How to classify things?

Noé de Rancourt

Université Paris VII, IMJ-PRG

INRIA, March 20, 2018
Let’s go back to primary school!

Problem 1

*Léa buys six cacti. Each cactus costs eight euros. How much does she pay?*

Problem 2

*I rent a spacious apartment in the center of Paris. It is three meters wide and four meters long. What is the surface of my apartment?*
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To solve these two very different concrete problems, we use the same abstract mathematical notions: numbers and multiplication.
Mathematics are a story of structures
First example: Euclidean rings

Theorem
Every positive integer can be decomposed, in an essentially unique way, as a product of prime numbers.

To prove this, we use:

Lemma (Existence of the Euclidean division)
For every $a \in \mathbb{N}$ and $b \in \mathbb{N}^*$, there exist $q \in \mathbb{N}$ and $0 \leq r < b$ with $a = bq + r$. 
Recall that a **polynomial** is an expression of the form 
\[ P(X) = a_nX^n + \ldots + a_1X + a_0, \]
and that its **degree** is \( \text{deg}(P) = n \). A polynomial \( P \) is **irreducible** if for every \( Q, R \) such that \( P(X) = Q(X)R(X) \), then either \( Q \) or \( R \) is constant.
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**Theorem**

*Every nonzero polynomial can be decomposed, in an essentially unique way, as a product of irreducible polynomials.*

To prove this, we use:

**Lemma (Existence of the Euclidean division)**

*For every polynomials $A$ and $B \neq 0$, there exist polynomials $Q$ and $R$ with $A = BQ + R$ and $\text{deg}(R) < \text{deg}(B)$.*

The proof of the theorem from the lemma is exactly the same in both cases.
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In these structures, we can define a notion of irreducible elements, and prove that every nonzero element of the structure can be decomposed, in an essentially unique way, as a product of irreducibles.
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The set of integers, and the set of polynomials, are particular cases of Euclidean rings. From this fact immediately follow the two previous theorems.
Mathematics are a story of structures
Second example: vector spaces

- **Vectors of the plane** can be summed and multiplied by a real number.

  If $R$ is a rotation of the plane, $\vec{u}$ and $\vec{v}$ two vectors, and $\lambda$ a real number, then $R(\vec{u} + \vec{v}) = R(\vec{u}) + R(\vec{v})$, and $R(\lambda \vec{u}) = \lambda R(\vec{u})$. 

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- **$C^\infty$ functions** $\mathbb{R} \to \mathbb{R}$ can be summed and multiplied by a real number.

  We can derivate these functions. If $f$ and $g$ two such functions, and $\lambda$ a real number, then $(f + g)' = f' + g'$, and $(\lambda f)' = \lambda f'$.
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We can define the abstract notion of a **vector space**, a set with two operation: an internal addition and a multiplication by scalar numbers; and the notion of a **linear mapping**, a mapping between vector spaces that preserves these operations. The two last examples are particular cases of vector spaces and of linear mappings.
Sometimes, two structures arising in different contexts are “the same”. That doesn’t mean that their elements are the same, but rather that, if we forget the nature of their elements, they behave exactly in the same way. We say that they are isomorphic.
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- \(( \mathbb{R}, + \) \) and \(((0, +\infty), \times)\) are isomorphic, because the exponential function maps bijectively \( \mathbb{R} \) to \((0, +\infty)\) and it “transforms” the addition into the multiplication.

- As vector spaces, the plane and the set of solutions of the differential equation \( f'' = -f \) are isomorphic.
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If we prove a theorem for one of these structure, then it is also true for the other one.
What is classification?

Classifying a certain type of structures, it’s finding a list of structures $\mathcal{L}$ of this type such that:

- the structures in $\mathcal{L}$ are pairwise non-isomorphic;
- every structure of this type is isomorphic to a structure in $\mathcal{L}$;
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It enables to:

- prove easily that properties are true for all structures;
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Examples of classification
First example: orientable compact surfaces

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Orientable compact surfaces are surfaces that can be embedded in the euclidean space.

They can be classified by their genus, i.e. the number of holes. Two surfaces are isomorphic if and only if they have the same genus.
Finite-dimensional vector spaces (over $\mathbb{R}$) can be classified by their dimension. Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.
Examples of classification

Example 2: finite-dimensional vector spaces

- **Finite-dimensional vector spaces (over $\mathbb{R}$)** can be classified by their dimension. Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

- All **standard Borel spaces with a non-atomic probability measure** are isomorphic. Thus, when you want to prove a result on these spaces, you only need to prove it on your favourite example, $[0, 1]$ with the Lebesgue measure.
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In general, classifying a class of structures means associating to each structure a **characteristic** (which is a real number or a sequence of real numbers), such that two structures are isomorphic iff they have the same characteristic.
An example of bad classification

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To such a graph, we can associate a real number between 0 and 1 in the following way: its \( n^{\text{th}} \) digit (in base 3) is 1 if the \( n^{\text{th}} \) pair of vertices is linked by an edge, and 0 otherwise.

For this graph we get 0, 1100101110\ldots
An example of bad classification

Then it is easy to get a classification:

1. Gather graphs in isomorphism classes;

0.110101101001...

0.00010111101001...

0.011010001010...
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1. Gather graphs in **isomorphism classes**;
2. Pick one graph in each class;
An example of bad classification

Then it is easy to get a classification:

1. Gather graphs in **isomorphism classes**;
2. Pick one graph in each class;
3. Associate to the whole class the number of this graph.
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This seems to be impossible for countably infinite graphs. How to prove it?
A formalism

To prove something, we need to formalise it.

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An **equivalence relation** is a binary relation which is reflexive, symmetric and transitive. **The equality** is an equivalence relation, and **being isomorphic** is another one.

The set of elements that are equivalent to a given element $x$ is called an **equivalence class**.

![Diagram of equivalence classes]

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![Equivalence classes](image)

Here, we consider equivalence relations on standard Borel spaces. These are spaces where we can define a continuous analogue of calculability. The morphisms between these spaces are called Borel mappings; they can be seen as computable functions, in a continuous way. Classes of infinite structures can often be endowed with a structure of standard Borel space.
A formalism

We say that an equivalence relation $E$ on a standard Borel space $X$ is **reducible** to an equivalence relation $F$ on a standard Borel space $Y$ if there exists a mapping $f : X \to Y$ that maps each class of $E$ to exactly one class of $F$.

The idea is that if we know $F$, then we can compute $E$. 

![Diagram](image-url)
Then a class of structures is classifiable if and only if the isomorphism relation on this class is reducible to the equality on real numbers.

There exists an equivalence relation $E_0$ which is not reducible to the equality on $\mathbb{R}$. We say that two real numbers $x$ and $y$ are $E_0$-equivalents if and only if they have the same writing in basis 2 from some rank.

Theorem (Folklore)
The isomorphism relation between countably infinite graphs has the highest possible complexity among all isomorphism relations between countably infinite structures. In particular, it is not reducible to the equality on the real numbers.
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Thank you for your attention!
“C’est bien plus beau lorsque c’est inutile !”
“It’s much more beautiful when it’s useless!”

_Cyrano de Bergerac, acte V, scène 6_