit{\(d\)-tight} if every \(d\)-large Banach space is tight in it.
- A Banach space \(X\) is \textit{\(d\)-tight} if it admits a \(d\)-better, \(d\)-tight FDD.
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- A FDD $(F_i)$ is **$d$-tight** if every *d*-large Banach space is tight in it.
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**Theorem (The third dichotomy, local version)**
*Every* *d*-large Banach space either has a *d*-minimal subspace, or has a *d*-tight subspace.*
Lemma

Fix $d$ a local degree, and $X$ a $d$-minimal Banach space. Then there is a mapping $\Gamma: \mathbb{N} \to \mathbb{R}^+$, tending to infinity, such that for every $d$-large subspace $Y$ of $X$ and every $n$, we have:

$$\sup_{\substack{F \subseteq Y \\ \text{dim}(F) = n}} d(F) \geq \Gamma(n).$$

In other words, $d$-large subspaces of a $d$-minimal space are uniformly $d$-large.

If $X$ is not minimal (or equivalently, is saturated with $d$-small subspaces), then there is a gap in the possible growth rates of the function $\sup_{\substack{F \subseteq Y \\ \text{dim}(F) = n}} d(F)$, when $Y$ ranges over all subspaces of $X$: either it is bounded, or it grows at least at the same rate as $\Gamma(n)$.

This is a very surprising local property.
Growth gap for $d$-minimal spaces

**Lemma**

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W. Cuellar Carrera, N. de Rancourt, V. Ferenczi

Local Banach-space dichotomies
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Growth gap for $d$-minimal spaces

Question

Does there exist local degrees $d$ such that all $d$-minimal subspaces are minimal? Does there exist some for which $d$-minimality does not coincide with minimality?
In all generality, our local dichotomies can be extended to \textit{D-families}, a more general class of families of Banach spaces.
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Unlike degrees, D-families have good closure properties: they are closed under countable intersection and finite unions. So, given a sequence $(d_n)$ of degrees, we can prove local dichotomies for the family of all spaces that are $d_n$-large for every $n$. 

For instance, if $1 \leq p < 2$, we can localize dichotomies to the class of spaces that don't have any type $> p$. Such a dichotomy could be used on spaces having type $p$. 

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The Hilbertian degree

We study in more details the Hilbertian degree $d_2(F) = d_{BM}(F, \ell_2^{\dim(F)})$, for which $d$-small spaces are Hilbertian spaces.
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**Theorem**

Let $X$ be a non-Hilbertian Banach space. Then $X$ has a non-Hilbertian subspace $Y$ satisfying one of the following properties:

1. $Y$ is $d_2$-minimal and has an unconditional basis;
2. $Y$ has a $d_2$-better, $d_2$-tight UFDD;
3. $Y$ is $d_2$-HI and $d_2$-minimal;
4. $Y$ is $d_2$-HI and $d_2$-tight.

This theorem gives no new information when $X$ is not $\ell_2$-saturated. However, in the case of $\ell_2$-saturated spaces, this can be seen as a Gowers list for non-Hilbertian, $\ell_2$-saturated Banach spaces.
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Lemma

Consider $X = \bigoplus_{n \in \mathbb{N}} \ell^k_{p_n} \ell_2$, where $p_n \to 2$ and $k_n \to \infty$ are well-chosen. Then $X$ is $\ell_2$-saturated and has a subspace in class (2) of the latter list.
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This class (2) is the only one we know to be nonempty, when we restrict our attention to $\ell^2$-saturated Banach spaces.
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Question

Does there exist a Banach space which is simultaneously $d_2$-minimal and $d_2$-HI?
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Question

Does there exist a Banach space which is simultaneously \( d_2 \)-minimal and \( d_2 \)-HI?

Such a space should be \( \ell_2 \)-saturated. We don’t even know whether \( \ell_2 \) saturated \( d_2 \)-minimal spaces, or \( \ell_2 \)-saturated \( d_2 \)-HI spaces, exist.
The Hilbertian degree

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Does James’ space belong to one of the classes?
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Does James’ space belong to one of the classes? If not, in what classes does it have subspaces?
The Hilbertian degree

Question

*Does James’ space belong to one of the classes? If not, in what classes does it have subspaces? What about other classical $\ell_2$-saturated spaces?*
A few words on ergodicity

Definition (Ferenczi–Rosendal, ’05)
A separable Banach space $X$ is said to be ergodic if $E_0$ reduces to the isomorphism relation between its subspaces.

This is a little bit stronger than saying that $X$ has continuum-many pairwise non-isomorphic subspaces.
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Conjecture (Ferenczi–Rosendal, ’05)

*Every non-Hilbertian separable Banach space is ergodic.*
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*Let $X$ be a counterexample to the above conjecture. Then $X$ has a non-Hilbertian subspace $Y$ (that is itself a counterexample) that is $d_2$-minimal.*
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**Conjecture (Ferenczi–Rosendal, ’05)**

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**Theorem**

*Let $X$ be a counterexample to the above conjecture. Then $X$ has a non-Hilbertian subspace $Y$ (that is itself a counterexample) that is $d_2$-minimal. Moreover, it can be ensured that $Y$ either has an unconditional basis, or is $d_2$-HI.*
Thank you for your attention!