1878

- beginning of the Second Anglo-Afghan War
- fist public demonstrations of the telephone
- Tchaikovsky writes his Fourth Symphony

Georg Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre* (1878)

Is it true that there is no set whose cardinality is strictly between that of the integers and that of the real numbers?

In other words, is $2^{\aleph_0} = \aleph_1$? (Continuum Hypothesis).

Felix Hausdorff, *Grundzüge einer Theorie der geordneten Mengen* (1908)

(Generalized Continuum Hypothesis) $\forall\kappa\ 2^\kappa = \kappa^+$.

κ is *weakly inaccessible* iff κ is a regular limit cardinal.

In another article, Hausdorff adds: ”the least among them has such an exorbitant magnitude that it will hardly ever come into consideration for the usual purposes of set theory”.

Main Question

How large is the power set of a set?

Definition

Let κ be a cardinal. Then

- $\mathrm{cof}(\kappa)$ (the cofinality of κ) is the smallest cardinality of an unbounded subset of κ;
- κ is *regular* iff $\mathrm{cof}(\kappa)=\kappa$. Examples: $\aleph_0$, $\aleph_1$, all successor cardinals.
- κ is *singular* iff it is not regular. Example: $\aleph_\omega$ has cofinality $\aleph_0$.

First impressions:

- $2^\kappa > \kappa$;
- if $\kappa < \lambda$ then $2^\kappa \leq 2^\lambda$;
- even better: $\mathrm{cof}(2^\kappa) > \kappa$.

Example: $2^{\aleph_0}\aleph_\omega$.

Theorem (Gödel, 1937)

$\mathrm{Con}(ZF) \Rightarrow \mathrm{Con}(ZFC + GCH)$.

Theorem (Cohen, 1963)

$\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + \neg CH)$.

Theorem (Easton, 1970)

We say that E is an Easton function if

- if $\kappa < \lambda$ then $E(\kappa) \leq E(\lambda)$;
- $\text{cof}(E(\kappa)) > \kappa$.

Then $\text{Con}(ZFC) \Rightarrow \text{Con}(ZFC + \forall\kappa \text{ regular } 2^\kappa = E(\kappa))$.

So, any behaviour of the power function on the regular cardinals that satisfies the basic combinatoric rules is consistent.

The situation for singular cardinals is not so clear cut. It is in fact dependent from large cardinals.

Definition (1930)

Let κ be a cardinal. Then

- κ is *strong limit* iff $\forall\gamma,\eta<\kappa\ \gamma^\eta<\kappa$.
- κ is *(strongly) inaccessible* iff uncountable, regular and strong limit.

Theorem (1930)

If κ is inaccessible then $V_\kappa\vDash$ ZFC.

Corollary of the Second Gödel Theorem

ZFC $\nvdash\exists\kappa$ inaccessible.

At the beginning, large cardinals were seen as extensions of ZFC, with the aim of describing better the universe. It is easy to see why.

ω is regular and strong limit: n^m is always a finite number, and the supremum of a finite sequence of finite numbers is finite. This means that if the only things we know are finite numbers, it is impossible to reach infinity. This is reflected by the necessity of the Infinity axiom.

If there is exactly one inaccessible, the ordinals are divided in three parts: finite, small infinite and large infinite (above and below the inaccessible). The boundary between small and large infinity is as strong as the one between finite and infinite. What if there are two inaccessible? Three? Infinite? Unboundedly?

In any case, a universe with an inaccessible can imagine a universe without, but not viceversa, and this is why ZFC + inaccessible is considered stronger than ZFC.

In the post-modern non-platonistic view, this aspect is less important, yet large cardinals are important in a different way.

Many large cardinals are now defined, and they are unexpectedly ordered in a linear way. This create a ruler, with which is possible to "measure" how strong is any hypothesis that is stronger than ZFC. For almost all the hypotheses, this measure has been done.

For example:

Theorem

Con(inaccessible) $\leftrightarrow$ Con(every projective set of reals is Lebesgue-measurable.

Definition

Let κ be a cardinal. Then κ is *measurable* iff there is a κ-complete ultrafilter over κ.

Remark

Any measurable cardinal is inaccessible, but not viceversa.

Theorem (Solovay)

Con(measurable) $\leftrightarrow$ Con(every Borelian measure on $\mathcal{B}([0,1])$ can be extended to $\mathcal{P}([0,1])$).

Theorem

Con(measurable) $\leftrightarrow$ Con(there exists a non-trivial homeomorphism $h : \mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa} \to \mathbb{Z}$.

Definition

Let M, N be sets or classes. Then $j : M \to N$ is an *elementary embedding* iff for any formula $\varphi(v_0, \ldots, v_n)$ and for any $x_0, \ldots, x_n \in M$,

$$M \vDash \varphi(x_0, \ldots, x_n) \text{ iff } N \vDash \varphi(j(x_0), \ldots, j(x_n)).$$

In particular, if ψ is a sentence, $M \vDash \psi$ iff $N \vDash \psi$. So one is a model of ZFC, the other one also is.

It is possible to prove that $j(\mathrm{rnk}(x)) = \mathrm{rnk}(j(x))$, so restricting to the ordinals gives a satisfactory view on j. In particular if $\mathrm{rnk}(x)$ is not a fixed point, neither is x.

We call $\mathrm{crt}(j)$, the *critical point* of j, the least ordinal that is not a fixed point of j. If M satisfies enough of ZFC and $j \neq \mathrm{id}$, it exists.

Theorem (Keisler, 1962)

κ is measurable iff there exists $j : V \prec M$ with $\mathrm{crt}(j) = \kappa$.

Remark

If $j : V \prec M$ and $\mathrm{crt}(j) = \kappa$, then ${}^{<\kappa}M \subseteq M$.

Definition (late 60's)

Let κ and γ be cardinals. Then κ is *γ-supercompact* iff there is a $j : V \prec M$ with $\mathrm{crt}(j) = \kappa$, $\gamma < j(\kappa)$ and ${}^{\gamma}M \subseteq M$. If κ is γ-supercompact for any γ, then κ is supercompact.

Definition (Kunen, 1972)

Let κ be a cardinal. Then κ is *huge* iff there is a $j : V \prec M$ with $\mathrm{crt}(j) = \kappa$, ${}^{j(\kappa)}M \subseteq M$.

Definition

Let $j : V \prec M$ with $\mathrm{crt}(j) = \kappa$. We define the critical sequence $\langle \kappa_0, \kappa_1, \ldots \rangle$ as $\kappa_0 = \kappa$ and $j(\kappa_n) = \kappa_{n+1}$.

Definition

Let κ be a cardinal. Then κ is *n-huge* iff there is a $j : V \prec M$ with $\mathrm{crt}(j) = \kappa_0$, ${}^{\kappa_n}M \subseteq M$.

Definition (Reinhardt, 1970)

Let κ be a cardinal. Then κ is *Reinhardt* iff there is a $j : V \prec M$ with $\mathrm{crt}(j) = \kappa_0$, ${}^{\lambda}M \subseteq M$, with $\lambda = \sup_{n \in \omega} \kappa_n$. Equivalently, if there is a $j : V \prec V$, with $\kappa = \mathrm{crt}(j)$.

Theorem (Kunen, 1971)

There is no Reinhardt cardinal.

The power of a limit cardinal depends partially on the powers of the cardinals below.

Let λ be a limit cardinal. Then $2^\lambda \geq \sup_{\kappa<\lambda} 2^\kappa$. Because of this it is sometimes sensible to restrict the research to strong limit cardinals.

Theorem (Silver, 1974)

Let λ be a singular cardinal of uncountable cofinality. If for all $\kappa < \lambda$ $2^\kappa = \kappa^+$, then $2^\lambda = \lambda^+$.

Theorem (Solovay, 1974)

Let κ be supercompact. For all $\lambda > \kappa$ strong limit singular, $2^\lambda = \lambda^+$.

Theorem (Galvin-Hajnal, 1975)

Let $\aleph_\alpha$ be a strong limit singular of uncountable cofinality, Then $2^{\aleph_\alpha} < \aleph_\gamma$, where $\gamma = (2^{|\alpha|})^+$.

Theorem (Shelah)

If $\aleph\omega$ is a strong limit cardinal then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Theorem (Gitik)

Con(there exists a measurable cardinal κ such that $2^\kappa > \kappa^+$) $\leftrightarrow$ Con(there exists a strong limit singular cardinal κ such that $2^\kappa > \kappa^+$).

Theorem (Woodin, Gitik, 1989)

Con(here exists a strong limit singular cardinal κ such that $2^\kappa > \kappa^+$) $\rightarrow$ Con($2^{\aleph_n} = \aleph_{n+1} \wedge 2^{\aleph_\omega} = \aleph_{\omega+2}$).

Theorem (Woodin, Cummings, 1992)

Con(there exists a supercompact cardinal) $\rightarrow$ Con($\forall\kappa\ \ 2^\kappa = \kappa^{++}$).

The previous research indicates a natural question:

Main Question

How large is the power set of a set under large cardinals? In other words, is Con(large cardinal $+ 2^\kappa = \dots$)?

In the specific case of GCH, this is related to the Outer Model Program.

Theorem (Scott, 1961)

Let κ be measurable and U its measure. Then if $2^\kappa > \kappa^+$, then $\{\gamma : 2^\gamma > \gamma^+\} \in U$.

Theorem (Levy, Solovay, 1967)

Let κ be measurable and E Easton function such that there exists $\gamma < \kappa\ \forall \eta > \gamma\ E(\eta) = 2^\eta$. Then Con(measurable $+ \forall \eta\ E(\eta) = 2^\eta$).

The same is true for supercompact.

Theorem

Let E be an Easton function such that $E(\kappa) < 2^\kappa$ and γ an inaccessible cardinals. Then it is consistent $\forall\kappa$ regular, $2^\kappa = E(\kappa)$. In particular Con(inaccessible) $\rightarrow$ Con(inaccessible $+$ GCH).

Theorem

Con(supercompact) $\rightarrow$ Con(supercompact $+$ GCH).

Let's go back to the Reinhardt cardinal.

Theorem (Kunen, 1971)

There is no $j : V \prec V$.

The proof uses a well-ordering of $V_{\lambda+1}$, with $\lambda = \sup_{n\in\omega} \kappa_n$. So

Corollary

There is no $j : V_{\lambda+2} \prec V_{\lambda+2}$.

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$L(V_{\lambda+1})$ is the smallest model of ZF that contains $V_{\lambda+1}$ and all the ordinals. Under I0 it cannot satisfy AC.
There are even stronger hypotheses, all of the form "there exists a $j : L(V_{\lambda+1}, X) \prec L(V_{\lambda+1}, X)$, with $\text{crt}(j) < \lambda$", with $X \subseteq V_{\lambda+1}$, called $E^0_\alpha(V_{\lambda+1})$.
Under such hypotheses, $\text{crt}(j)$ is inaccessible, measurable, n-huge for any n, and $V_\lambda \vDash \text{crt}(j)$ is supercompact.

Theorem (Hamkins, 1994)

Suppose I1 and let E be an Easton function such that $E(\kappa) < 2^\kappa$. Then Con(I1+$\forall\kappa$ regulars $2^\kappa = E(\kappa)$).

Theorem (Friedman, 2006)

$\text{Con}(I2) \to \text{Con}(I2 + GCH)$.

Theorem (Corazza, 2007)

Suppose I3 and let E be an Easton function such that $E(\kappa)$ is less than the least inaccessible above κ. Then Con(I3+$\forall\kappa$ regulars $2^\kappa = E(\kappa)$).

Theorem (D., Friedman, 2013)

Let I* be either I3, I2, I1 or I0. Suppose I*(λ) and let E be an Easton function closed and definable under λ. Then Con(I*+$\forall\kappa$ regulars $2^\kappa = E(\kappa)$).

The singular cardinals case is, again, much more difficult. Solovay's Theorem gives us:

Remark

Suppose I*(λ) and crt(j) $= \kappa$. Then for all $\kappa < \gamma < \lambda$ strong limit, $2^\gamma = \gamma^+$.

With previous techniques, it is easy to see that we can change the power function below crt(j) and above $\lambda + 1$ without problems. This leaves out λ.

Theorem (Gitik, 2002)

$\text{Con}(I3) \to \text{Con}(I3(\lambda) + 2^\lambda > \lambda^+)$.

Theorem (Cummings, Foreman, 2010)

$\text{Con}(\exists\lambda\, \exists j : V_{\lambda+1} \prec_3 V_{\lambda+1}) \to \text{Con}(I2(\lambda) + 2^\lambda > \lambda^+)$.

Theorem (D., Friedman, 2013)(Independently by Woodin)

$\mathrm{Con}(I0) \rightarrow \mathrm{Con}(I1(\lambda) + 2^\lambda > \lambda^+)$.

- Can we lower the consistency of I1(λ) + $2^\lambda > \lambda^+$)?
- What about I0 + $2^\lambda > \lambda^+$)?
- Where does inconsistency sits in the hierarchy of large cardinals?
- Is there an easy way to describe te possible behaviour of the power function at singular cardinals?