Very Large Cardinals and Combinatorics

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These are all question non-answerable in ZFC.

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The right point of view for this seminar is "set theory as a mathematical branch" and not "set theory as foundation of mathematics".
A good mental image is the multiverse, a collection of universes that satisfy ZFC. We want to know what can happen in those universes, and what cannot.

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- Forcing constructions permit to pass from one universe to another;
- Large cardinals hypotheses enlarge our multiverse (more universes!)
- $V = L$ has very nice structural properties;
- it is also interesting to go the other way, and investigating properties opposed to those in $V = L$;
- combinatorial properties can be local (regarding one cardinal) or global (regarding all cardinals, or at least a class).

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- $\square$ at small cofinalities is a weaker version of $\square$ (ad hoc to avoid inconsistencies with large cardinals);
- TP_κ (Tree Property) is König's Lemma for κ. $TP_{\kappa^{++}}$ is both a stronger failure of the local GCH and a failure of $\square$.

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Let κ be a cardinal. Then

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Let M, N be sets or classes. Then $j : M \to N$ is an *elementary embedding* iff for any formula $\varphi(v_0, \ldots, v_n)$ and for any $x_0, \ldots, x_n \in M$,

$$M \vDash \varphi(x_0, \ldots, x_n) \text{ iff } N \vDash \varphi(j(x_0), \ldots, j(x_n)).$$

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Theorem (Keisler, 1962)

κ is measurable iff there exists $j : V \prec M$ with $\mathrm{crt}(j) = \kappa$. This implies ${}^{<\kappa}M \subseteq M$.

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- I0 For some λ there exists a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, with $\mathrm{crt}(j) < \lambda$

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- Special case: local case exactly at the large cardinal.

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Let κ be supercompact.For all $\lambda > \kappa$ strong limit singular, $2^\lambda = \lambda^+$.

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If κ is λ^+-supercompact, then $\square_\lambda$ fails. If there exists a subcompact, then $\square$ fails.

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- $\diamondsuit$
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- etc...

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Shi-Trang actually raised this to I0, starting with a hypothesis stronger than I0.

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Or some that is equiconsistent?

Thanks for your attention.