An abstract formalism for strategical Ramsey theory

Noé de Rancourt

Université Paris VII, IMJ-PRG

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Infinite-dimensional Ramsey theory is about coloring infinite sequences of objects, and finding monochromatic subspaces.
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**Theorem (Silver)**

Let $\mathcal{X}$ be an analytic set of infinite subsets of $\mathbb{N}$. Then there exists $M \subseteq \mathbb{N}$ infinite such that:

- either for every infinite $A \subseteq M$, we have $A \in \mathcal{X}$;
- or for every infinite $A \subseteq M$, we have $A \notin \mathcal{X}$.
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Here, the set $M$ is generally viewed as a element of a forcing poset, whereas the set $A$ is viewed as an increasing sequence of integers.
Fix $k$ an at most countable field. Let $E = k^{(\mathbb{N})}$ be the countably infinite-dimensional vector space over $k$, with canonical basis $(e_i)_{i \in \mathbb{N}}$. Recall that a block-sequence of $E$ is a sequence $(x_n)_{n \in \mathbb{N}}$ of nonzero successive vectors of $E$, i.e. such that $\text{supp}(x_0) < \text{supp}(x_1) < \ldots$ (where $\text{supp}(\sum_{i \in \mathbb{N}} a_i e_i) = \{ i \in \mathbb{N} \mid a_i \neq 0 \}$).
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**Theorem (Milliken)**

Suppose $k = \mathbb{F}_2$. Let $\mathcal{X}$ be an analytic set of block-sequences of $E$. Then there exists an infinite-dimensional subspace $F$ of $E$ such that:

- either every block-sequence of $F$ belongs to $\mathcal{X}$;
- or every block-sequence of $F$ belongs to $\mathcal{X}^c$. 
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The pigeonhole principle associated to Milliken’s theorem is:

**Theorem (Hindman)**

Suppose $k = \mathbb{F}_2$. For every $A \subseteq E \setminus \{0\}$, there exists an infinite-dimensional subspace $F$ of $E$ such that either $F \setminus \{0\} \subseteq A$, or $F \setminus \{0\} \subseteq A^c$. 
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\[ \text{Suppose } k = \mathbb{F}_2. \text{ For every } A \subseteq E \setminus \{0\}, \text{ there exists an infinite-dimensional subspace } F \text{ of } E \text{ such that either } F \setminus \{0\} \subseteq A, \text{ or } F \setminus \{0\} \subseteq A^c. \]

Can we still get something interesting without pigeonhole principle?
The formalism of Gowers spaces

Let \( P \) be a set (the set of subspaces) and \( \leq \) and \( \leq^* \) be two quasi-orderings on \( P \), satisfying:

1. for every \( p, q \in P \), if \( p \leq q \), then \( p \leq^* q \);
2. for every \( p, q \in P \), if \( p \leq^* q \), then there exists \( r \in P \) such that \( r \leq p \), \( r \leq q \) and \( p \leq^* r \);
3. for every \( \leq \)-decreasing sequence \( (p_i)_{i \in \mathbb{N}} \) of elements of \( P \), there exists \( p^* \in P \) such that for all \( i \in \mathbb{N} \), we have \( p^* \leq^* p_i \);

Write \( p \preceq q \) for \( p \leq q \) and \( q \leq^* p \).
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Let $X$ be an at most countable set (the set of points) and $\triangleleft \subseteq X \times P$ a binary relation, satisfying:

4. for every $p \in P$, there exists $x \in X$ such that $x \triangleleft p$.
5. for every $x \in X$ and every $p, q \in P$, if $x \triangleleft p$ and $p \leq q$, then $x \triangleleft q$.
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The quintuple $G = (P, X, \leq, \leq^*, \triangleleft)$ is called a Gowers space.
The formalism of Gowers spaces

Two examples

The Silver space:

- $X = \mathbb{N}$;
- $P$ is the set of infinite subsets of $\mathbb{N}$;
- $\leq$ is the inclusion;
- $\leq^*$ is the inclusion-by-finite;
- $\triangleleft$ the membership relation.
The formalism of Gowers spaces
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   - $\leq$ is the inclusion;
   - $\leq^*$ is the inclusion-by-finite;
   - $\leq$ the membership relation.

2. The Rosendal space over an at most countable field $k$:
   - $X = E$ is a countably-infinite-dimensional vector space over $k$;
   - $P$ is the set of infinite-dimensional subspaces of $E$;
   - $\leq$ is the inclusion;
   - $\leq^*$ is the inclusion up to finite dimension ($F \leq^* G$ iff $F \cap G$ has finite codimension in $F$);
   - $\leq$ is the membership relation.
The formalism of Gowers spaces

The pigeonhole principle

Definition

The space $\mathcal{G}$ is said to satisfy the **pigeonhole principle** if for every $A \subseteq X$ and every $p \in P$, there exists $q \leq p$ such that either for all $x \leq q$, we have $x \in A$, or for all $x > q$, we have $x \in A^c$. 
Asymptotic games

Definition

Let $p \in P$. The asymptotic game below $p$, denoted by $F_p$, is the following two-players game:

$\begin{align*}
\text{I} & & p_0 \preceq p & & p_1 \preceq p & & \ldots \\
\text{II} & & x_0 \lhd p_0 & & x_1 \lhd p_1 & & \ldots,
\end{align*}$

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^\mathbb{N}$. 
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Definition

Let $p \in P$. The asymptotic game below $p$, denoted by $F_p$, is the following two-players game:

I

\[
\begin{align*}
p_0 & \lesssim p \\
p_1 & \lesssim p \\
& \vdots
\end{align*}
\]

II

\[
\begin{align*}
x_0 & \triangleleft p_0 \\
x_1 & \triangleleft p_1 \\
& \vdots
\end{align*}
\]

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^\mathbb{N}$.

Saying that I has a strategy to reach $\mathcal{X} \subseteq X^\mathbb{N}$ in $F_p$ means that “almost every” sequence below $p$ belongs to $\mathcal{X}$.
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<table>
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<tr>
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</tr>
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The outcome of the game is the sequence \((x_i)_{i \in \mathbb{N}} \in X^\mathbb{N}\).

Saying that I has a strategy to reach \( X \subseteq X^\mathbb{N} \) in \( F_p \) means that “almost every” sequence below \( p \) belongs to \( X \).

In the Silver space, we have the following:

Proposition

If \( X \subseteq \mathbb{N}^\mathbb{N} \) is such that I has a strategy to reach \( X \) in \( F_M \), then there exists \( N \subseteq M \) infinite such that every increasing sequence of elements of \( N \) belongs to \( X \).
The abstract Silver’s theorem

So this is an equivalent formulation of Silver’s theorem:

<table>
<thead>
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<td>For every analytic $\mathcal{X} \subseteq \mathbb{N}^\mathbb{N}$, there exists $M \subseteq \mathbb{N}$ infinite such that:</td>
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<td>- either I has a strategy in $F_M$ to reach $\mathcal{X}^c$;</td>
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**Theorem**

For every analytic $\mathcal{X} \subseteq \mathbb{N}^\mathbb{N}$, there exists $M \subseteq \mathbb{N}$ infinite such that:

- either $I$ has a strategy in $F_M$ to reach $\mathcal{X}^c$;
- or $I$ has a strategy in $F_M$ to reach $\mathcal{X}$.

In general, we have:

**Theorem (Abstract Silver’s)**

Suppose that the space $G$ satisfies the pigeonhole principle. Let $p \in P$ and $\mathcal{X} \subseteq X^\mathbb{N}$ be analytic. Then there exists $q \leq p$ such that:

- either $I$ has a strategy in $F_q$ to reach $\mathcal{X}^c$;
- or $I$ has a strategy in $F_q$ to reach $\mathcal{X}$.
Definition

Let \( p \in P \). Gowers’ game below \( p \), denoted by \( G_p \), is the following two-players game:

I \[ p_0 \preceq p \quad p_1 \preceq p \quad \ldots \]

II \[ x_0 \preceq p_0 \quad x_1 \preceq p_1 \quad \ldots \]

The outcome of the game is the sequence \( (x_i)_{i \in \mathbb{N}} \in X^\mathbb{N} \).
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p_0 &\leq p \\
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We have the following implication: if I has a strategy to reach $X$ in $F_p$, then II has a strategy to reach $X$ in $G_p$. 
Gowers’ games and the abstract Rosendal’s theorem

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Let \( p \in P \). Gowers’ game below \( p \), denoted by \( G_p \), is the following two-players game:

- \( I \) \( p_0 \leq p \), \( p_1 \leq p \), \( \ldots \)
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We have the following implication: if \( I \) has a strategy to reach \( X \) in \( F_p \), then \( II \) has a strategy to reach \( X' \) in \( G_p \). Under the pigeonhole principle, the converse is true up to taking a subspace.
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Theorem (Abstract Rosendal’s)

Let $p \in P$ and $\mathcal{X} \subseteq X^\mathbb{N}$ be analytic. Then there exists $q \leq p$ such that:

- either I has a strategy in $F_q$ to reach $\mathcal{X}^c$;
- or II has a strategy in $G_q$ to reach $\mathcal{X}$.
Local Ramsey theory in Gowers spaces

Gowers spaces are great for doing local Ramsey theory. If $X$ is an (algebraic) structure with a natural notion of subspaces, then you can define a Gowers space by taking for $P$ more or less any subfamily of the family of subspaces provided we can diagonalize among this subfamily.
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**Definition**

Let $\mathcal{F}$ be a nonempty family of infinite subsets of $\mathbb{N}$. We say that:

- $\mathcal{F}$ is a *$p$-family* if it is $E_0$-invariant and if for every decreasing sequence $(A_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{F}$, there exists $A^* \in \mathcal{F}$ such that for every $n \in \mathbb{N}$, $A^* \subseteq^* A_n$;

- $\mathcal{F}$ is *selective* if it is a $p$-family and if moreover, the set $A^*$ can be choosen in such a way that for every $n \in A^*$, $A^*/n \subseteq A_n$ (where $A^*/n = \{ k \in A^* \mid k > n \}$).
Fix $\mathcal{F}$ a $p$-family of subsets of $\mathbb{N}$. Then $(\mathcal{F}, \mathbb{N}, \subseteq, \subseteq^*, \varepsilon)$ is a Gowers space.

Corollary Let $X \subseteq \mathbb{N}$ be analytic. Then there exists $M \in P \mathcal{F}$ such that:

either $I$ has a strategy in $F_M$ to reach $X^c$;

or $II$ has a strategy in $G_M$ to reach $X$.

Moreover, if $F$ is selective, then the first possible conclusion can be replaced by "$r_Ms_8 X^c$".

Beware, here in $G_M$, player $I$ can only play elements of $F$!

Corollary (Mathias) Let $H$ be a selective coideal on $\mathbb{N}$, and $X \subseteq r\mathbb{N}s_8$ be analytic. Then there exists $M \in P H$ such that either $r_Ms_8 X^c$, or $r_Ms_8 X$.
Fix $\mathcal{F}$ a $p$-family of subsets of $\mathbb{N}$. Then $(\mathcal{F}, \mathbb{N}, \subseteq, \subseteq^*, \in)$ is a Gowers space.

**Corollary**

Let $\mathcal{X} \subseteq \mathbb{N}^\mathbb{N}$ be analytic. Then there exists $M \in \mathcal{F}$ such that:

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**Corollary (Mathias)**

Let $\mathcal{H}$ be a selective coideal on $\mathbb{N}$, and $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ be analytic. Then there exists $M \in \mathcal{H}$ such that either $[M]^\infty \subseteq \mathcal{X}^c$, or $[M]^\infty \subseteq \mathcal{X}$. 
What about Banach spaces?

What follows is part of a common work with W. Cuellar Carrera and V. Ferenczi.
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Fix $X$ a Banach space. We denote by $\text{Sub}(X)$ the set of closed infinite-dimensional subspaces of $X$. We endow $\text{Sub}(X)$ with the slice topology, i.e. the topology such that $(Y_\lambda)$ converges to $Y$ iff for every equivalent norm $\| \cdot \|$ and for every $x \in X$, the norm of $x$ in the quotient $(X, \| \cdot \|)/Y_\lambda$ converges to the norm of $x$ in the quotient $(X, \| \cdot \|)/Y$. 

Theorem

Let $P$ "Sub$p$X$q$ be a slice-$G_\delta$ subset, invariant under finite-dimensional modifications. Then $P$ is an (uncountable) Gowers space.
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**Theorem**

Let \( P \subseteq \text{Sub}(X) \) be a \textit{slice-\( G_\delta \) subset}, invariant under finite-dimensional modifications. Then \((P, S_X, \subseteq, \subseteq^*, \in)\) is an \textit{(uncountable) Gowers space}.
What about Banach spaces?

**Definition**

A **finite-dimensional decomposition (FDD)** of a Banach space $Y$ is a sequence $(F_i)_{i \in \mathbb{N}}$ of finite-dimensional subspaces of $Y$ such that every $x \in Y$ can be written in a unique way as a sum $x = \sum_{i=0}^{\infty} x_i$, where for every $i$, $x_i \in F_i$.

A **block-sequence** of the FDD $(F_i)$ is a sequence $(x_n)_{n \in \mathbb{N}}$ of normalized successive vectors for this FDD (i.e. there exists $A_0 < A_1 < A_2 < \ldots$ sets of integers such that for every $n$, $x_n \in \bigoplus_{i \in A_n} F_i$).
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Definition

Given $\mathcal{X} \subseteq (S_X)_{\mathbb{N}}$ and $\Delta = (\Delta_n)_{n \in \mathbb{N}}$ a sequence of positive real numbers, we let $(\mathcal{X})_\Delta = \{ (y_n) \in (S_X)_{\mathbb{N}} \mid \exists (x_n) \in \mathcal{X} \forall n \| x_n - y_n \| \leq \Delta_n \}$. 
Corollary

Let $P \subseteq \text{Sub}(X)$ be a slice-$G_\delta$ subset, invariant under finite-dimensional modifications. Let $\mathcal{X} \subseteq (S_X)^\mathbb{N}$ be analytic, and let $\Delta$ be a sequence of positive real numbers. Then there exists $Y \in P$ such that:

- either $Y$ has a FDD $(F_n)$ such that every subsequence of $(F_n)$ generates an element of $P$, and such that every block-sequence of $(F_n)$ is in $\mathcal{X}^c$;
- or II has a strategy in $G_Y$ to reach $(\mathcal{X})_\Delta$ (where in $G_Y$, player I is only allowed to play elements of $P$).
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**Lemma**

A Banach space $X$ is non-Hilbertian iff for every $n \in \mathbb{N}$, there exists a finite-dimensional subspace $F \subseteq X$ that is not $n$-isomorphic to a Euclidean space.
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**Lemma**

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**Question**

Does there exist similar examples in other areas of mathematics?
Thank you for your attention!