Non proper elementary embeddings beyond $L(V_{\lambda+1})$

Vincenzo Dimonte

19 June 2009
ACHTUNG!

The following seminar talk will not contain forcing. We apologize for the inconvenience.
Non proper elementary embeddings beyond $L(V_{\lambda+1})$

Vincenzo Dimonte

Large Cardinals Map

Introduction

Higher Determinacy Axiom

Main Results

Totally Non-proper Ordinals

Partially non-proper ordinal

Both

Implications and open problems

Chart of Cardinals

The arrows indicate direct implications or relative consistency implications, often both.

\[
\begin{align*}
0 = 1 &\rightarrow 10-13 &\rightarrow n\text{-huge} \\
&\rightarrow \text{huge} &\rightarrow \text{superhuge} \\
&\rightarrow \text{Vapnik's Principle} &\rightarrow \text{almost huge} \\
&\rightarrow \text{extendible} &\rightarrow \text{supercompact} \rightarrow \text{strongly compact} \\
&\rightarrow \text{superstrong} &\rightarrow \text{Woodin} &\rightarrow \text{strong} \\
&\rightarrow \text{0'' exists} &\rightarrow \text{measurable} &\rightarrow \text{Ramsey} \\
&\rightarrow \text{Jensen} &\rightarrow \text{Rowbottom} \\
\kappa \rightarrow (\omega_1)_{\omega_0} &\rightarrow \forall \alpha \in \text{``}(\alpha'' \text{ exists)} \\
&\rightarrow 0'' \text{ exists} &\rightarrow \kappa \rightarrow (\omega_0)_{\omega_0} &\rightarrow \text{indescribable} \\
&\rightarrow \text{weakly compact} &\rightarrow \text{Mahlo} &\rightarrow \alpha\text{-inaccessible} \\
&\rightarrow \text{inaccessible} &\rightarrow \text{weakly inaccessible}
\end{align*}
\]
Chart of Cardinals

The arrows indicate direct implications or relative consistency implications, often both.

You are here

0 = 1 \rightarrow 10-13 \rightarrow n-huge \rightarrow superhuge

downward

Vapnik–Chervonenkis Principle

extendible \rightarrow supercompact \rightarrow strongly compact

Woodin

strong \rightarrow 0^\# exists \rightarrow measurable \rightarrow Ramsey

\kappa \rightarrow (\omega_1)_{\kappa^+} \rightarrow \forall \alpha \in {}^{\omega_0}(\omega_0^\alpha \text{ exists}) \rightarrow 0^\# exists \rightarrow \kappa \rightarrow (\omega_1)_{\kappa^+} \rightarrow \text{indescribable}

\alpha\text{-Mahlo} \rightarrow \text{weakly compact} \rightarrow \text{weakly inaccessible}

\text{Weakly inaccessible} \rightarrow \alpha\text{-inaccessible}
Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$. 
Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It’s a natural strengthening of the hypotheses with a $j : V \prec M$. 
Reinhardt Hypothesis: there exists an elementary embedding \( j : V \prec V \).
It’s a natural strengthening of the hypotheses with a \( j : V \prec M \).

**Theorem (Kunen, 1971)**

If \( j : V \prec M \), then \( M \neq V \).
Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It’s a natural strengthening of the hypotheses with a $j : V \prec M$.

**Theorem (Kunen, 1971)**

If $j : V \prec M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:
Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.
It’s a natural strengthening of the hypotheses with a $j : V \prec M$.

**Theorem (Kunen, 1971)**

If $j : V \prec M$, then $M \neq V$.

The critical sequence has an important role in the proof:

**Definition**

$k_0 = \text{crit}(j), \ k_{n+1} = j(k_n)$,
Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It’s a natural strengthening of the hypotheses with a $j : V \prec M$.

**Theorem (Kunen, 1971)**

If $j : V \prec M$, then $M \neq V$.

The **critical sequence** has an important role in the proof:

**Definition**

$$\kappa_0 = \text{crit}(j), \quad \kappa_{n+1} = j(\kappa_n), \quad \lambda = \sup_{n \in \omega} \kappa_n.$$
Reinhardt Hypothesis: there exists an elementary embedding $j : V \preceq V$.
It’s a natural strengthening of the hypotheses with a $j : V \preceq M$.

**Theorem (Kunen, 1971)**

If $j : V \preceq M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:

**Definition**

$k_0 = \text{crit}(j), \ k_{n+1} = j(k_n), \ \lambda = \sup_{n \in \omega} k_n$.

Kunen’s proof uses a choice function that is in $V_{\lambda+2}$. So

**Corollary**

There is no $j : V_{\eta} \preceq V_{\eta}$,
Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.
It’s a natural strengthening of the hypotheses with a $j : V \prec M$.

**Theorem (Kunen, 1971)**

If $j : V \prec M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:

**Definition**

$k_0 = crit(j), k_{n+1} = j(k_n), \lambda = \sup_{n \in \omega} k_n$.

Kunen’s proof uses a choice function that is in $V_{\lambda+2}$. So

**Corollary**

There is no $j : V_\eta \prec V_\eta$, with $\eta \geq \lambda + 2$. 
It’s quite natural to define the following Axioms
It’s quite natural to define the following Axioms

**Definition**

- **I3**: There exists an elementary embedding $j : V_\lambda \prec V_\lambda$. 
It’s quite natural to define the following Axioms

**Definition**

- **I3:** There exists an elementary embedding \( j : V_\lambda \prec V_\lambda \).
- **I1:** There exists an elementary embedding \( j : V_{\lambda+1} \prec V_{\lambda+1} \).

Technical note: if \( j, k : V_{\lambda+1} \prec V_{\lambda+1} \) and \( j \upharpoonright V_\lambda = k \upharpoonright V_\lambda \), then \( j = k \).
It’s quite natural to define the following Axioms

**Definition**

- I3: There exists an elementary embedding $j : V_\lambda \prec V_\lambda$.
- I1: There exists an elementary embedding $j : V_{\lambda+1} \prec V_{\lambda+1}$.

Technical note: if $j, k : V_{\lambda+1} \prec V_{\lambda+1}$ and $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$, then $j = k$. 
Woodin proposed an even stronger axiom:

**Definition**

I0: There exists an elementary embedding \( j : L(V_{\lambda+1}) \prec L(V_{\lambda+1}) \) with \( \text{crt}(j) < \lambda \).
Woodin proposed an even stronger axiom:

**Definition**

I0: There exists an elementary embedding \( j : L(V_{\lambda+1}) \prec L(V_{\lambda+1}) \) with \( \text{crt}(j) < \lambda \).

This axiom is more interesting, since it produces a structure on \( L(V_{\lambda+1}) \) that is strikingly similar to the structure of \( L(\mathbb{R}) \) under AD.
Woodin proposed an even stronger axiom:

**Definition**

I0: There exists an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$.

This axiom is more interesting, since it produces a structure on $L(V_{\lambda+1})$ that is strikingly similar to the structure of $L(\mathbb{R})$ under AD.

Since $\lambda$ has cofinality $\omega$, $V_\lambda$ is similar to $V_\omega$, so $V_{\lambda+1}$ is similar to $\mathbb{R}$. 
First degree analogies (without I0 and AD):
First degree analogies (without I0 and AD):

Let $\Theta^L(V_{\lambda+1})$ be the supremum of the $\alpha$'s such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \rightarrow \alpha$. 

\[ \begin{align*} 
\Theta^L(V_{\lambda+1}) & \text{ is regular} \\
V_{\lambda+1} & \text{ is regular} \\
\text{DC} & \text{ holds.} \\
\text{In fact these analogies hold for every model of HOD} \\
V_{\lambda+1} & \text{.} 
\end{align*} \]
First degree analogies (without I0 and AD):

Let $\Theta^L(V_{\lambda+1})$ be the supremum of the $\alpha$'s such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \rightarrow \alpha$.

\[
\begin{array}{c|c}
L(\mathbb{R}) & L(V_{\lambda+1}) \\
\end{array}
\]
First degree analogies (without I0 and AD):

Let $\Theta_{V_{\lambda+1}}^{L(V_{\lambda+1})}$ be the supremum of the $\alpha$’s such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \rightarrow \alpha$.

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$</th>
<th>$L(V_{\lambda+1})$</th>
</tr>
</thead>
</table>
| $\Theta$ is regular

Non proper elementary embeddings beyond $L(V_{\lambda+1})$

Vincenzo Dimonte

Large Cardinals Map

Introduction

Higher Determinacy Axiom

Main Results

Totally Non-proper Ordinals

Partially non-proper ordinal

Both

Implications and open problems
First degree analogies (without I0 and AD):

Let $\Theta_{V_{\lambda+1}}^{L(V_{\lambda+1})}$ be the supremum of the $\alpha$'s such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \rightarrow \alpha$.

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$</th>
<th>$L(V_{\lambda+1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta$ is regular</td>
<td>$\Theta_{V_{\lambda+1}}^{L(V_{\lambda+1})}$ is regular</td>
</tr>
</tbody>
</table>
First degree analogies (without I0 and AD):

Let $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ be the supremum of the $\alpha$'s such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \rightarrow \alpha$.

\[
\begin{array}{ccc}
L(\mathbb{R}) & L(V_{\lambda+1}) \\
\Theta \text{ is regular} & \Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}} \text{ is regular} \\
\text{DC holds} & \\
\end{array}
\]
First degree analogies (without I0 and AD):

Let \( \Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}} \) be the supremum of the \( \alpha \)'s such that in \( L(V_{\lambda+1}) \) there exists a surjection \( \pi : V_{\lambda+1} \rightarrow \alpha \).

\[
\begin{array}{c|c}
L(\mathbb{R}) & L(V_{\lambda+1}) \\
\hline
\Theta \text{ is regular} & \Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}} \text{ is regular} \\
\text{DC holds} & \text{DC}_\lambda \text{ holds.}
\end{array}
\]
First degree analogies (without I0 and AD):

Let $\Theta_{V_{\lambda+1}}^{L(V_{\lambda+1})}$ be the supremum of the $\alpha$’s such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \rightarrow \alpha$.

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$</th>
<th>$L(V_{\lambda+1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta$ is regular</td>
<td>$\Theta_{V_{\lambda+1}}^{L(V_{\lambda+1})}$ is regular</td>
</tr>
<tr>
<td>DC holds</td>
<td>DC$_\lambda$ holds.</td>
</tr>
</tbody>
</table>

In fact these analogies hold for every model of HOD$_{V_{\lambda+1}}$. 
Second degree analogies (under I0 and AD):
Second degree analogies (under I0 and AD):

\[
\begin{array}{c|c}
L(\mathbb{R}) \text{ under AD} & L(V_{\lambda+1}) \text{ under I0} \\
\end{array}
\]
Second degree analogies (under I0 and AD):

\[
\begin{array}{c|c}
L(\mathbb{R}) \text{ under AD} & L(V_{\lambda+1}) \text{ under I0} \\
\omega_1 \text{ is measurable} & \\
\end{array}
\]
Second degree analogies (under $\text{I}_0$ and AD):

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$ under AD</th>
<th>$L(V_{\lambda+1})$ under $\text{I}_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$ is measurable</td>
<td>$\lambda^+$ is measurable</td>
</tr>
</tbody>
</table>
Second degree analogies (under I0 and AD):

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$ under AD</th>
<th>$L(V_{\lambda+1})$ under I0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$ is measurable</td>
<td>$\lambda^+$ is measurable</td>
</tr>
<tr>
<td>the Coding Lemma holds</td>
<td></td>
</tr>
</tbody>
</table>

$\omega_1$ is measurable

$\lambda^+$ is measurable

the Coding Lemma holds
Second degree analogies (under I0 and AD):

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$ under AD</th>
<th>$L(V_{\lambda+1})$ under I0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$ is measurable</td>
<td>$\lambda^+$ is measurable</td>
</tr>
<tr>
<td>the Coding Lemma holds</td>
<td>the Coding Lemma holds.</td>
</tr>
</tbody>
</table>
Second degree analogies (under I0 and AD):

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$ under AD</th>
<th>$L(V_{\lambda+1})$ under I0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$ is measurable</td>
<td>$\lambda^+$ is measurable</td>
</tr>
<tr>
<td>the Coding Lemma holds</td>
<td>the Coding Lemma holds</td>
</tr>
</tbody>
</table>

The most immediate corollary for the Coding Lemma is:
For every $\alpha < \Theta$
Second degree analogies (under I0 and AD):

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$ under AD</th>
<th>$L(V_{\lambda+1})$ under I0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$ is measurable</td>
<td>$\lambda^+$ is measurable</td>
</tr>
<tr>
<td>the Coding Lemma holds</td>
<td>the Coding Lemma holds</td>
</tr>
</tbody>
</table>

The most immediate corollary for the Coding Lemma is:
For every $\alpha < \Theta$ there exists a surjection $\pi : \mathbb{R} \to P(\alpha)$. 
Second degree analogies (under I0 and AD):

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$ under AD</th>
<th>$L(V_{\lambda+1})$ under I0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$ is measurable</td>
<td>$\lambda^+$ is measurable</td>
</tr>
<tr>
<td>the Coding Lemma holds</td>
<td>the Coding Lemma holds</td>
</tr>
</tbody>
</table>

The most immediate corollary for the Coding Lemma is:
For every $\alpha < \Theta$ there exists a surjection $\pi : \mathbb{R} \to \mathcal{P}(\alpha)$.

Bonus result: Let $S^\lambda_\delta$ be the set of the ordinals in $\lambda^+$ with cofinality $\delta$. 
Second degree analogies (under I0 and AD):

<table>
<thead>
<tr>
<th>$L(\mathbb{R})$ under AD</th>
<th>$L(V_{\lambda+1})$ under I0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$ is measurable</td>
<td>$\lambda^+$ is measurable</td>
</tr>
<tr>
<td>the Coding Lemma holds</td>
<td>the Coding Lemma holds</td>
</tr>
</tbody>
</table>

The most immediate corollary for the Coding Lemma is:
For every $\alpha < \Theta$ there exists a surjection $\pi : \mathbb{R} \to \mathcal{P}(\alpha)$.

Bonus result: Let $S_\delta^{\lambda^+}$ be the set of the ordinals in $\lambda^+$ with cofinality $\delta$. Then there exists a partition $\langle S_\alpha : \alpha < \eta \rangle$ of $S_\delta^{\lambda^+}$ in $\eta < \lambda$ stationary sets such that for every $\alpha < \eta$ the club filter of $\lambda^+$ on $S_\alpha$ is an ultrafilter.
Third degree analogy:

**Theorem**
Third degree analogy:

**Theorem**

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$. 
Third degree analogy:

**Theorem**

Suppose that there exists \( j : L(V_{\lambda+1}) \prec L(V_{\lambda+1}) \) with \( \text{crt}(j) < \lambda \). Then \( \Theta \) is a limit of \( \gamma \) such that:

- \( \gamma \) is weakly inaccessible in \( L(V_{\lambda+1}) \);
Third degree analogy:

**Theorem**

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$. Then $\Theta$ is a limit of $\gamma$ such that:

- $\gamma$ is weakly inaccessible in $L(V_{\lambda+1})$;
- $\gamma = \Theta^L_{\gamma}(V_{\lambda+1})$ and $j(\gamma) = \gamma$;
Third degree analogy:

**Theorem**

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$. Then $\Theta$ is a limit of $\gamma$ such that:

- $\gamma$ is weakly inaccessible in $L(V_{\lambda+1})$;
- $\gamma = \Theta^{L(\gamma)}(V_{\lambda+1})$ and $j(\gamma) = \gamma$;
- for all $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(V_{\lambda+1}) \in L(\gamma)(V_{\lambda+1})$;
Third degree analogy:

**Theorem**

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$. Then $\Theta$ is a limit of $\gamma$ such that:

- $\gamma$ is weakly inaccessible in $L(V_{\lambda+1})$;
- $\gamma = \Theta^{L_\gamma(V_{\lambda+1})}$ and $j(\gamma) = \gamma$;
- for all $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(V_{\lambda+1}) \in L_\gamma(V_{\lambda+1})$;
- for cofinally $\kappa < \gamma$, $\kappa$ is a measurable cardinal in $L(V_{\lambda+1})$ and this is witnessed by the club filter on a stationary set;
Third degree analogy:

**Theorem**

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$. Then $\Theta$ is a limit of $\gamma$ such that:

- $\gamma$ is weakly inaccessible in $L(V_{\lambda+1})$;
- $\gamma = \Theta^{L(\gamma)}(V_{\lambda+1})$ and $j(\gamma) = \gamma$;
- for all $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(V_{\lambda+1}) \in L(\gamma)(V_{\lambda+1})$;
- for cofinally $\kappa < \gamma$, $\kappa$ is a measurable cardinal in $L(V_{\lambda+1})$ and this is witnessed by the club filter on a stationary set;
- $L(\gamma)(V_{\lambda+1}) \prec L(\Theta)(V_{\lambda+1})$. 
I0 is called Higher Determinacy Axiom, because it has consequences similar to Determinacy, but in a larger model.
I0 is called Higher Determinacy Axiom, because it has consequences similar to Determinacy, but in a larger model. Is it possible to find stronger Higher Determinacy Axioms?
I0 is called Higher Determinacy Axiom, because it has consequences similar to Determinacy, but in a larger model. Is it possible to find stronger Higher Determinacy Axioms? We will consider elementary embeddings between two kind of models:

- \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \), with \( X \subset V_{\lambda+1} \);
I0 is called Higher Determinacy Axiom, because it has consequences similar to Determinacy, but in a larger model. Is it possible to find stronger Higher Determinacy Axioms? We will consider elementary embeddings between two kind of models:

- \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \), with \( X \subset V_{\lambda+1} \);
- \( j : L(N) \prec L(N) \), with \( V_{\lambda+1} \subset N \subset V_{\lambda+2} \) and \( N = L(N) \cap V_{\lambda+2} \).
In the first case, the first and second degree analogies hold.
In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses.
In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses.

Let \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \) with \( X \subseteq V_{\lambda+1} \).
In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses. Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then

$$U_j = \{Z \in L(X, V_{\lambda+1}) \cap V_{\lambda+2} : j \upharpoonright V_{\lambda} \in j(Z)\}$$

generates an elementary embedding $j_U$. 
In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses.

Let \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \) with \( X \subseteq V_{\lambda+1} \).

Then

\[
U_j = \{ Z \in L(X, V_{\lambda+1}) \cap V_{\lambda+2} : j \upharpoonright V_\lambda \in j(Z) \}
\]

generates an elementary embedding \( j_U \), and there exists a \( k_U : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \) with \( \text{crt}(k_U) > \Theta \) such that \( j = k_U \circ j_U \).
In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses. Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then

$$U_j = \{ Z \in L(X, V_{\lambda+1}) \cap V_{\lambda+2} : j \upharpoonright V_{\lambda} \in j(Z) \}$$

generates an elementary embedding $j_U$, and there exists a $k_U : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $\text{crt}(k_U) > \Theta$ such that $j = k_U \circ j_U$. So the “important part” of $j$ is under $L_\Theta(X, V_{\lambda+1})$. 
We all know about the richness of life when we reach twenty. I think after that things we learn about life do not add up much to it, and I think that once we formed [...] a map of humanity in our mind, [...] understanding of humanity doesn’t change much.

Orhan Pamuk
We all know about the richness of life when we reach twenty. I think after that things we learn about life do not add up much to it, and I think that once we formed [...] a map of humanity in our mind, [...] understanding of humanity doesn’t change much.

Orhan Pamuk

Passé la puberté, tout le reste n’est qu’un épilogue.
(From puberty onwards, life is just an epilogue)

Amélie Nothomb, Le sabotage amoureux
We all know about the richness of life when we reach twenty. I think after that things we learn about life do not add up much to it, and I think that once we formed [...] a map of humanity in our mind, [...] understanding of humanity doesn’t change much.

Orhan Pamuk

Passé la puberté, tout le reste n’est qu’un épilogue. (From puberty onwards, life is just an epilogue)

Amélie Nothomb, Le sabotage amoureux (maybe)
We all know about the richness of life when we reach twenty. I think after that things we learn about life do not add up much to it, and I think that once we formed [...] a map of humanity in our mind, [...] understanding of humanity doesn’t change much.

Orhan Pamuk

Passé la puberté, tout le reste n’est qu’un épilogue.
(From puberty onwards, life is just an epilogue)

Amélie Nothomb, Le sabotage amoureux (maybe)

Proof by Woodin.

If \( j, k : L(V_{\lambda+1}) \preceq L(V_{\lambda+1}) \) and \( j \upharpoonright V_{\lambda} = k \upharpoontright V_{\lambda} \), then\[ j \upharpoonright L_\Theta(V_{\lambda+1}) = k \upharpoontright L_\Theta(V_{\lambda+1}). \]
**Definition**

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then $j$ is weakly proper iff $j = j_U$. 
Definition
Let \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \) with \( X \subseteq V_{\lambda+1} \). Then \( j \) is weakly proper iff \( j = j_U \).

Definition
Let \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \) with \( X \subseteq V_{\lambda+1} \). Then \( j \) is proper if it is weakly proper.
Non proper elementary embeddings beyond $L(V_{\lambda+1})$.

Vincenzo Dimonte

Large Cardinals Map

Introduction

Higher Determinacy Axiom

Main Results

Totally Non-proper Ordinals

Partially non-proper ordinal

Both

Implications and open problems

**Definition**

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then $j$ is **weakly proper** iff $j = j_U$.

**Definition**

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then $j$ is **proper** if it is weakly proper and $\langle X, j(X), j(j(X)), \ldots \rangle \in L(X, V_{\lambda+1})$.
**Definition**

Let \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \) with \( X \subseteq V_{\lambda+1} \). Then \( j \) is weakly proper iff \( j = j_U \).

**Definition**

Let \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \) with \( X \subseteq V_{\lambda+1} \). Then \( j \) is proper if it is weakly proper and \( \langle X, j(X), j(j(X)), \ldots \rangle \in L(X, V_{\lambda+1}) \).

If \( j \) is proper, then the third degree analogies hold.
Definition

Let \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \) with \( X \subseteq V_{\lambda+1} \). Then \( j \) is weakly proper iff \( j = j^U \).

Definition

Let \( j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1}) \) with \( X \subseteq V_{\lambda+1} \). Then \( j \) is proper if it is weakly proper and \( \langle X, j(X), j(j(X)), \ldots \rangle \in L(X, V_{\lambda+1}) \).

If \( j \) is proper, then the third degree analogies hold.
The second case is more complicated. It can be even that the first degree analogy doesn’t hold.
Non proper elementary embeddings beyond $L(V_{\lambda+1})$

Vincenzo Dimonte

Large Cardinals Map

Introduction

Higher Determinacy Axiom

Main Results

Totally Non-proper Ordinals

Partially non-proper ordinal

Both

Implications and open problems

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then $j$ is weakly proper iff $j = j_U$.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then $j$ is proper if it is weakly proper and $\langle X, j(X), j(j(X)), \ldots \rangle \in L(X, V_{\lambda+1})$.

If $j$ is proper, then the third degree analogies hold. The second case is more complicated. It can be even that the first degree analogy doesn’t hold. But if we have that $L(N) \models V = \text{HOD}_{V_{\lambda+1}}$, then the first and second degree analogy hold.
A similar ultrapower theorem exists, and we define similarly weakly proper embeddings
A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

**Definition**

Let $j : L(N) \prec L(N)$ with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$. Then $j$ is *proper* if it is weakly proper and for every $X \in N$ \(\langle X, j(X), j(j(X)), \ldots \rangle \in L(N).\)
A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

**Definition**

Let $j : L(N) \prec L(N)$ with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$. Then $j$ is *proper* if it is weakly proper and for every $X \in N$ \langle X, j(X), j(j(X)), \ldots \rangle \in L(N)$.

And if $j$ is proper, then the third degree analogy hold.
A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

**Definition**

Let $j : L(N) \prec L(N)$ with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$. Then $j$ is proper if it is weakly proper and for every $X \in N \langle X, j(X), j(j(X)), \ldots \rangle \in L(N)$.

And if $j$ is proper, then the third degree analogy hold.

Now we have to define new axioms of this kind, with the ultimate purpose of finding an analogous of $AD_{\mathbb{R}}$. 
A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

**Definition**

Let \( j : L(N) \prec L(N) \) with \( V_{\lambda+1} \subset N \subset V_{\lambda+2} \) and \( N = L(N) \cap V_{\lambda+2} \). Then \( j \) is **proper** if it is weakly proper and for every \( X \in N \)

\[ \langle X, j(X), j(j(X)), \ldots \rangle \in L(N). \]

And if \( j \) is proper, then the third degree analogy hold.

Now we have to define new axioms of this kind, with the ultimate purpose of finding an analogous of \( \text{AD}_\mathbb{R} \).

There is no evident elementary embedding form...
A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

**Definition**

Let \(j : L(N) \prec L(N)\) with \(V_{\lambda+1} \subset N \subset V_{\lambda+2}\) and \(N = L(N) \cap V_{\lambda+2}\). Then \(j\) is proper if it is weakly proper and for every \(X \in N\)
\[
\langle X, j(X), j(j(X)), \ldots \rangle \in L(N).
\]

And if \(j\) is proper, then the third degree analogy hold.

Now we have to define new axioms of this kind, with the ultimate purpose of finding an analogous of \(\text{AD}_{\mathbb{R}}\).

There is no evident elementary embedding form... so the way chose by Woodin is defining an analogous of the minimum model of \(\text{AD}_{\mathbb{R}}\).
Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on $\alpha$:
Non proper elementary embeddings beyond $L(V_{\lambda+1})$

Vincenzo Dimonte

Large Cardinals Map

Introduction

Higher Determinacy Axiom

Main Results

Totally Non-proper Ordinals

Partially non-proper ordinal

Both

Implications and open problems

Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on $\alpha$:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;
Definition

Define a sequence of $\Gamma_{\alpha} \subseteq \mathcal{P}(\mathbb{R})$ by induction on $\alpha$:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;
- If $\alpha$ is a limit ordinal then $\Gamma_{\alpha} = L((\bigcup_{\beta < \alpha} \Gamma_{\beta})^\omega) \cap \mathcal{P}(\mathbb{R})$;
Non proper elementary embeddings beyond $L(V_{\lambda+1})$

Vincenzo Dimonte

Large Cardinals Map

Introduction

Higher Determinacy Axiom

Main Results

Totally Non-proper Ordinals

Partially non-proper ordinal

Both

Implications and open problems

Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on $\alpha$:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;
- If $\alpha$ is a limit ordinal then $\Gamma_\alpha = L((\bigcup_{\beta < \alpha} \Gamma_\beta)^\omega) \cap \mathcal{P}(\mathbb{R})$;
- If $\text{cof}(\Theta L(\Gamma_\alpha)) = \omega$, then $\Gamma_{\alpha+1} = L((\Gamma_\alpha)^\omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$, where $F$ is the $\omega$-club filter in $\Theta L(\Gamma_\alpha)$.

The sequence stops when $L(\Gamma_\alpha) \not\models \text{AD}$ or $\Gamma_\alpha = \Gamma_{\alpha+1}$.
Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on $\alpha$:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;
- If $\alpha$ is a limit ordinal then $\Gamma_\alpha = L((\bigcup_{\beta < \alpha} \Gamma_\beta)^\omega) \cap \mathcal{P}(\mathbb{R})$;
- If $\text{cof}(\Theta^L(\Gamma_\alpha)) = \omega$, then $\Gamma_{\alpha+1} = L((\Gamma_\alpha)^\omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$, otherwise $\Gamma_{\alpha+1} = L(\Gamma_\alpha)[\mathcal{F}] \cap \mathcal{P}(\mathbb{R})$, where $\mathcal{F}$ is the $\omega$-club filter in $\Theta^L(\Gamma_\alpha)$. 

The sequence stops when $L(\Gamma_\alpha) \not\models \text{AD}$ or $\Gamma_\alpha = \Gamma_\alpha + 1$. 

**Definition**
Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on $\alpha$:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;
- If $\alpha$ is a limit ordinal then $\Gamma_\alpha = L((\bigcup_{\beta < \alpha} \Gamma_\beta)^\omega) \cap \mathcal{P}(\mathbb{R})$;
- If $\text{cof}(\Theta^L(\Gamma_\alpha)) = \omega$, then $\Gamma_{\alpha+1} = L((\Gamma_\alpha)^\omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$, otherwise $\Gamma_{\alpha+1} = L(\Gamma_\alpha) [\mathcal{F}] \cap \mathcal{P}(\mathbb{R})$, where $\mathcal{F}$ is the $\omega$-club filter in $\Theta^L(\Gamma_\alpha)$.

The sequence stops when $L(\Gamma_\alpha) \not\models \text{AD}$ or $\Gamma_\alpha = \Gamma_{\alpha+1}$.
Definition

The sequence

\[ \langle E_\alpha^0(V_{\lambda+1}) : \alpha < \Upsilon V_{\lambda+1} \rangle \]

is defined as:

- \( E_0^0(V_{\lambda+1}) = L(V_{\lambda+1}) \cap V_{\lambda+2}; \)
- for \( \alpha \) limit, \( E_\alpha^0(V_{\lambda+1}) = L(\bigcup_{\beta < \alpha} E_\beta^0(V_{\lambda+1})) \cap V_{\lambda+2}; \)
- for \( \alpha \) limit,
  - if \( (\text{cof}(\Theta E_\alpha^0(V_{\lambda+1})) < \lambda)^L(E_\alpha^0(V_{\lambda+1})) \) then
    \( E_{\alpha+1}^0(V_{\lambda+1}) = L((E_\alpha^0(V_{\lambda+1}))^\lambda) \cap V_{\lambda+2}; \)
  - if \( (\text{cof}(\Theta E_\alpha^0(V_{\lambda+1}))^L(E_\alpha^0(V_{\lambda+1})) > \lambda \) then
    \( E_{\alpha+1}^0(V_{\lambda+1}) = L(\mathcal{E}(E_\alpha^0(V_{\lambda+1}))) \cap V_{\lambda+2}; \)
Definition

- for $\alpha = \beta + 2$, if there exists $X \subseteq V_{\lambda+1}$ such that $E^0_{\beta+1}(V_{\lambda+1}) = L(X, V_{\lambda+1}) \cap V_{\lambda+2}$ and $E^0_\beta(V_{\lambda+1}) < X$, then
  
  $$E^0_{\beta+2}(V_{\lambda+1}) = L((X, V_{\lambda+1})^\#) \cap V_{\lambda+2}$$

  otherwise we stop the sequence.

- $\forall \alpha < \gamma V_{\lambda+1}$ $\exists X \subseteq V_{\lambda+1}$ such that $E^0_\alpha(V_{\lambda+1}) < X$ and $\exists j: L(X, V_{\lambda+1}) \rightarrow L(X, V_{\lambda+1})$ proper;

- $\forall \alpha$ limit $\alpha + 1 < \gamma V_{\lambda+1}$ iff

  $$(\cof(\Theta E^0_\alpha(V_{\lambda+1}))) L(E^0_\alpha(V_{\lambda+1})) > \lambda \rightarrow$$

  $$\exists Z \in E^0_\alpha(V_{\lambda+1}) L(E^0_\alpha(V_{\lambda+1})) = (HOD_{V_{\lambda+1} \cup \{Z\}}) L(E^0_\alpha(V_{\lambda+1})).$$
Let $N = L(\bigcup \{ E_\alpha^0(V_{\lambda+1}) : \alpha < \gamma V_{\lambda+1} \}) \cap V_{\lambda+2}$. Suppose that

- $\text{cof}(\Theta^N) > \lambda$;
- for all $Z \in N$ $L(N) \neq (\text{HOD}_{V_{\lambda+1} \cup \{Z\}})^{L(N)}$;
- there is an elementary embedding $j : L(N) \prec L(N)$ with $\text{crt}(j) < \lambda$. 

**Definition**
Definition

Let \( N = L(\bigcup \{ E_\alpha^0(V_{\lambda+1}) : \alpha < \gamma_{V_{\lambda+1}} \}) \cap V_{\lambda+2} \). Suppose that

- \( \text{cof}(\Theta^N) > \lambda \);
- for all \( Z \in N \) \( L(N) \neq (\text{HOD}_{V_{\lambda+1}} \cup \{ Z \})^{L(N)} \);
- there is an elementary embedding \( j : L(N) \prec L(N) \) with \( \text{crt}(j) < \lambda \).

Then \( E_\infty^0(V_{\lambda+1}) \) exists and \( E_\infty^0(V_{\lambda+1}) = N \).
Three important facts:

**Theorem**

- If $\alpha < \beta < \gamma$, then $\Theta^{E_0_\alpha} < \Theta^{E_0_\beta}$.
Three important facts:

**Theorem**

- If $\alpha < \beta < \gamma$, then $\Theta E_\alpha^0 < \Theta E_\beta^0$.
- The $E_\alpha^0$ sequence is absolute, i.e. for every $M$ such that $L(M) \cap V_{\lambda + 2}, V_{\lambda + 1} \subseteq M$ for every $\alpha < \gamma^M$, $(\langle E_\beta^0 : \beta < \alpha \rangle)_M = \langle E_\beta^0 : \beta < \alpha \rangle$. 
Three important facts:

**Theorem**

- If $\alpha < \beta < \gamma$, then $\Theta E_\alpha^0 < \Theta E_\beta^0$.
- The $E_\alpha^0$ sequence is absolute, i.e. for every $M$ such that $L(M) \cap V_{\lambda+2}, V_{\lambda+1} \subseteq M$ for every $\alpha < \gamma^M$, $(\langle E_\beta^0 : \beta < \alpha \rangle)^M = \langle E_\beta^0 : \beta < \alpha \rangle$.
- If $\alpha < \gamma$, then there exists an elementary embedding $j : L(E_\alpha^0) \prec L(E_\alpha^0)$. 
We’ve seen that properness is quite important for establishing Determinacy results.
We’ve seen that properness is quite important for establishing Determinacy results.
But is it really a property?
We’ve seen that properness is quite important for establishing Determinacy results. But is it really a property?

**Theorem**

Suppose $\alpha < \gamma$. If

- $\alpha = 0$,

then every weakly proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is proper.
We’ve seen that properness is quite important for establishing Determinacy results. But is it really a property?

**Theorem**

Suppose $\alpha < \gamma$. If
- $\alpha = 0$, or
- $\alpha$ is a successor ordinal,
then every weakly proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is proper.
We’ve seen that properness is quite important for establishing Determinacy results. But is it really a property?

**Theorem**

Suppose $\alpha < \gamma$. If
- $\alpha = 0$, or
- $\alpha$ is a successor ordinal, or
- $\alpha$ is a limit ordinal with cofinality $> \omega$

then every weakly proper elementary embedding $j : L(E_\alpha^0) \prec L(E_\alpha^0)$ is proper.
We can think of two possible scenarios
We can think of two possible scenarios:

**Definition**

- $\alpha$ is partially non-proper if there exist $j, k : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that $j$ is proper and $k$ is not proper;
We can think of two possible scenarios:

**Definition**

- $\alpha$ is partially non-proper if there exist $j, k : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that $j$ is proper and $k$ is not proper;
- $\alpha$ is totally non-proper if every elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is not proper.
We can think of two possible scenarios:

**Definition**

- \( \alpha \) is partially non-proper if there exist \( j, k : L(E_\alpha^0) \prec L(E_\alpha^0) \) such that \( j \) is proper and \( k \) is not proper;
- \( \alpha \) is totally non-proper if every elementary embedding \( j : L(E_\alpha^0) \prec L(E_\alpha^0) \) is not proper.

We will prove that both exist.
We can think of two possible scenarios:

**Definition**

- \( \alpha \) is partially non-proper if there exist \( j, k : L(E_\alpha^0) \prec L(E_\alpha^0) \) such that \( j \) is proper and \( k \) is not proper;
- \( \alpha \) is totally non-proper if every elementary embedding \( j : L(E_\alpha^0) \prec L(E_\alpha^0) \) is not proper.

We will prove that both exist. The key Lemma is the following:

**Lemma**

Suppose \( \alpha < \gamma \) and \( \Theta^{E_\alpha^0} \) is regular in \( L(E_\alpha^0) \). If \( j : L(E_\alpha^0) \prec L(E_\alpha^0) \) is proper then the set of fixed points of \( j \) is cofinal in \( \Theta^{E_\alpha^0} \).
Informal definition of $X^\#$:
Suppose that there exists a class $I$ of indiscernibles of $(L(X), \in, \{a : a \in X\}, X)$ such that every cardinal $> |X|$ is in $I$. Then $X^\#$ is the theory of the indiscernibles in the language \{\in\} \cup \{a : a \in X\} \cup \{X\}$, i.e.

$$X^# = \{\varphi(a_1, \ldots, a_n, X, i_1, \ldots, i_n) : a_1, \ldots, a_n \in X, \quad L(X) \models \varphi(a_1, \ldots, a_n, X, i_1, \ldots, i_n) \text{ for some (any)} \quad \text{indiscernibles } i_1 < \cdots < i_n \in I\}$$
Informal definition of $X^\#$:

Suppose that there exists a class $I$ of indiscernibles of $(L(X), \in, \{a : a \in X\}, X)$ such that every cardinal $\geq |X|$ is in $I$. Then $X^\#$ is the theory of the indiscernibles in the language $\{\in\} \cup \{a : a \in X\} \cup \{X\}$, i.e.

$$X^\# = \{ \varphi(a_1, \ldots, a_n, X, i_1, \ldots, i_n) : a_1, \ldots, a_n \in X, \quad L(X) \models \varphi(a_1, \ldots, a_n, X, i_1, \ldots, i_n) \text{ for some (any) indiscernibles } i_1 < \cdots < i_n \in I \}$$

$X^\#$ contains the “truth“ of $L(X)$, so it cannot be in $L(X)$. 
In our case, for every $\alpha$, $(E^{0}_\alpha)\# \notin L(E^{0}_\alpha)$. 
In our case, for every $\alpha$, $(E_\alpha^0)^# \notin L(E_\alpha^0)$.
If $\alpha$ is a limit and $E_\alpha^0 = \bigcup E_\beta^0$, then we can slice $(E_\alpha^0)^#$ in smaller pieces, digestible by $L(E_\alpha^0)$.
In our case, for every $\alpha$, $(E^0_\alpha)^\# \not\in L(E^0_\alpha)$. If $\alpha$ is a limit and $E^0_\alpha = \bigcup E^0_\beta$, then we can slice $(E^0_\alpha)^\#$ in smaller pieces, digestible by $L(E^0_\alpha)$:

$$(E^0_\alpha)^\#_{\beta,n} = (E^0_\alpha)^\# \cap (\{\in\} \cup \{a : a \in E^0_\beta\} \cup \{X\} \cup \{i_1, \ldots, i_n\})$$
In our case, for every $\alpha$, $(E_\alpha^0)^\# \not\in L(E_\alpha^0)$. If $\alpha$ is a limit and $E_\alpha^0 = \bigcup E_\beta^0$, then we can slice $(E_\alpha^0)^\#$ in smaller pieces, digestible by $L(E_\alpha^0)$:

$$(E_\alpha^0)^\#_{\beta,n} = (E_\alpha^0)^\# \cap (\{\in\} \cup \{a : a \in E_\beta^0\} \cup \{X\} \cup \{i_1, \ldots, i_n\})$$

For every $\beta < \alpha$, $n \in \omega$ $(E_\alpha^0)^\#_{\beta,n} \in E_\alpha^0$, but $L(E_\alpha^0)$ doesn’t know that they are sharp fragments.
In our case, for every \( \alpha \), \((E_\alpha^0)^\# \notin L(E_\alpha^0)\).
If \( \alpha \) is a limit and \( E_\alpha^0 = \bigcup E_\beta^0 \), then we can slice \((E_\alpha^0)^\#\) in smaller pieces, digestible by \( L(E_\alpha^0) \):

\[
(E_\alpha^0)^\#_{\beta,n} = (E_\alpha^0)^\# \cap (\{\in\} \cup \{a : a \in E_\beta^0\} \cup \{X\} \cup \{i_1, \ldots, i_n\})
\]

For every \( \beta < \alpha \), \( n \in \omega \) \((E_\alpha^0)^\#_{\beta,n} \in E_\alpha^0\), but \( L(E_\alpha^0) \) doesn’t know that they are sharp fragments.
So if \( k : E_\beta^0 \prec E_\alpha^0 \), \( k(\text{sharp fragment}) \) can be anything.
Definition

We say that $k : E^0_\beta \prec E^0_\alpha$ is sharp-friendly if it maps sharp fragments to sharp fragments.
Definition
We say that $k : E^0_\beta \prec E^0_\alpha$ is sharp-friendly if it maps sharp fragments to sharp fragments.

Lemma
If $k : E^0_\beta \prec E^0_\alpha$ is sharp-friendly, then it’s possible to extend it to $\hat{k} : L(E^0_\beta) \prec L(E^0_\alpha)$. 
In this subsection we work in
\[ I = \{ \beta < \gamma : \forall \gamma < \beta \ L(E_\gamma^0) \models V = \text{HOD}_{V_{\lambda+1}} \}. \]
In this subsection we work in
\[ I = \{ \beta < \gamma : \forall \gamma < \beta \ L(E^0_\gamma) \models V = \text{HOD}_{\mathcal{V}_{\lambda+1}} \}. \]

Beyond \((E^0_\beta)^\#_{\gamma, n}\), we can define also \((E^0_\beta)^\#_{\gamma}\), that it’s a theory in
the language with constants from \(E^0_\gamma\).
In this subsection we work in
\[ I = \{ \beta < \gamma : \forall \gamma < \beta \ L(E_0^\gamma) \models V = \text{HOD}_{V_{\lambda+1}} \} \].
Beyond \((E_0^\beta)^\#\), we can define also \((E_0^\beta)^\#\gamma\), that it’s a theory in
the language with constants from \(E_0^\gamma\).
But \((E_0^\gamma)^\#\) is also a theory in that language. What if they are
equal, i.e. what if the sharp reflects on \(\gamma\)?
In this subsection we work in
\[ I = \{ \beta < \Upsilon : \forall \gamma < \beta \ L(E_0^\gamma) \models V = \text{HOD}_{V_{\lambda+1}} \}. \]
Beyond \((E_0^\beta)^\#_{\gamma,n}\), we can define also \((E_0^\beta)^\#_{\gamma}\), that it’s a theory in
the language with constants from \(E_0^\gamma\).
But \((E_0^\gamma)^\#\) is also a theory in that language. What if they are
equal, i.e. what if the sharp reflects on \(\gamma\)?
This is something less than asking that \(L(E_0^\gamma) \prec L(E_0^\beta)\), but
something more than \(E_0^\gamma \prec E_0^\beta\).
In this subsection we work in
\[ I = \{ \beta < \gamma : \forall \gamma < \beta \; L(E^0_\gamma) \models V = \text{HOD}_{V_{\lambda+1}} \}. \]

Beyond \((E^0_\gamma)^\#_\gamma, n\), we can define also \((E^0_\beta)^\#_\gamma\), that it’s a theory in
the language with constants from \(E^0_\gamma\).

But \((E^0_\gamma)^\#\) is also a theory in that language. What if they are equal, i.e. what if the sharp reflects on \(\gamma\)?

This is something less than asking that \(L(E^0_\gamma) \prec L(E^0_\beta)\), but
something more than \(E^0_\gamma \prec E^0_\beta\).

In fact, it’s equivalent to the sharp-friendliness of the identity
from \(E^0_\gamma\) to \(E^0_\beta\).
Let $\beta \in I$. 
Let $\beta \in I$. Define $I_{\beta}$ as the set of all $\gamma$’s such that the sharp in $\beta$ reflects on $\gamma$. 
Let $\beta \in I$. Define $I_\beta$ as the set of all $\gamma$’s such that the sharp in $\beta$ reflects on $\gamma$.

**Lemma**

For every $\beta \in I$:

- If $I_\beta \neq \emptyset$ then $\beta$ is a limit and $\beta = \Theta^{E_\beta^0} = \sup_{\gamma < \beta} \Theta^{E_\gamma^0}$;
Let $\beta \in I$. Define $I_\beta$ as the set of all $\gamma$’s such that the sharp in $\beta$ reflects on $\gamma$.

**Lemma**

For every $\beta \in I$:

- If $I_\beta \neq \emptyset$ then $\beta$ is a limit and $\beta = \Theta^E_\beta = \sup_{\gamma < \beta} \Theta^E_\gamma$;
- if $\gamma \in I_\beta$, then $I_\beta \cap \gamma = I_\gamma$.
Let $\beta \in I$. Define $I_\beta$ as the set of all $\gamma$'s such that the sharp in $\beta$ reflects on $\gamma$.

**Lemma**

For every $\beta \in I$:
- If $I_\beta \neq \emptyset$ then $\beta$ is a limit and $\beta = \Theta^{E_\beta} = \sup_{\gamma < \beta} \Theta^{E_\gamma}$;
- if $\gamma \in I_\beta$, then $I_\beta \cap \gamma = I_\gamma$;
- $I_\beta$ is closed.
The following Lemma is a key point:
The following Lemma is a key point:

**Lemma**

For every $\gamma < \beta \in I$, 
- for every $j : L(E^0_\beta) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\beta$ is sharp-friendly;
The following Lemma is a key point:

**Lemma**

For every $\gamma < \beta \in I$,
- for every $j : L(E_\beta^0) \prec L(E_\beta^0)$, $j \upharpoonright E_\beta^0$ is sharp-friendly;
- for every $j : L(E_\gamma^0) \prec L(E_\beta^0)$, $j \upharpoonright E_\gamma^0$ is sharp-friendly.
The following Lemma is a key point:

**Lemma**

For every $\gamma < \beta \in \mathcal{I}$,
- for every $j: L(E^0_\gamma) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\gamma$ is sharp-friendly;
- for every $j: L(E^0_\gamma) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\gamma$ is sharp-friendly.

So every $j: L(E^0_\beta) \prec L(E^0_\beta)$ maps in a good way the initial segments of $\mathcal{I}_\beta$, 

(Note that $j(\mathcal{I}_\gamma) = \mathcal{I}_{j(\gamma)}$.)
The following Lemma is a key point:

**Lemma**

For every $\gamma < \beta \in I$,

- for every $j : L(E_\beta^0) \prec L(E_\beta^0)$, $j \upharpoonright E_\beta^0$ is sharp-friendly;
- for every $j : L(E_\gamma^0) \prec L(E_\beta^0)$, $j \upharpoonright E_\gamma^0$ is sharp-friendly.

So every $j : L(E_\beta^0) \prec L(E_\beta^0)$ maps in a good way the initial segments of $I_\beta$,
i.e. for every $\gamma \in I_\beta$ $j(\gamma) \in I_\beta$. 
The following Lemma is a key point:

Lemma

For every $\gamma < \beta \in I$,
- for every $j : L(E_\beta^0) \prec L(E_\beta^0)$, $j \upharpoonright E_\beta^0$ is sharp-friendly;
- for every $j : L(E_\gamma^0) \prec L(E_\beta^0)$, $j \upharpoonright E_\gamma^0$ is sharp-friendly.

So every $j : L(E_\beta^0) \prec L(E_\beta^0)$ maps in a good way the initial segments of $I_\beta$,
i.e. for every $\gamma \in I_\beta$ $j(\gamma) \in I_\beta$. (Note that $j(I_\gamma) = I_{j(\gamma)}$).
Theorem

Let $\beta \in I$ such that $\text{ot}(I_\beta) = \lambda$. 

Proof. Since $I_\beta$ is closed, $\sup I_\beta = \beta = \Theta_{E_0}$. Define $\gamma_n$ as the $\kappa_n$-th element of $I_\beta$. So $j(\gamma_n) = \gamma_n + 1$ (we can see $\gamma_n$ as the $\kappa_n$-th element of $I_{\gamma_n + 2}$), and $j$ cannot be proper.
Theorem

Let $\beta \in I$ such that $\text{ot}(I_\beta) = \lambda$. Then $\beta$ is totally non-proper.
**Theorem**

Let $\beta \in I$ such that $\text{ot}(I_\beta) = \lambda$. Then $\beta$ is totally non-proper.

**Proof.**

Since $I_\beta$ is closed, $\sup I_\beta = \beta = \Theta^{E_0}_\beta$. 
Theorem

Let $\beta \in I$ such that $\ot(I_\beta) = \lambda$. Then $\beta$ is totally non-proper.

Proof.

Since $I_\beta$ is closed, $\sup I_\beta = \beta = \Theta^{E_\beta}$. Define $\gamma_n$ as the $\kappa_n$-th element of $I_\beta$. 
Theorem

Let $\beta \in I$ such that $\ot(l\beta) = \lambda$. Then $\beta$ is totally non-proper.

Proof.

Since $l\beta$ is closed, $\sup l\beta = \beta = \Theta^{E\beta}_\beta$. Define $\gamma_n$ as the $\kappa_n$-th element of $l\beta$. So $j(\gamma_n) = \gamma_{n+1}$ (we can see $\gamma_n$ as the $\kappa_n$-th element of $l\gamma_{n+2}$), and $j$ cannot be proper.
By the Lemma above, we only have to find an $\alpha$ such that we know that there exists a proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$,
By the Lemma above, we only have to find an $\alpha$ such that we know that there exists a proper elementary embedding $j : L(E_\alpha^0) \prec L(E_\alpha^0)$, and a sharp-friendly elementary embedding $k : E_\alpha^0 \prec E_\alpha^0$ whose extension is not proper.
Define the game $G_{\alpha}$ in $L((E_0^0)^\#)$:

$I$ \(\langle k_0, \beta_0 \rangle \quad \langle k_1, \beta_1 \rangle \quad \langle k_2, \beta_2 \rangle \quad \ldots\)

$II$ \(\eta_0 \quad \eta_1\)

with the following rules:

- $k_0 = \emptyset$;
- $k_{i+1} : E_0^0_{\beta_i} \prec E_0^0_{\beta_{i+1}}$;
- for every $\gamma < \beta_i$, $k_{i+1}((E_0^0)_{\gamma}^\#, n) = (E_0^0)_{k_{i+1}(\gamma)}^\#, n$;
- $\beta_i, \eta_i < \alpha$;
- $\beta_{i+1} > \eta_i$;
- $k_i \subseteq k_{i+1}$ and $k_{i+1}(\beta_i) = \beta_{i+1}$;
- II wins if and only if I at a certain point can’t play anymore.
So we have to find an $\alpha$ such that $\alpha = \Theta^{E_0^{\alpha}}$, $\text{cof}(\alpha) = \omega$ and $G_\alpha$ is determined for I.
So we have to find an $\alpha$ such that $\alpha = \Theta^{E_\alpha^0}$, $\text{cof}(\alpha) = \omega$ and $G_\alpha$ is determined for I. Let $\xi < \gamma$ and define the closed initial segment

$$H_\xi = \{ \gamma \leq \xi : E_\gamma^0 \subseteq (\text{HOD}_{V_{\lambda+1}})^{L(E_{\xi}^0)} \}.$$
So we have to find an $\alpha$ such that $\alpha = \Theta^{E_0^\alpha}$, $\text{cof}(\alpha) = \omega$ and $G_\alpha$ is determined for $I$. Let $\xi < \gamma$ and define the \textit{closed} initial segment

$$H_\xi = \{\gamma \leq \xi : E_\gamma^0 \subseteq (\text{HOD}_{\mathcal{V}_{\lambda+1}})^{L(E_\xi^0)}\}.$$

**Lemma**

Let $\xi < \gamma$. Let $\eta = \sup H_\xi$. 
So we have to find an $\alpha$ such that $\alpha = \Theta^{E^0_\alpha}$, $\text{cof}(\alpha) = \omega$ and $G_\alpha$ is determined for I. Let $\xi < \Upsilon$ and define the \textit{closed} initial segment

$$H_\xi = \{ \gamma \leq \xi : E^0_\gamma \subseteq (\text{HOD}_{V_{\lambda+1}})^{L(E^0_\xi)} \}.$$ 

**Lemma**

Let $\xi < \Upsilon$. Let $\eta = \sup H_\xi$. If $\eta < \xi$, then $\eta$ is a limit ordinal.
So we have to find an $\alpha$ such that $\alpha = \Theta^{E_\alpha^0}$, $\text{cof}(\alpha) = \omega$ and $G_\alpha$ is determined for I. Let $\xi < \Upsilon$ and define the *closed* initial segment

\[ H_\xi = \{ \gamma \leq \xi : E_\gamma^0 \subseteq (\text{HOD}_{V_{\lambda+1}})^{L(E_\xi^0)} \}. \]

**Lemma**

Let $\xi < \Upsilon$. Let $\eta = \sup H_\xi$. If $\eta < \xi$, then $\eta$ is a limit ordinal and

\[ \eta = \Theta^{E_\eta^0} = \Theta((\text{HOD}_{V_{\lambda+1}})^{E_\xi^0}). \]
So we have to find an $\alpha$ such that $\alpha = \Theta^{E_0^\alpha}$, $\text{cof}(\alpha) = \omega$ and $G_\alpha$ is determined for I. Let $\xi < \gamma$ and define the closed initial segment

$$H_\xi = \{\gamma \leq \xi : E_\gamma^0 \subseteq (\text{HOD} V_{\lambda+1})^{L(E_0^{\xi})}\}.$$ 

**Lemma**

Let $\xi < \gamma$. Let $\eta = \sup H_\xi$. If $\eta < \xi$, then $\eta$ is a limit ordinal and

- $\eta = \Theta^{E_0^\eta} = \Theta^{(\text{HOD} V_{\lambda+1})^{E_0^\eta}}$,
- $L((E_0^\eta)^#) \cap V_{\lambda+2} = E_0^\eta$. 

Notation

From now on, we call $\alpha$ the minimum ordinal such that $L((E^0_\alpha)^\#) \cap V_{\lambda+2} = E^0_\alpha$. 
Notation

From now on, we call \( \alpha \) the minimum ordinal such that \( L((E^0_\alpha)^\#) \cap V_{\lambda+2} = E^0_\alpha \).

Then both \( L(E^0_\alpha) \) and \( L((E^0_\alpha)^\#) \) have good qualities, and \( \alpha \) is “large” in \( L((E^0_\alpha)^\#) \).
Non proper elementary embeddings beyond $L(V_{\lambda+1})$

Vincenzo Dimonte

Large Cardinals Map

Introduction

Higher Determinacy Axiom

Main Results

Totally Non-proper Ordinals

Partially non-proper ordinal

Both

Implications and open problems

---

**Notation**

From now on, we call $\alpha$ the minimum ordinal such that $L((E_\alpha^0)^\#) \cap V_{\lambda+2} = E_\alpha^0$.

Then both $L(E_\alpha^0)$ and $L((E_\alpha^0)^\#)$ have good qualities, and $\alpha$ is “large” in $L((E_\alpha^0)^\#)$.

**Lemma**

- $\alpha = \Theta^{E_\alpha^0} = \Theta((E_\alpha^0)^\#)$;
From now on, we call $\alpha$ the minimum ordinal such that $L((E^0_\alpha)^\#) \cap V_{\lambda+2} = E^0_\alpha$.

Then both $L(E^0_\alpha)$ and $L((E^0_\alpha)^\#)$ have good qualities, and $\alpha$ is “large” in $L((E^0_\alpha)^\#)$.

**Lemma**

- $\alpha = \Theta^{E^0_\alpha} = \Theta(E^0_\alpha)^\#$; 
- $L(E^0_\alpha), L((E^0_\alpha)^\#) \models V = \text{HOD}_{V_{\lambda+1}}$;
Non proper elementary embeddings beyond $L(V_{\lambda+1})$

Vincenzo Dimonte

Large Cardinals Map

Introduction

Higher Determinacy Axiom

Main Results

Totally Non-proper Ordinals

Partially non-proper ordinal

Both

Implications and open problems

Notation

From now on, we call $\alpha$ the minimum ordinal such that $L((E^0_\alpha)^\#) \cap V_{\lambda+2} = E^0_\alpha$.

Then both $L(E^0_\alpha)$ and $L((E^0_\alpha)^\#)$ have good qualities, and $\alpha$ is “large” in $L((E^0_\alpha)^\#)$.

Lemma

- $\alpha = \Theta^{E^0_\alpha} = \Theta(E^0_\alpha)^\#$;
- $L(E^0_\alpha), L((E^0_\alpha)^\#) \models V = \text{HOD}_{V_{\lambda+1}}$;
- $\alpha$ is regular in $L((E^0_\alpha)^\#)$.
Notation

From now on, we call $\alpha$ the minimum ordinal such that $L((E_\alpha^0)\#) \cap V_{\lambda+2} = E_\alpha^0$.

Then both $L(E_\alpha^0)$ and $L((E_\alpha^0)\#)$ have good qualities, and $\alpha$ is “large” in $L((E_\alpha^0)\#)$.

Lemma

- $\alpha = \Theta^{E_\alpha^0} = \Theta(E_\alpha^0)\#$;
- $L(E_\alpha^0), L((E_\alpha^0)\#) \models V = \text{HOD}_{V_{\lambda+1}}$;
- $\alpha$ is regular in $L((E_\alpha^0)\#)$.

Note that, since there exists $j : L(E_{\alpha+2}^0) \prec L(E_{\alpha+2}^0)$, $j \upharpoonright L((E_\alpha^0)\#)$ is an elementary embedding, so in $L((E_\alpha^0)\#)$ the first and second degree analogies hold.
Theorem

In \( L((E_0^0)^\#) \) II cannot have a winning strategy for the game \( G_\alpha \).
Theorem

In $L((E_0^0)^\#)$ II cannot have a winning strategy for the game $G_\alpha$.

Proof.

We fix $j : L((E_0^0)^\#) \prec L((E_0^0)^\#)$. 
Theorem

In $L(((E_0^0)\#))$ II cannot have a winning strategy for the game $G_\alpha$.

Proof.

We fix $j : L(((E_0^0)\#)) \prec L(((E_0^0)\#))$.

Claim. For every $\beta_n < \alpha$, there is a surjection in $L(((E_0^0)\#))$ from $V_{\lambda+1}$ to the set of all the $k_n$ such that $\langle k_n, \beta_n \rangle$ is a legal move for I.
Non proper elementary embeddings beyond $L(V_{\lambda+1})$.

Vincenzo Dimonte

Large Cardinals Map

Introduction

Higher Determinacy Axiom

Main Results

Totally Non-proper Ordinals

Partially non-proper ordinal

Both

Implications and open problems

Theorem

In $L((E_\alpha^0)^\#)$ II cannot have a winning strategy for the game $G_\alpha$.

Proof.

We fix $j : L((E_\alpha^0)^\#) \prec L((E_\alpha^0)^\#)$.

Claim. For every $\beta_n < \alpha$, there is a surjection in $L((E_\alpha^0)^\#)$ from $V_{\lambda+1}$ to the set of all the $k_n$ such that $\langle k_n, \beta_n \rangle$ is a legal move for I.

This is because for every $\beta < \alpha$ every element of $E_\beta^0$ is definable with parameters from $\Theta^{E_\beta^0} \cup V_{\lambda+1}$. 
Theorem

In $L((E^0_\alpha)^\#)$ II cannot have a winning strategy for the game $G_\alpha$.

Proof.

We fix $j : L((E^0_\alpha)^\#) \prec L((E^0_\alpha)^\#)$.

Claim. For every $\beta_n < \alpha$, there is a surjection in $L((E^0_\alpha)^\#)$ from $V_{\lambda+1}$ to the set of all the $k_n$ such that $\langle k_n, \beta_n \rangle$ is a legal move for I.

This is because for every $\beta < \alpha$ every element of $E^0_\beta$ is definable with parameters from $\Theta^{E^0_\beta} \cup V_{\lambda+1}$. So every elementary embedding $k : E^0_{\beta_n-1} \prec E^0_{\beta_n}$ is defined by its behaviour on $\Theta^{E^0_{\beta_n-1}}$, and we have few of this behaviours because of the Coding Lemma.
Proof.

If \( II \) had a winning strategy \( \tau \), it would be definable.
Proof.

If II had a winning strategy $\tau$, it would be definable. So we can define the set $C$ of the ordinals closed under $\tau$, i.e. of the ordinals $\eta$ such that if $\beta_n < \eta$, then for every $k_n$ $\tau(\langle k_n, \beta_n \rangle) < \eta$. 

By the first claim $C$ is a club. Since $C$ is definable and $\alpha$ is regular, $C$ has ordertype $\alpha$. But then if I plays the $\kappa_n$-th element of $C$ as $\beta_n$ and $j \upharpoonright E_0^{\alpha+2}$ as $k_n$, I wins, and that's a contradiction. Since $\text{cof}(\alpha) = \omega$ and for every $j$: $L(E_0^{\alpha+2}) \preceq L(E_0^{\alpha+2}) j \upharpoonright L(E_0^{\alpha})$ is proper, then $\alpha$ is a partially non-proper ordinal.
Proof.

If II had a winning strategy $\tau$, it would be definable. So we can define the set $C$ of the ordinals closed under $\tau$, i.e. of the ordinals $\eta$ such that if $\beta_n < \eta$, then for every $k_n$ $\tau(\langle k_n, \beta_n \rangle) < \eta$. By the first claim $C$ is a club.
Proof.

If II had a winning strategy $\tau$, it would be definable. So we can define the set $C$ of the ordinals closed under $\tau$, i.e. of the ordinals $\eta$ such that if $\beta_n < \eta$, then for every $k_n$ $\tau(\langle k_n, \beta_n \rangle) < \eta$. By the first claim $C$ is a club. Since $C$ is definable and $\alpha$ is regular, $C$ has ordertype $\alpha$. 
Proof.

If II had a winning strategy $\tau$, it would be definable. So we can define the set $C$ of the ordinals closed under $\tau$, i.e. of the ordinals $\eta$ such that if $\beta_n < \eta$, then for every $k_n$ $\tau(\langle k_n, \beta_n \rangle) < \eta$. By the first claim $C$ is a club. Since $C$ is definable and $\alpha$ is regular, $C$ has ordertype $\alpha$. But then if I plays the $\kappa_n$-th element of $C$ as $\beta_n$ and $j \upharpoonright E^0_{\beta_n - 1}$ as $k_n$, I wins, and that’s a contradiction.
Proof.

If II had a winning strategy $\tau$, it would be definable. So we can define the set $C$ of the ordinals closed under $\tau$, i.e. of the ordinals $\eta$ such that if $\beta_n < \eta$, then for every $k_n$ $\tau(\langle k_n, \beta_n \rangle) < \eta$. By the first claim $C$ is a club. Since $C$ is definable and $\alpha$ is regular, $C$ has ordertype $\alpha$. But then if I plays the $\kappa_n$-th element of $C$ as $\beta_n$ and $j \upharpoonright E_0^{\alpha+2}$ as $k_n$, I wins, and that’s a contradiction. Since $\text{cof}(\alpha) = \omega$ and for every $j : L(E_{\alpha+2}^0) \prec L(E_{\alpha+2}^0)$ $j \upharpoonright L(E_{\alpha}^0)$ is proper, then $\alpha$ is a partially non-proper ordinal.
What is the correlation between $\alpha$ and $\beta$?
What is the correlation between $\alpha$ and $\beta$?

**Lemma**
The ordertype of $I_\alpha$ is $\alpha$, so there exists an $\alpha_0 < \alpha$ such that $\text{ot}(I_{\alpha_0}) = \lambda$. 
What is the correlation between $\alpha$ and $\beta$?

The ordertype of $I_\alpha$ is $\alpha$, so there exists an $\alpha_0 < \alpha$ such that $\text{ot}(I_{\alpha_0}) = \lambda$.

Proof. This is because in $(E_0)^\#$ there are few partial Skolem functions.

Lemma

Let $L(V_{\lambda+1})$ be the elementary embeddings beyond $L(V_{\lambda+1})$.

Introduction

Large Cardinals Map

Vincenzo Dimonte

Totally Non-proper Ordinals

Partially Non-proper Ordinal

Higher Determinacy

Axiom
What is the correlation between $\alpha$ and $\beta$?

**Lemma**

The ordertype of $I_\alpha$ is $\alpha$, so there exists an $\alpha_0 < \alpha$ such that $\text{ot}(I_{\alpha_0}) = \lambda$.

**Proof.**

This is because in $(E_\alpha^0)^\#$, there are few partial Skolem functions. Let $\gamma < \alpha$. Then $H = H(E_\alpha^0)^\#((E_\alpha^0)^\# \cap E_\gamma^0)$ is small, so the least $\eta$ such that $H \subseteq E_\eta^0$ is less than $\alpha$. 
What is the correlation between $\alpha$ and $\beta$?

**Lemma**

The ordertype of $I_\alpha$ is $\alpha$, so there exists an $\alpha_0 < \alpha$ such that $\text{ot}(I_{\alpha_0}) = \lambda$.

**Proof.**

This is because in $(E^0_\alpha)^#$ there are few partial Skolem functions. Let $\gamma < \alpha$. Then $H = H((E^0_\alpha)^#((E^0_\alpha)^# \cap E^0_\gamma))$ is small, so the least $\eta$ such that $H \subseteq E^0_\eta$ is less than $\alpha$.

So we can build a club of $\gamma$'s such that $(E^0_\alpha)^# \cap \bigcup_{\eta < \gamma} E^0_\eta \prec (E^0_\alpha)^#$. 
What is the correlation between $\alpha$ and $\beta$?

**Lemma**

The ordertype of $I_\alpha$ is $\alpha$, so there exists an $\alpha_0 < \alpha$ such that $\text{ot}(I_{\alpha_0}) = \lambda$.

**Proof.**

This is because in $(E^0_\alpha)^#$ there are few partial Skolem functions.

Let $\gamma < \alpha$. Then $H = H((E^0_\alpha)^#)((E^0_\alpha)^# \cap E^0_\gamma)$ is small, so the least $\eta$ such that $H \subseteq E^0_\eta$ is less than $\alpha$.

So we can build a club of $\gamma$’s such that $(E^0_\alpha)^# \cap \bigcup_{\eta < \gamma} E^0_\eta \prec (E^0_\alpha)^#$. Since “being a sharp” is a local property, this means that $(E^0_\alpha)^#$ reflects in $\gamma$, i.e. $\gamma \in I_\alpha$. 
What is the correlation between $\alpha$ and $\beta$?

**Lemma**

The ordertype of $I_\alpha$ is $\alpha$, so there exists an $\alpha_0 < \alpha$ such that $\text{ot}(I_{\alpha_0}) = \lambda$.

**Proof.**

This is because in $(E_\alpha^0)^#$ there are few partial Skolem functions. Let $\gamma < \alpha$. Then $H = H^{(E_\alpha^0)^#}((E_\alpha^0)^# \cap E^0)$ is small, so the least $\eta$ such that $H \subseteq E_\eta^0$ is less than $\alpha$.

So we can build a club of $\gamma$’s such that $(E_\alpha^0)^# \cap \bigcup_{\eta < \gamma} E_\eta^0 \prec (E_\alpha^0)^#$. Since “being a sharp” is a local property, this means that $(E_\alpha^0)^#$ reflects in $\gamma$, i.e. $\gamma \in I_\alpha$.

This proves that $I_\alpha$ is a club in $\alpha$. Since $I_\alpha \in L((E_\alpha^0)^#)$ and $\alpha$ is regular in $L((E_\alpha^0)^#)$, we’re done.
Are there other differences?
Are there other differences?

**Lemma**

Let $\alpha$ and $\beta$ as above.
Are there other differences?

**Lemma**

Let $\alpha$ and $\beta$ as above.

- Let $j : L(E_\alpha^0) \prec L(E_\alpha^0)$ weakly proper.
Are there other differences?

Lemma

Let $\alpha$ and $\beta$ as above.

- Let $j : L(E_0^\alpha) \prec L(E_0^\alpha)$ weakly proper. Then there exist at least $2^\lambda$ different weakly proper non-proper (proper) elementary embeddings $k : L(E_0^\alpha) \prec L(E_0^\alpha)$ such that $k \upharpoonright V_\lambda = j \upharpoonright V_\lambda$. 
Are there other differences?

**Lemma**

Let $\alpha$ and $\beta$ as above.

- Let $j : L(E_\alpha^0) \prec L(E_\alpha^0)$ weakly proper. Then there exist at least $2^\lambda$ different weakly proper non-proper (proper) elementary embeddings $k : L(E_\alpha^0) \prec L(E_\alpha^0)$ such that $k \upharpoonright V_\lambda = j \upharpoonright V_\lambda$.

- For every $j, k : L(E_\beta^0) \prec L(E_\beta^0)$ weakly proper if $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$, then $j = k$. 
Proof.

- Remember the game $G_\alpha$. 
Proof.

- Remember the game $G_\alpha$. In $L((E_\alpha^0)^\#)$ I has a winning quasistrategy, so the set of all the winning successor moves is in $L((E_\alpha^0)^\#)$ and it’s cofinal in $\alpha$. 
Proof.

- Remember the game $G_\alpha$. In $L((E_\alpha^0)^\#)$ I has a winning quasistrategy, so the set of all the winning successor moves is in $L((E_\alpha^0)^\#)$ and it’s cofinal in $\alpha$. This means that the possible different winning plays for I are at least $|\alpha^2| > 2^\lambda$. 
Proof.

- Remember the game $G_\alpha$. In $L((E^0_\alpha)^\#)$ I has a winning quasistrategy, so the set of all the winning successor moves is in $L((E^0_\alpha)^\#)$ and it’s cofinal in $\alpha$. This means that the possible different winning plays for I are at least $|\alpha 2| > 2^\lambda$.

- It’s possible to prove that every element of $E^0_\beta$ is definable with parameters from $I_\beta$ and $V_{\lambda+1}$. 
Proof.

- Remember the game $G_\alpha$. In $L((E_\alpha^0)^\#)$ I has a winning quasistrategy, so the set of all the winning successor moves is in $L((E_\alpha^0)^\#)$ and it’s cofinal in $\alpha$. This means that the possible different winning plays for I are at least $|\alpha^2| > 2^\lambda$.

- It’s possible to prove that every element of $E_\beta^0$ is definable with parameters from $I_\beta$ and $V_{\lambda+1}$. So the behaviours of $j$ and $k$ depend only on their behaviours on $I_\beta$ and $V_\lambda$, that in turn depend on their behaviour on $\lambda$ and $V_\lambda$, that are equal.
The structure of the previous proofs is the following:
The structure of the previous proofs is the following:

- If $E_0^0$ exists, then $I \subsetneq \gamma$;
The structure of the previous proofs is the following:

- If $E_0^0$ exists, then $I \subseteq \Upsilon$;
- If $I \not\subseteq \Upsilon$, then there exists $\eta$ such that $L((E_\eta^0)^\#) \cap V_{\lambda+2} = E_\eta^0$, and we can define $\alpha$;
The structure of the previous proofs is the following:

- If $E^0_\infty$ exists, then $I \subsetneq \gamma$;
- if $I \subsetneq \gamma$, then there exists $\eta$ such that $L((E^0_\eta)^\#) \cap V_{\lambda+2} = E^0_\eta$, and we can define $\alpha$;
- $\alpha$ is a partially non-proper ordinal, and there exists a totally non-proper ordinal below it.
The structure of the previous proofs is the following:

- If $E_0^0$ exists, then $I \subsetneq \gamma$;
- if $I \subsetneq \gamma$, then there exists $\eta$ such that $L((E_0^0)'') \cap V_{\lambda+2} = E_{\eta}^0$, and we can define $\alpha$;
- $\alpha$ is a partially non-proper ordinal, and there exists a totally non-proper ordinal below it

Some of these implications cannot be reversed.
There are plenty of open problems, and most of them seems very difficult:
There are plenty of open problems, and most of them seem very difficult:

- Are there other partially or totally non-proper ordinals?
There are plenty of open problems, and most of them seems very difficult:

- Are there other partially or totally non-proper ordinals?
- Is it possible to have consistency-like results?
There are plenty of open problems, and most of them seems very difficult:

- Are there other partially or totally non-proper ordinals?
- Is it possible to have consistency-like results?
- Are there non-proper elementary embeddings between models like $L(X, V_{\lambda+1})$?
There are plenty of open problems, and most of them seem very difficult:

- Are there other partially or totally non-proper ordinals?
- Is it possible to have consistency-like results?
- Are there non-proper elementary embeddings between models like $L(X, V_{\lambda+1})$?
- Is the existence of $E_0^\infty$ inconsistent?