The hyper-weak distributive law and a related game in Boolean algebras

J. Cummings

Carnegie Mellon University, Department of Mathematical Sciences, Wean Hall, Room 6113, Pittsburgh, PA 15213-3890

N. Dobrinen *

The Pennsylvania State University, Department of Mathematics, 218 McAllister Building, University Park, PA 16802, USA

Abstract

The hyper-weak (κ, λ)-distributive law, formulated by Prikry, is a very weak generalization of the $(\kappa, \lambda, < \mu)$-distributive law. We define a related infinitary, two-player game, called $\mathcal{G}^{\kappa}_{\lambda-1}$, and give connections between the hyper-weak (κ, λ)-distributive law and the existence of winning strategies for the two players of $\mathcal{G}^{\kappa}_{\lambda-1}$, obtaining a game-theoretic characterization of the hyper-weak (κ, λ)-distributive law for many pairs of cardinals κ, λ, under GCH. We then construct κ^+-Suslin algebras for every infinite cardinal κ in which, for each infinite cardinal $\lambda \leq \kappa$, neither player has a winning strategy for $\mathcal{G}^{\kappa}_{\lambda-1}$. This shows that the gap between the strengths of the properties "II wins $\mathcal{G}^{\kappa}_{1}(\infty)$ in $\mathbb{B}$" and "the (κ, ∞)-distributive law holds in $\mathbb{B}$" is consistently even larger than was previously known.

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* Corresponding author.
Email addresses: `jcumming@andrew.cmu.edu` (J. Cummings), `dobrinen@math.psu.edu` (N. Dobrinen).
URLs: `http://www.math.cmu.edu/users/jcumming/` (J. Cummings), `http://www.math.psu.edu/dobrinen/` (N. Dobrinen).

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1 Introduction

The hyper-weak (ω,ω)-distributive law was formulated by Prikry as a generalization of the weak (ω,ω)-distributive law. His motivation was an open problem of von Neumann, whether it is consistent with ZFC that the countable chain condition and the weak (ω,ω)-distributive law completely characterize measurable Boolean algebras among Boolean σ-algebras [1]. Consistent counter-examples to von Neumann's proposed characterization of measurable algebras were obtained by Maharam [2], Jensen [3], Glówczyński [4], and Veličković [5]. Prikry's idea was to try to find in ZFC a complete, non-measurable Boolean algebra satisfying the countable chain condition (c.c.c.) in which some weaker form of the weak (ω,ω)-distributive law holds. This would give a type of lower bound on von Neumann's problem within ZFC. Specifically, Prikry asked the following question.

Open Problem 1 (Prikry) *Can one find in ZFC a complete c.c.c. Boolean algebra in which the hyper-weak (ω,ω)-distributive law holds, but the weak (ω,ω)-distributive law fails everywhere?*

We shall call such a Boolean algebra a *P-algebra*. Finding a P-algebra in ZFC turns out to be harder than it might seem at first glance. No Boolean algebra in which a c.c.c. "Suslin" (i.e. $\mathbf{\Sigma}^1_1$) forcing embeds as a dense subset can be a P-algebra, for Shelah has recently shown that for each c.c.c. Suslin forcing $\mathbb{P}$, the weak (ω,ω)-d.l. holds in r.o.$(\mathbb{P})$ iff $\mathbb{P}$ does not add a Cohen real [6]. Since the hyper-weak (ω,ω)-d.l. is weaker than the weak (ω,ω)-d.l. and implies that no Cohen reals are added, Shelah's result implies every c.c.c. Suslin forcing which is hyper-weakly (ω,ω)-distributive is also weakly (ω,ω)-distributive. In [7], Dobrinen investigated two families of Suslin and one family of non-Suslin, c.c.c. forcings which give rise to non-measurable Boolean algebras, and found that each of these algebras adds a Cohen real. Further, Błaszczyk and Shelah have shown that the Cohen algebra embeds into each σ-centered complete Boolean algebra if and only if there are no nowhere dense ultrafilters over ω [8]. Since Shelah has shown that the existence of nowhere dense ultrafilters is independent of ZFC [9], such Boolean algebras cannot be shown to be P-algebras in ZFC. At least the existence of a P-algebra is consistent with ZFC, for the regular open algebra of Mathias forcing with all tails in some Ramsey ultrafilter is a c.c.c., hyper-weakly (ω,ω)-distributive algebra in which the weak (ω,ω)-d.l. fails everywhere [10].

Although the problem of finding a P-algebra in ZFC remains open, the hyper-weak (ω,ω)-distributive law and its generalizations for larger cardinals have proved useful in the realm of games. Let us give a bit of background into the connections between distributive laws and games in Boolean algebras.

Jech investigated the (ω, λ)-d.l., weak(ω, λ)-d.l., and $(\omega, \lambda, \omega)$-d.l., and related infinitary two-player games in [11]. Among other things, he gave a game-theoretic characterization of the (ω, λ)-d.l. Dobrinen generalized this to the $(\kappa, \lambda, < \mu)$-d.l. for many triples of cardinals in [12]. Kamburelis solved an open problem from [11] giving a best possible result connecting the weak (ω, λ)-d.l. and its related game [13]. In §3 of this paper, we give connections between the hyper-weak (κ, λ)-d.l. and its related game, obtaining a game-theoretic characterization of the hyper-weak (κ, λ)-d.l. for many pairs of cardinals κ, λ, under GCH.

For each distributive law and its related game, the property "II has a winning strategy" implies that that distributive law holds. In [11], Jech used $\diamondsuit$ to construct a Suslin algebra in which the game $\mathcal{G}^{\omega}_{1}(2)$ (related to the $(\omega, 2)$-d.l.) is undetermined. Dobrinen generalized that result in [12,14] by using $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$ and $\kappa^{<\kappa} = \kappa$ to construct a κ^+-Suslin algebra containing a $< \kappa$-closed dense subset, and in which the game $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ is undetermined for all λ and all $2 \leq \mu \leq \min(\lambda, \kappa)$. This showed that the property "II has a winning strategy for $\mathcal{G}^{\kappa}_{1}(\infty)$ in $\mathbb{B}$" is consistently strictly stronger than the property "the (κ, ∞)-d.l. holds in $\mathbb{B}$", since there is a κ^+-Suslin algebra in which the (κ, ∞)-d.l. holds, yet II does not even have a winning strategy for the game $\mathcal{G}^{\kappa}_{<\kappa}(\kappa)$, which is considerably easier for II to win than $\mathcal{G}^{\kappa}_{1}(\infty)$.

In §5, we improve on this result in several ways. For every infinite cardinal κ and each infinite cardinal $\nu \leq \mathrm{cf}(\kappa)$, we use $\Box_\kappa$ and $\diamondsuit_{\kappa^+}(S)$ for all stationary subsets $S \subseteq \mathrm{cof}(\kappa)$ to construct a κ^+-Suslin algebra which contains a $< \nu$-closed dense subset (thus, II wins $\mathcal{G}^{\rho}_{1}(\infty)$ for each $\rho < \nu$) and in which II does not have a winning strategy for $\mathcal{G}^{\nu}_{\kappa-1}$. In particular, CON(ZFC + for each infinite cardinal κ there is a Boolen algebra in which the (κ, ∞)-d.l. holds, but in which II does not even have a winning strategy for $\mathcal{G}^{\omega}_{\kappa-1}$). $\mathcal{G}^{\omega}_{\kappa-1}$ is much easier for II to win than $\mathcal{G}^{\kappa}_{<\kappa}(\kappa)$. Hence we have shown the gap between the (κ, ∞)-d.l. and II winning $\mathcal{G}^{\kappa}_{1}(\infty)$ to be consistently even wider than was previously known. In the special case when $\kappa^{<\kappa} = \kappa$, we can drop the $\Box_\kappa$ assumption and use $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$ to construct a κ^+-Suslin algebra which contains a $< \kappa$-closed dense subset and in which II does not have a winning strategy for $\mathcal{G}^{\kappa}_{\kappa-1}$. Useful in our construction is the concept of $\geq \nu$-club and $\geq \nu$-stationary sets, which form a hierarchy of strengths between clubness and stationarity. Those ideas are discussed in §4.

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on games and distributive laws and suggesting the idea of generalizing Jech's Suslin algebra example to more general distributive laws and related games.

2 Generalized distributive laws: definitions and basic facts

We start by reviewing the three-parameter distributive law, which subsumes all the conventional generalized distributive laws. Throughout this paper, we let $\mathbb{B}$ denote a Boolean algebra and $\mathbb{B}^+$ denote $\mathbb{B} \setminus \{\mathbf{0}\}$.

Definition 2 [15] For cardinals κ, λ, μ with $2 \leq \mu \leq \lambda$, a Boolean algebra $\mathbb{B}$ satisfies the $(\kappa, \lambda, < \mu)$-*distributive law* ($(\kappa, \lambda, < \mu)$-d.l.) if for each family $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda}$ of elements of $\mathbb{B}$,

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} = \bigvee_{f:\kappa\to[\lambda]^{<\mu}} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}, \tag{1}$$

provided that $\bigvee_{\beta<\lambda} b_{\alpha\beta}$ for each $\alpha < \kappa$, $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta}$, and $\bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}$ for each $f : \kappa \to [\lambda]^{<\mu}$ exist in $\mathbb{B}$. $\mathbb{B}$ satisfies the $(\kappa, \infty, < \mu)$-*d.l.* if the $(\kappa, \lambda, < \mu)$-d.l. holds in $\mathbb{B}$ for all λ. We say that the $(\kappa, \lambda, < \mu)$-d.l. *fails everywhere* in $\mathbb{B}$ if there exists a family $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda} \subseteq \mathbb{B}$ such that $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} = \mathbf{1}$ and $\bigvee_{f:\kappa\to[\lambda]^{<\mu}} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta} = \mathbf{0}$.

Remark 3 The $(\kappa, \lambda, < 2)$-d.l. is usually referred to as the (κ, λ)-d.l., and the $(\kappa, \lambda, < \omega)$-d.l. is usually referred to as the weak (κ, λ)-d.l. Saying that "the $(\kappa, \lambda, < \mu)$-d.l. fails everywhere in $\mathbb{B}$" is equivalent to saying that "$\mathbb{B}$ is $(\kappa, \lambda, < \mu)$-nowhere distributive" in Koppelberg's terminology [15].

Prikry formulated the following weakening of the $(\kappa, \lambda, < \mu)$-d.l.

Definition 4 (Prikry) [16] For κ, λ cardinals with $\lambda \geq \omega$, a Boolean algebra $\mathbb{B}$ satisfies the *hyper-weak* (κ, λ)-*distributive law* (hyper-weak (κ, λ)-d.l.) if for each family $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda}$ of elements of $\mathbb{B}$,

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} = \bigvee_{f:\kappa\to\lambda} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} b_{\alpha\beta}, \tag{2}$$

provided that $\bigvee_{\beta<\lambda} b_{\alpha\beta}$ for each $\alpha < \kappa$, $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta}$, and $\bigwedge_{\alpha<\kappa} \bigvee\{b_{\alpha\beta} : \beta \in \lambda \setminus \{f(\alpha)\}\}$ for each $f : \kappa \to \lambda$ exist in $\mathbb{B}$. $\mathbb{B}$ satisfies the *hyper-weak* (κ, ∞)-*d.l.* if the hyper-weak (κ, λ)-d.l. holds in $\mathbb{B}$ for all $\lambda \geq \omega$. We say that the hyper-weak (κ, λ)-d.l. *fails everywhere* in $\mathbb{B}$ if there exists $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda} \subseteq \mathbb{B}$ such that $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} = \mathbf{1}$ and $\bigvee_{f:\kappa\to\lambda} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} b_{\alpha\beta} = \mathbf{0}$.

In this paper we shall usually work with Boolean algebras which are sufficiently complete so that all relevant infima and suprema exist.

The next fact follows naturally from the definitions.

Fact 5 *For each Boolean algebra $\mathbb{B}$, the following hold.*

1. *For all cardinals $\kappa_0 \leq \kappa_1$, $2 \leq \mu_0 \leq \mu_1$, and $\lambda_0 \leq \lambda_1$, if $\mathbb{B}$ satisfies the $(\kappa_1, \lambda_1, < \mu_0)$-d.l., then $\mathbb{B}$ satisfies the $(\kappa_0, \lambda_0, < \mu_1)$-d.l.*
2. *For all cardinals $\kappa_0 \leq \kappa_1$ and $\omega \leq \lambda_0 \leq \lambda_1$, if $\mathbb{B}$ satisfies the hyper-weak (κ_1, λ_0)-d.l., then $\mathbb{B}$ satisfies the hyper-weak (κ_0, λ_1)-d.l. Hence, the hyper-weak (κ, ∞)-d.l. is equivalent to the hyper-weak (κ, ω)-d.l.*
3. *For all cardinals $\kappa, \lambda, \mu, \nu$ with $2 \leq \mu \leq \lambda$ and $\nu \geq \max(\omega, \mu)$, if $\mathbb{B}$ satisfies the $(\kappa, \lambda, < \mu)$-d.l., then $\mathbb{B}$ satisfies the hyper-weak (κ, ν)-d.l.*

Examples 6 (1) For cardinals κ, λ with $\lambda \geq \omega$, the hyper-weak (κ, λ)-d.l. holds in each Boolean algebra satisfying the λ-chain condition.

(2) For cardinals κ, λ, μ with κ regular and $\omega \leq \mu \leq \kappa$, the regular open algebra of $\mathrm{Fn}(\kappa, \lambda, \mu)$ satisfies the (ρ, ∞)-d.l. for each $\rho < \kappa$, but the hyper-weak (κ, ν)-d.l. fails everywhere, where $\nu = \max(\mu, \lambda)$. In particular, in the Collapsing algebra $\mathrm{Col}(\kappa, \lambda)$, for each $\rho < \kappa$, the (ρ, ∞)-d.l. holds, but the hyper-weak (κ, ν)-d.l. fails everywhere, where $\nu = \max(\kappa, \lambda)$.

(3) In each free Boolean algebra on infinitely many generators, the hyper-weak (ω, ω)-d.l. fails everywhere, but the hyper-weak (ω, ω_1)-d.l. holds.

(4) In Laver, Mathias, and Miller forcings, the hyper-weak (ω, ω)-d.l. holds, but the weak (ω, ω)-d.l. fails everywhere.

Definition 7 [15] For any cardinal κ and $b \in \mathbb{B}^+$, a collection $\{b_\alpha : \alpha < \kappa\} \subseteq \mathbb{B}$ is a *quasi-partition* of b if each $b_\alpha \leq b$, $\{b_\alpha : \alpha < \kappa\}$ is pairwise disjoint, and $\bigvee_{\alpha<\kappa} b_\alpha = b$. $\{b_\alpha : \alpha < \kappa\} \subseteq \mathbb{B}$ is a *partition* of b if it is a quasi-partition of b where each $b_\alpha > \mathbf{0}$.

In a λ-complete Boolean algebra $\mathbb{B}$, whether the hyper-weak (κ, λ)-d.l. holds in $\mathbb{B}$ can be determined by looking only at quasi-partitions or partitions of unity.

Fact 8 *For all cardinals κ, λ with $\lambda \geq \omega$, for each λ-complete Boolean algebra $\mathbb{B}$, the following are equivalent.*

1. *The hyper-weak (κ, λ)-d.l. holds in $\mathbb{B}$.*
2. *$\forall b \in \mathbb{B}^+$, equation(2) of Definition 4 holds for all families $\{b_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, of (quasi-)partitions of b in $\mathbb{B}$.*
3. *Equation (2) of Definition 4 holds for all families $\{b_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, of (quasi-)partitions of unity in $\mathbb{B}$.*

The following is a characterization of the hyper-weak (κ, λ)-d.l. for a partial ordering $(\mathbb{P}, \leq)$ via its Boolean completion r.o.$(\mathbb{P})$. It holds whether or not $\mathbb{P}$ is separative.

Fact 9 *Given a partial ordering* $(\mathbb{P}, \leq)$*, the following are equivalent.*

(1) The hyper-weak (κ, λ)*-d.l. holds in r.o.*$(\mathbb{P})$*.*
(2) If $\mathcal{W}_\alpha = \{P_{\alpha\beta} : \beta < \lambda\}$*,* $\alpha < \kappa$*, is a family such that for each* $\alpha < \kappa$*,*
  (a) $\beta \neq \beta' \longrightarrow P_{\alpha\beta} \cap P_{\alpha\beta'} = \emptyset$*,*
  (b) $\bigcup_{\beta<\lambda} P_{\alpha\beta}$ *is a maximal antichain in* $\mathbb{P}$*,*
  then there exists a maximal antichain $\mathcal{Q} \subseteq \mathbb{P}$ *such that* $\forall q \in \mathcal{Q}$*,* $\forall \alpha < \kappa$*,* $\exists \beta < \lambda$ *such that* $\forall p \in P_{\alpha\beta}$*,* $p \perp q$*.*

Remark 10 Prikry observed that taking suprema over all but one element of λ on the right hand side of (2) in Definition 4 of the hyper-weak (κ, λ)-d.l. is equivalent to taking suprema over subsets of λ whose complements have cardinality λ; i.e. replacing the right hand side of (2) with $\bigvee_{f:\kappa\to\mathcal{S}} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha\beta}$, where $\mathcal{S} = \{X \subseteq \lambda : |\lambda \setminus X| = \lambda\}$. (See [17].)

We let α, β denote ordinals in V and κ, λ denote cardinals in V. For $x \in V$ $\check{x}$ denotes the canonical $\mathbb{B}$-name for x. We use $\dot{x}$ to denote general $\mathbb{B}$-names. The following is a well-known forcing equivalent of general distributive laws.

Theorem 11 (Folklore) *For each complete Boolean algebra* $\mathbb{B}$*, the* $(\kappa, \lambda, <\mu)$*-d.l. holds in* $\mathbb{B}$ *iff for each* $\mathbb{B}$*-name* $\dot{g}$ *for a function from* $\check{\kappa}$ *to* $\check{\lambda}$*,* $V^{\mathbb{B}} \models (\exists f : \kappa \to [\lambda]^{<\mu}$ *in* V *such that* $\forall \check{\alpha} < \check{\kappa},\ \dot{g}(\check{\alpha}) \in \check{f}(\check{\alpha}))$*.*

In particular, a complete Boolean algebra $\mathbb{B}$ satisfies the (κ, λ)-d.l. iff forcing with $\mathbb{B}^+$ adds no new functions from $\check{\kappa}$ to $\check{\lambda}$. The following is the analog for the hyper-weak (κ, λ)-d.l.

Theorem 12 *For each complete Boolean algebra* $\mathbb{B}$*, the hyper-weak* (κ, λ)*-d.l. holds in* $\mathbb{B}$ *iff for each* $\mathbb{B}$*-name* $\dot{g}$ *for a function from* $\check{\kappa}$ *to* $\check{\lambda}$*,* $V^{\mathbb{B}} \models (\exists f : \kappa \to \lambda$ *in* V *such that* $\forall \check{\alpha} < \check{\kappa},\ \dot{g}(\check{\alpha}) \neq \check{f}(\check{\alpha}))$*.*

PROOF. Suppose $\mathbb{B}$ satisfies the hyper-weak (κ, λ)-d.l. (in V). Let $\dot{g}$ be a $\mathbb{B}$-name for a function from $\check{\kappa}$ to $\check{\lambda}$. For every $\alpha < \kappa$ and every $\beta < \lambda$, define $b_{\alpha\beta} = \|\dot{g}(\check{\alpha}) = \check{\beta}\|$. Then for every $\alpha < \kappa$, $\{b_{\alpha\beta} : \beta < \lambda\}$ is a quasi-partition of unity. For each $f : \kappa \to \lambda$ in V, define $b_f = \bigwedge_{\alpha<\kappa} \bigvee\{b_{\alpha\beta} : \beta \in \lambda \setminus \{f(\alpha)\}\}$. Then $b_f = \|\forall \check{\alpha} < \check{\kappa},\ \dot{g}(\check{\alpha}) \neq \check{f}(\check{\alpha})\|$. Since the hyper-weak (κ, λ)-d.l. holds in $\mathbb{B}$, $\mathbf{1} = \bigvee_{f:\kappa\to\lambda} b_f = \|\exists \check{f} : \check{\kappa} \to \check{\lambda}$ in V such that $\forall \check{\alpha} < \check{\kappa},\ \dot{g}(\check{\alpha}) \neq \check{f}(\check{\alpha})\|$.

To prove the converse, suppose that the hyper-weak (κ, λ)-d.l. fails in $\mathbb{B}$. Then by Fact 8, there exists a $b \in \mathbb{B}^+$ and partitions of b, $\{b_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, such that,

$$\bigvee_{f:\kappa\to\lambda}\bigwedge_{\alpha<\kappa}\bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} b_{\alpha\beta} = \mathbf{0}. \tag{3}$$

Let $\dot g$ be a $\mathbb{B}$-valued name for a function from $\check\kappa$ to $\check\lambda$ such that for all $\alpha,\beta<\kappa$, $b\wedge\|\dot g(\check\alpha)=\check\beta\| = b_{\alpha\beta}$. Let $f:\kappa\to\lambda$ be a function in V. Then $\bigwedge_{\alpha<\kappa}\bigvee_{\beta\in\lambda\setminus\{f(\alpha)\}} b_{\alpha\beta}=\mathbf{0}$, by (3); hence, $\mathbf{0}<b=\bigvee_{\alpha<\kappa} b_{\alpha,f(\alpha)} = \|\exists\check\alpha<\check\kappa$ such that $\dot g(\check\alpha)=\check f(\check\alpha)\|$.

□

3 A game related to the hyper-weak distributive law

We begin by reviewing the following game investigated in [12], which generalizes a game of Jech in [11]. This game is related to the $(\kappa,\lambda,<\mu)$-d.l. (See Theorem 16 (1) below.)

Definition 13 [12] Given cardinals κ,λ,μ with $2\le\mu\le\lambda$, the game $\mathcal{G}^\kappa_{<\mu}(\lambda)$ is played between two players in a $\max(\kappa^+,\mu)$-complete Boolean algebra $\mathbb{B}$ as follows: At the beginning of the game, player I fixes some $a\in\mathbb{B}^+$. For $\alpha<\kappa$, the α-th round is played as follows: player I chooses a partition W_α of a such that $|W_\alpha|\le\lambda$; then player II chooses some $E_\alpha\in[W_\alpha]^{<\mu}$. In this manner, the two players construct a sequence of length κ

$$\langle a, W_0, E_0, W_1, E_1, \dots, W_\alpha, E_\alpha, \dots\ :\alpha<\kappa\rangle \tag{4}$$

called a *play* of the game. I *wins the play* (4) if and only if

$$\bigwedge_{\alpha<\kappa}\bigvee E_\alpha\ =\ \mathbf{0}. \tag{5}$$

$\mathcal{G}^\kappa_{<2}(\lambda)$ is usually denoted as $\mathcal{G}^\kappa_1(\lambda)$. $\mathcal{G}^\kappa_{<\mu}(\infty)$ is the game played just as $\mathcal{G}^\kappa_{<\mu}(\lambda)$, except now player I can choose partitions of any size.

Definition 14 [18] Let ν be an infinite cardinal. A partial order $\mathbb{P}$ is $<\nu$-*closed* if for every ordinal $\rho<\nu$, for each decreasing sequence $(p_\alpha)_{\alpha<\rho}$ in $\mathbb{P}$ there is some $q\in\mathbb{P}$ satisfying $q\le p_\alpha$ for all $\alpha<\rho$. A Boolean algebra $\mathbb{B}$ *contains a* $<\nu$*-closed dense subset* if there is some $D\subseteq\mathbb{B}^+$ such that D is $<\nu$-closed, and for each $b\in\mathbb{B}^+$ there is some $d\in D$ such that $d\le b$.

Remark 15 If $\mathbb{B}$ has a $<\nu$-closed dense subset, then for each $\rho < \nu$, player II has a winning strategy for $\mathcal{G}_1^\rho(\infty)$; moreover, II even has a winning strategy for the harder game $G^{\mathrm{I}}_{\rho^+}$ invented by Foreman (see [19]).

In [12,14] we found the following.

Theorem 16 (Dobrinen) [12,14]

1. *For* $\mathbb{B}$ *a* $\max(\kappa^+, \mu)$*-complete Boolean algebra, the* $(\kappa, \lambda, <\mu)$*-d.l. fails in* $\mathbb{B} \Longrightarrow$ *I has a winning strategy for* $\mathcal{G}^\kappa_{<\mu}(\lambda)$ *in* $\mathbb{B} \Longrightarrow$ *the* $((\lambda^{<\mu})^{<\kappa}, \lambda, <\mu)$*-d.l. fails in* $\mathbb{B}$.
2. $\kappa^{<\kappa} = \kappa$ *and* $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa)) \Longrightarrow$ *there is a* κ^+*-Suslin algebra which has a* $<\kappa$*-closed dense subset and in which neither player has a winning strategy for* $\mathcal{G}^\kappa_{<\mu}(\lambda)$ *for all* λ, μ *with* $2 \le \mu \le \min(\lambda, \kappa)$.

The following game corresponds naturally to the hyper-weak (κ, λ)-distributive law. (See Fact 18 and Theorems 28 and 29 below.)

Definition 17 (Dobrinen-Prikry) [17] Given cardinals κ, λ with $\lambda \ge \omega$, the game $\mathcal{G}^\kappa_{\lambda-1}$ is played between two players in a κ^+-complete Boolean algebra $\mathbb{B}$ as follows: At the beginning of the game, player I fixes some $a \in \mathbb{B}^+$. For $\alpha < \kappa$, the α-th round is played as follows: player I chooses a quasi-partition W_α of a such that $|W_\alpha| = \lambda$; then player II chooses one $b_\alpha \in W_\alpha$ to "leave out" and "plays" $\bigvee(W_\alpha \setminus \{b_\alpha\})$, equivalently $a \setminus b_\alpha$. In this manner, the two players construct a sequence of length κ

$$\langle a, W_0, b_0, W_1, b_1, \ldots, W_\alpha, b_\alpha, \ldots\ : \alpha < \kappa\rangle \tag{6}$$

called a *play* of the game. I *wins the play* (6) if and only if

$$\bigwedge_{\alpha<\kappa} a \setminus b_\alpha \ = \ \mathbf{0}, \tag{7}$$

which happens if and only if $\bigvee_{\alpha<\kappa} b_\alpha = a$. We named this game $\mathcal{G}^\kappa_{\lambda-1}$, because II plays "all but one" piece, or "λ minus 1-many" pieces, from each quasi-partition.

Fact 18 *For each* κ^+*-complete Boolean algebra* $\mathbb{B}$*, if II has a winning strategy for* $\mathcal{G}^\kappa_{\lambda-1}$ *in* $\mathbb{B}$*, then* $\mathbb{B}$ *satisfies the hyper-weak* (κ, λ)*-d.l.*

Remark 19 Note that in the game $\mathcal{G}^\kappa_{\lambda-1}$, we require player I to choose quasi-partitions of size exactly λ. If we did not, then I would always choose partitions of size ω, since this maximizes I's chances of winning.

The following fact relates the various games.

Fact 20 *For all* $\kappa_0 \leq \kappa_1$, $2 \leq \mu_0 \leq \mu_1 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2$, *and* $\omega \leq \lambda_1$, *the following diagram shows the implications for the existence of a winning strategy for the two players.*

$$\begin{tikzcd}
\mathcal{G}^{\kappa_1}_{<\mu_0}(\lambda_1) \arrow[r, shift left, "\mathrm{II}"] \arrow[d, shift left, "\mathrm{II}"] & \mathcal{G}^{\kappa_1}_{\lambda_1-1} \arrow[l, shift left, "\mathrm{I}"] \arrow[d, shift left, "\mathrm{II}"] \\
\mathcal{G}^{\kappa_0}_{<\mu_1}(\lambda_0) \arrow[r, shift left, "\mathrm{II}"] \arrow[u, shift left, "\mathrm{I}"] & \mathcal{G}^{\kappa_0}_{\lambda_2-1} \arrow[l, shift left, "\mathrm{I}"] \arrow[u, shift left, "\mathrm{I}"]
\end{tikzcd} \tag{8}$$

For example, $\mathcal{G}^{\kappa_1}_{<\mu_0}(\lambda_1) \xrightarrow{\mathrm{II}} \mathcal{G}^{\kappa_1}_{\lambda_1-1}$ *means that for each Boolean algebra* $\mathbb{B}$ *in which both games are defined, if II has a winning strategy for* $\mathcal{G}^{\kappa_1}_{<\mu_0}(\lambda_1)$ *in* $\mathbb{B}$, *then II also has a winning strategy for* $\mathcal{G}^{\kappa_1}_{\lambda_1-1}$ *in* $\mathbb{B}$.

Examples 21 (1) If $\mathbb{B}$ satisfies the λ-chain condition, then for every infinite κ, II wins $\mathcal{G}^{\kappa}_{\lambda-1}$ in $\mathbb{B}$.

(2) For $\mu \leq \kappa$ and $\theta = \max(\lambda, \mu)$, I has a winning strategy for $\mathcal{G}^{\kappa}_{\theta-1}$ in r.o.$(\mathrm{Fn}(\kappa, \lambda, \mu))$. If μ is regular, then $\mathrm{Fn}(\kappa, \lambda, \mu)$ is $< \mu$-closed; so for every $\rho < \mu$, II has a winning strategy for $\mathcal{G}^{\rho}_{1}(\infty)$ in r.o.$(\mathrm{Fn}(\kappa, \lambda, \mu))$. As a special case, for κ regular and λ any cardinal, for every $\rho < \kappa$, II wins $\mathcal{G}^{\rho}_{1}(\infty)$ in the collapsing algebra $\mathrm{Col}(\kappa, \lambda)$, but I wins $\mathcal{G}^{\kappa}_{\theta-1}$, where $\theta = \max(\kappa, \lambda)$.

(3) In the Cohen algebra, for every κ, I wins $\mathcal{G}^{\kappa}_{\omega-1}$ but II wins $\mathcal{G}^{\kappa}_{\omega_1-1}$.

(4) In Laver, Mathias, and Miller forcings, II wins $\mathcal{G}^{\omega}_{\omega-1}$, but I wins $\mathcal{G}^{\omega}_{\mathrm{fin}}(\omega)$.

Laver, Mathias, and Miller forcings are specific cases of a more general class of forcings $\mathbb{P}$ in which II has a winning strategy for $\mathcal{G}^{\omega}_{\omega-1}$ in r.o.$(\mathbb{P})$. We review the following definitions.

Definition 22 [20] A partial ordering $(\mathbb{P}, \leq)$ satisfies *Axiom A* if there exists a sequence of partial orderings $\leq_n$, $n < \omega$, on $\mathbb{P}$ satisfying the following:

(1) $\leq_0$ is $\leq$, and for all $n < \omega$, $q \leq_{n+1} p \longrightarrow q \leq_n p$;

(2) For each sequence $(p_n)_{n<\omega}$ in $\mathbb{P}$ satisfying $p_{n+1} \leq_n p_n$ for all $n < \omega$, there is some $q \in \mathbb{P}$ such that $q \leq_n p_n$ for all $n < \omega$;

(3) For each $p \in \mathbb{P}$, for each pairwise incompatible set $A \subseteq \mathbb{P}$, for each $n < \omega$ there is some $q \leq_n p$ such that q is compatible with at most countably many elements of A.

The following Property P_f generalizes the Axiom A version of the Laver property, called L_f in [20]. Bartoszyński and Judah proved that the property P_f implies that no Cohen reals are added, and moreover, that the countable support iteration of partial orderings satisfying P_f does not add Cohen reals. (See [20].)

Definition 23 [20] Let $(\mathbb{P}, \leq)$ be a partial ordering satisfying Axiom A and let $f : \omega \to \omega$. $\mathbb{P}$ *satisfies Property* P_f if $\forall p \in \mathbb{P}$, $\forall n, k < \omega$, $\forall A \in [\omega]^{<\omega}$, if $p \Vdash (\dot{B} \subseteq A$ and $|\dot{B}| \leq k)$, then $\exists C \subseteq A$ such that $|C| \leq k \cdot f(n)$ and $\forall c \notin C$, $\exists q \leq_n p$ such that $q \Vdash c \notin \dot{B}$.

The following fact can be proved by an argument analogous to an argument given by Prikry [16] giving a Boolean algebraic equivalent of the property L_f.

Fact 24 *Let $\mathbb{P}$ be a separative partial ordering satisfying Axiom A, and let $f : \omega \to \omega$. Then Property P_f holds in $\mathbb{P} \iff \forall p \in \mathbb{P}$, $\forall n, k < \omega$, $\forall A \in [\omega]^{<\omega}$, if $\dot{B}$ is an r.o.$(\mathbb{P})$-name of the form $\langle \{\dot{a}_1, \ldots, \dot{a}_k\}, e(p)\rangle$, where the $\dot{a}_i$ are r.o.$(\mathbb{P})$-names for integers, and $e(p) \leq \|\dot{B} \subseteq A\|$, then $\exists C \subseteq A$ such that $|C| \leq k \cdot f(n)$ and $\forall c \notin C$, $\exists q \leq_n p$ such that $e(q) \leq \bigwedge_{1 \leq i \leq k} \|c \neq \dot{a}_i\|$.*

Note: if P_f holds, then f must actually have its range in $\omega \setminus \{0\}$.

The following proof is similar to one given by Prikry [16] that L_f implies the hyper-weak (ω, ω)-d.l.

Proposition 25 *For each separative partial ordering $(\mathbb{P}, \leq)$ which satisfies P_f for some $f : \omega \to \omega \setminus \{0\}$, player II has a winning strategy for $\mathcal{G}^{\omega}_{\omega-1}$ in r.o.$(\mathbb{P})$.*

PROOF. Suppose P_f holds in $\mathbb{P}$ for some function $f : \omega \to \omega \setminus \{0\}$. Suppose I fixes $a \in \text{r.o.}(\mathbb{P})^+$. Let $p_0 \in \mathbb{P}$ be such that $e(p_0) \leq a$. We show how II can choose at each round of the game to ensure a win.

Suppose we have I and II's choices up to stage n: For each round $i < n$, I has played $\{b_{ij} : j < \omega\}$, a quasi-partition of a, and we have chosen $p_{i+1} \in \mathbb{P}$ and $g(i) \in \omega$ such that $e(p_{i+1}) \leq \bigvee_{j \in \omega \setminus \{g(i)\}} b_{ij}$ and $p_0 \geq_0 p_1 \geq_1 \ldots \geq_{n-1} p_n$.

Round n: Suppose I plays $\{b_{nj} : j < \omega\}$, a quasi-partition of a. Let $A_n = \{0, \ldots, f(n)\}$. For $j < f(n)$, let $c_{nj} = e(p_n) \wedge b_{nj}$, and let $c_{n,f(n)} = e(p_n) \wedge (\bigvee_{f(n) \leq j < \omega} b_{nj})$. Let $\dot{a}_n = \{\langle \check{j}, c_{nj}\rangle : j \leq f(n)\}$ and $\dot{B}_n = \langle \{\dot{a}_n\}, e(p_n)\rangle$. $(k = 1$ here.) Then $\bigvee_{j \leq f(n)} c_{nj} = e(p_n) \leq \|\dot{B}_n \subseteq A_n\|$, so by P_f and Fact 24, $\exists C_n \subseteq A_n$ such that $|C_n| \leq 1 \cdot f(n)$ and $\forall c \notin C_n$, $\exists q \leq_n p_n$ such that $e(q) \leq \|c \neq \dot{a}_n\|$. Let $g(n)$ be the least element of $A_n \setminus C_n$ and choose a $p_{n+1} \leq_n p_n$ such that $e(p_{n+1}) \leq \|g(n) \neq \dot{a}_n\|$. Then

$$e(p_{n+1}) \leq \|g(n) \neq \dot{a}_n\| \leq \bigvee_{j \in \omega \setminus \{g(n)\}} b_{nj} = a \setminus b_{n,g(n)}. \tag{9}$$

Let II choose to leave out $b_{n,g(n)}$.

In this manner, we obtain a sequence $(p_n)_{n<\omega}$ and a function $g : \omega \to \omega$ such that $\forall n < \omega$, $p_{n+1} \leq_n p_n$ and $e(p_{n+1}) \leq a \setminus b_{n,g(n)}$. By (2) of Axiom A, $\exists q \in \mathbb{P}$ such that $\forall n < \omega$, $q \leq_n p_n$. By (1) of Axiom A, $\forall n < \omega$, $q \leq p_n$. Therefore, $\mathbf{0} < e(q) \leq \bigwedge_{n<\omega} a \setminus b_{n,g(n)}$, by (9). Hence, II has a winning strategy for $\mathcal{G}^{\omega}_{\omega-1}$ in r.o.($\mathbb{P}$).

□

Remark 26 In the above proof, property (3) of Axiom A was never used.

Examples 27 The following partial orderings satisfy Axiom A and P_f for some function $f : \omega \to \omega \setminus \{0\}$: Laver, Mathias, Miller, and Random real forcings. (See [20].) Hence, by Proposition 25, II has a winning strategy for $\mathcal{G}^{\omega}_{\omega-1}$ in the Boolean completions of these forcings.

Next, we relate the hyper-weak (κ, λ)-d.l. to the existence of a winning strategy for each of the two players.

Theorem 28 *For each κ^+-complete Boolean algebra, if the hyper-weak (κ, λ)-d.l. fails, then I has a winning strategy for $\mathcal{G}^{\kappa}_{\lambda-1}$.*

PROOF. Suppose the hyper-weak (κ, λ)-d.l. fails in $\mathbb{B}$. Then there is a family $(b_{\alpha\beta})_{\alpha<\kappa,\beta<\lambda} \subseteq \mathbb{B}$ such that $\bigvee_{\beta<\lambda} b_{\alpha\beta}$ for all $\alpha < \kappa$, $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta}$, and $\bigwedge_{\alpha<\kappa} \bigvee \{b_{\alpha\beta} : \beta \in \lambda \backslash \{f(\alpha)\}\}$ for all $f : \kappa \to \lambda$ exist in $\mathbb{B}$; and $\exists b \in \mathbb{B}^+$ such that $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta} > b$ and b is an upper bound for the family $\bigwedge_{\alpha<\kappa} \bigvee \{b_{\alpha\beta} : \beta \in \lambda \backslash \{f(\alpha)\}\}$, $f : \kappa \to \lambda$. Let $a = (\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha\beta}) \setminus b$ and let $a_{\alpha\beta} = a \wedge b_{\alpha\beta}$. Then $\{a_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \kappa$, are quasi-partitions of a, and $\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} a_{\alpha\beta} = a > \mathbf{0} = \bigvee_{f:\kappa\to\lambda} \bigwedge_{\alpha<\kappa} \bigvee \{a_{\alpha\beta} : \beta \in \lambda \setminus \{f(\alpha)\}\}$. The following is a winning strategy for I: I plays a at the beginning of the game, and on the α-th round, I plays $\{a_{\alpha\beta} : \beta < \lambda\}$.

□

It is not known whether the full converse of Theorem 28 holds in ZFC. Indeed, we conjecture that it does not. However, we do have the following partial converse.

Theorem 29 *For $\mathbb{B}$ a κ^+-complete Boolean algebra, if I has a winning strategy in $\mathcal{G}^{\kappa}_{\lambda-1}$, then the hyper-weak $(\lambda^{<\kappa}, \lambda)$-d.l. fails.*

PROOF. Suppose σ is a winning strategy for I. Let a be the non-zero element which I fixes according to σ, and let $W_{\langle\ \rangle} = \sigma(\langle\ \rangle)$, the quasi-partition of a of size λ which I plays on round 0 according to σ. Index the elements of $W_{\langle\ \rangle}$

as $\{b_{\langle s(0)\rangle} : s(0) < \lambda\}$. For each $s(0) < \lambda$, let $W_{\langle s(0)\rangle} = \sigma(\langle b_{\langle s(0)\rangle}\rangle)$, the quasi-partition of a which I chooses according to σ if II has just chosen to leave out $b_{\langle s(0)\rangle}$. In general, given $\alpha < \kappa$, $s \in (\lambda)^\alpha$, and W_s a quasi-partition of a of size λ, index the elements of W_s as $\{b_{s^\frown s(\alpha)} : s(\alpha) < \lambda\}$. For each $s(\alpha) < \lambda$, let

$$W_{s^\frown s(\alpha)} = \sigma(\langle b_{s\restriction(\beta+1)} : \beta \leq \alpha\rangle), \tag{10}$$

the quasi-partition of a which I chooses according to σ if II has just chosen to leave out $b_{s^\frown s(\alpha)}$. For limit ordinals $\mu < \kappa$ and $s \in (\lambda)^\mu$, let

$$W_s = \sigma(\langle b_{s\restriction(\alpha+1)} : \alpha < \mu\rangle). \tag{11}$$

Note that $\{W_s : s \in (\lambda)^{<\kappa}\}$ lists all the possible choices for I under σ.

Claim 30 *The hyper-weak $(\lambda^{<\kappa}, \lambda)$-d.l. fails for the quasi-partitions W_s, $s \in (\lambda)^{<\kappa}$, of a.*

Let $f : (\lambda)^{<\kappa} \to \lambda$ be given. Recursively define a sequence $t \in (\lambda)^\kappa$ by $t \restriction (\alpha+1) = t \restriction \alpha^\frown f(t \restriction \alpha)$ for each $\alpha < \kappa$. Then $\langle W_{t\restriction\alpha}, b_{t\restriction(\alpha+1)} : \alpha < \kappa\rangle$ is a play of $\mathcal{G}^\kappa_{\lambda-1}$ in which I follows the winning strategy σ. Thus,

$$\bigwedge_{s\in(\lambda)^{<\kappa}} a \setminus b_{s^\frown f(s)} \;\leq \bigwedge_{\alpha<\kappa} a \setminus b_{t\restriction(\alpha+1)} = \mathbf{0}. \tag{12}$$

Since f was arbitrary,

$$\bigvee_{f:(\lambda)^{<\kappa}\to\lambda} \; \bigwedge_{s\in(\lambda)^{<\kappa}} a \setminus b_{s^\frown f(s)} = \;\mathbf{0}\; < \; a \; = \bigwedge_{s\in(\lambda)^{<\kappa}} \; \bigvee_{j<\lambda} b_{s^\frown j}. \tag{13}$$

□

For some pairs of cardinals, Theorems 28 and 29 combine to yield a game-theoretic characterization of the hyper-weak (κ, λ)-d.l.

Corollary 31 *If $\mathbb{B}$ is κ^+-complete and $\lambda^{<\kappa} = \kappa$, then the hyper-weak (κ, λ)-d.l. holds in $\mathbb{B}$ iff I does not have a winning strategy for $\mathcal{G}^\kappa_{\lambda-1}$ played in $\mathbb{B}$.*

In particular, this yields a game-theoretic characterization of the hyper-weak (ω, ω)-d.l. for Boolean σ-algebras.

Corollary 32 (GCH) *Suppose $\mathbb{B}$ is κ^+-complete and either (a) $\lambda < \kappa$, or (b) $\lambda = \kappa$ and κ is regular. Then the hyper-weak (κ, λ)-d.l. holds in $\mathbb{B}$ iff I does not have a winning strategy for $\mathcal{G}^\kappa_{\lambda-1}$ played in $\mathbb{B}$.*

4 $\geq \nu$-club and $\geq \nu$-stationary subsets of κ

In this section, we introduce notions intermediate between clubness and stationarity. The main purpose is to highlight Proposition 36 and Fact 37 which will be used in a non-trivial way in Theorem 38. Throughout this and the next section, we say that a sequence of ordinals is "increasing" if it is non-decreasing.

Definition 33 Suppose ν, κ are regular cardinals with $\omega \leq \nu < \kappa$. We say that a set $C \subseteq \kappa$ is *$\geq \nu$-club* if C is unbounded in κ and for all regular cardinals $\nu \leq \mu < \kappa$, C is closed under strictly increasing sequences of length μ. We say that a set $S \subseteq \kappa$ is *$\geq \nu$-stationary* if $S \cap C \neq \emptyset$ for each $\geq \nu$-club $C \subseteq \kappa$.

Fact 34 *Suppose ν, κ are regular cardinals with $\omega \leq \nu < \kappa$.*

(1) $\geq \omega$-club equals club and $\geq \omega$-stationary equals stationary.
(2) For regular cardinals $\omega \leq \nu_0 \leq \nu_1 < \kappa$, $\geq \nu_0$-club $\Longrightarrow$ $\geq \nu_1$-club $\Longrightarrow$ $\geq \nu_1$-stationary $\Longrightarrow$ $\geq \nu_0$-stationary.

Remark 35 The varying degrees of $\geq \nu$-clubness and $\geq \nu$-stationarity are strict.

Proposition 36 *Suppose ν, κ are regular with $\omega \leq \nu < \kappa$. Let cof($\geq \nu$) denote $\{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \nu\}$. $C \subseteq \kappa$ is $\geq \nu$-club iff there exists a club set $D \subseteq \kappa$ such that $C \cap \mathrm{cof}(\geq \nu) = D \cap \mathrm{cof}(\geq \nu)$. It follows that for $S \subseteq \kappa$, if $S \cap \mathrm{cof}(\nu)$ is stationary, then S is a $\geq \nu$-stationary.*

PROOF. Suppose $C \subseteq \kappa$ is $\geq \nu$-club. Let D be the closure of C in κ; i.e. $D = \{\alpha < \kappa : \alpha \text{ is the limit of an increasing sequence in } C\}$. Then D is club, and $C \cap \mathrm{cof}(\geq \nu) = D \cap \mathrm{cof}(\geq \nu)$.

Suppose $D \subseteq \kappa$ is club and $C \cap \mathrm{cof}(\geq \nu) = D \cap \mathrm{cof}(\geq \nu)$. Since $D \cap \mathrm{cof}(\geq \nu)$ is stationary, C is unbounded. Suppose θ is regular, $\nu \leq \theta < \kappa$, and $(c_\alpha)_{\alpha<\theta}$ is a strictly increasing sequence in C. Then $\sup_{\alpha<\theta} c_\alpha \in D \cap \mathrm{cof}(\geq \nu) \subseteq C$.

$\square$

Using Proposition 36 and basic theorems for club and stationary sets, one can easily prove the following useful fact.

Fact 37 *Suppose ν, κ are regular with $\omega \leq \nu < \kappa$. The intersection of less than κ-many $\geq \nu$-club subsets of κ is a $\geq \nu$-club subset of κ. The diagonal intersection of κ-many $\geq \nu$-club subsets of κ is again a $\geq \nu$-club subset of κ.*

5 Constructions of κ^+-Suslin algebras in which many games are undetermined

In this section, we show that it is consistent with ZFC that for each infinite cardinal κ, for each regular ν with $\omega \leq \nu \leq \mathrm{cf}(\kappa)$, there is a κ^+-Suslin algebra $\mathbb{B}^\nu_\kappa$ with the following properties. $\mathbb{B}^\nu_\kappa$ contains a $< \nu$-closed dense subset, so II wins every game of length $< \nu$. For all ρ, λ with $\nu \leq \rho \leq \kappa$ and $\omega \leq \lambda \leq \kappa$, $\mathcal{G}^\rho_{\lambda-1}$ is undetermined in $\mathbb{B}^\nu_\kappa$. Hence, for all ρ, λ, μ with $\nu \leq \rho \leq \kappa$ and $2 \leq \mu \leq \min(\lambda, \kappa)$, $\mathcal{G}^\rho_{<\mu}(\lambda)$ is undetermined in $\mathbb{B}^\nu_\kappa$. To do this, we will construct a κ^+-Suslin algebra which has a $< \nu$-closed dense subset and in which II does not have a winning strategy for $\mathcal{G}^\nu_{\kappa-1}$. This improves on and extends to singular κ a result in [12,14], showing that the gap between the strengths of "II has a winning strategy for $\mathcal{G}^\kappa_1(\infty)$" and "the (κ, ∞)-d.l. holds" is consistently even larger than was previously known. Our construction is a variant of the usual construction of κ^+-Suslin trees found in [21]. We begin by giving some definitions, notation, and background which will be used for Theorem 38.

Let $\mathbb{B}$ denote a κ^+-Suslin algebra. Since the (κ, ∞)-d.l. holds in each κ^+-Suslin algebra, I does not have a winning strategy for $\mathcal{G}^\kappa_1(\infty)$ in $\mathbb{B}$, by Corollary 1.6 of [12]. Hence, by Fact 20, for each infinite cardinal $\rho \leq \kappa$, for every λ, μ with $2 \leq \mu \leq \lambda$, I does not have a winning strategy for $\mathcal{G}^\rho_{<\mu}(\lambda)$ in $\mathbb{B}$; and for every $\lambda \geq \omega$, I does not have a winning strategy for $\mathcal{G}^\rho_{\lambda-1}$ in $\mathbb{B}$. Let ν be an infinite regular cardinal with $\nu \leq \mathrm{cf}(\kappa)$. If we construct one κ^+-Suslin algebra in which II does not have a winning strategy for $\mathcal{G}^\nu_{\kappa-1}$, then Fact 20 implies that for each $\nu \leq \rho \leq \kappa$, for every $\omega \leq \lambda \leq \kappa$, II does not have a winning strategy for $\mathcal{G}^\rho_{\lambda-1}$ in $\mathbb{B}$. Hence, for every λ, μ with $2 \leq \mu \leq \min(\lambda, \kappa)$, II does not have a winning strategy for $\mathcal{G}^\rho_{<\mu}(\lambda)$ in $\mathbb{B}$. Moreover, if $\mathbb{B}$ contains a dense $< \nu$-closed subset, then II wins $\mathcal{G}^\theta_1(\infty)$ for all $\theta < \nu$, and II even wins Foreman's game $G^{\mathrm{I}}_{\theta^+}$ for all $\theta < \nu$ (see [19]).

For each infinite cardinal κ and each infinite regular cardinal $\nu \leq \mathrm{cf}(\kappa)$, we will construct a κ^+-Suslin tree $(T, \leq_T)$, $\leq_T$ denoting the tree ordering on T. For $t \in T$, $\mathrm{ht}(t)$ denotes $\mathrm{o.t.}(\{s \in T : s <_T t\})$. For $\alpha < \kappa^+$, let $\mathrm{Lev}(\alpha) = \{t \in T : \mathrm{ht}(t) = \alpha\}$, the *$\alpha$-th level* of T, and $T_\alpha = \{t \in T : \mathrm{ht}(t) < \alpha\}$. The nodes of T will be the ordinals in κ^+ arranged so that for each ordinal $\alpha \geq 2$, $\mathrm{Lev}(\alpha) = \{\text{ordinals } \beta : \kappa \cdot \alpha \leq \beta < \kappa \cdot (\alpha + 1)\}$. Let T^* denote T under the reverse partial ordering $\geq_T$, and let $e : T^* \to \mathrm{r.o.}(T^*)$ denote the canonical embedding of T^* into its Boolean completion. We will construct T so that every strategy for II for the game $\mathcal{G}^\nu_{\kappa-1}$ in $\mathrm{r.o.}(T^*)$ will fail to be winning when I plays some sequence of partitions of unity in $\mathrm{r.o.}(T^*)$.

We restrict player I to playing only partitions of unity in the game $\mathcal{G}^\nu_{\kappa-1}$ in $\mathrm{r.o.}(T^*)$. We then transfer this restricted game to the tree $(T, \leq_T)$ as follows. We say that $\langle W(\gamma) : \gamma < \kappa \rangle$ is a *partition of $\mathrm{Lev}(\alpha)$ into κ-many pieces* if

$\forall\gamma < \gamma' < \kappa, W(\gamma) \neq \emptyset, W(\gamma) \subseteq \mathrm{Lev}(\alpha), W(\gamma) \cap W(\gamma') = \emptyset$, and $\bigcup_{\gamma<\kappa} W(\gamma) = \mathrm{Lev}(\alpha)$. On round 0, I chooses a level $\alpha_0 < \kappa^+$ and a partition of $\mathrm{Lev}(\alpha_0)$ into κ-many pieces, say $\mathcal{Q}_{\alpha_0} = \langle Q(\alpha_0, \gamma) : \gamma < \kappa \rangle$. Then II chooses one $\gamma_0 < \kappa$. On round $i < \nu$, I chooses some $\alpha_i < \kappa^+$ such that $\alpha_i > \sup_{j<i} \alpha_j$ and some partition of $\mathrm{Lev}(\alpha_i)$ into κ-many pieces, say $\mathcal{Q}_{\alpha_i} = \langle Q(\alpha_i, \gamma) : \gamma < \kappa \rangle$. Then II chooses one $\gamma_i < \kappa$. Let $\alpha = \sup_{i<\nu} \alpha_i$. II wins the play iff there is some branch B in T of length $\alpha + 1$ such that for every $i < \nu$, $B \cap Q(\alpha_i, \gamma_i) = \emptyset$. Denote this game by $\mathcal{G}^{\nu}_{\kappa-1}(T)$. Note that if II does not have a winning strategy for $\mathcal{G}^{\nu}_{\kappa-1}(T)$, then II does not have a winning strategy for $\mathcal{G}^{\nu}_{\kappa-1}$ in r.o.(T^*).

Theorem 38 *Let κ be any infinite cardinal, and let ν be any regular cardinal such that $\omega \leq \nu \leq \mathrm{cf}(\kappa)$. Suppose that $\Box_\kappa$ holds and $\diamondsuit_{\kappa^+}(S)$ holds for every stationary set $S \subseteq \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \nu\}$. Then there is a κ^+-Suslin algebra which contains a $< \nu$-closed dense subset, and in which for each ρ, λ with $\nu \leq \rho \leq \mathrm{cf}(\kappa)$ and $\omega \leq \lambda \leq \kappa$ the game $\mathcal{G}^{\rho}_{\lambda-1}$ is undetermined. Hence, for every ρ, λ, μ with $\nu \leq \rho \leq \mathrm{cf}(\kappa)$ and $2 \leq \mu \leq \min(\kappa, \lambda)$, the game $\mathcal{G}^{\rho}_{<\mu}(\lambda)$ is undetermined.*

PROOF. Let $\mathrm{cof}(\nu)$ denote $\{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \nu\}$. For any set of ordinals X, let $\lim(X)$ denote the set of ordinals $\alpha \in X$ such that α is the limit of an infinite, strictly increasing sequence of elements of X. Using a $\Box_\kappa$-sequence, we construct in the usual manner another $\Box_\kappa$-sequence $\langle D_\alpha : \alpha \in \lim(\kappa^+) \rangle$ and a non-reflecting stationary set $S \subseteq \mathrm{cof}(\nu)$ such that $\forall\alpha \in \lim(\kappa^+)$,

(1) $D_\alpha \subseteq \alpha$ is club in α;
(2) $\mathrm{cf}(\alpha) < \kappa \longrightarrow \mathrm{o.t.}(D_\alpha) < \kappa$;
(3) $\gamma \in \lim(D_\alpha) \longrightarrow D_\gamma = D_\alpha \cap \gamma$;
(4) $\lim(D_\alpha) \cap S = \emptyset$.

We fix some definitions and notation. Fix a $\diamondsuit_{\kappa^+}(S)$-sequence $\langle A_\alpha : \alpha \in S \rangle$; i.e. a sequence such that $\forall\alpha \in S\ A_\alpha \subseteq \alpha$, and $\forall A \subseteq \kappa^+\ \{\alpha \in S : A_\alpha = A \cap \alpha\}$ is stationary. Let $\mathcal{L}$ denote the collection of all partitions of levels of T into κ-many pieces. $\mathcal{L} \subseteq ([\kappa^+]^{\leq\kappa})^\kappa$ and $2^\kappa = \kappa^+$ imply $|\mathcal{L}| = \kappa^+$. Fix an enumeration $\langle \mathcal{W}_\alpha : \alpha < \kappa^+ \rangle$ of $\mathcal{L}$, where each $\mathcal{W}_\alpha = \langle W(\alpha, \gamma) : \gamma < \kappa \rangle$. Then each strategy for II for $\mathcal{G}^{\nu}_{\kappa-1}(T)$ can be represented as a function $f : (\kappa^+)^{<\nu} \to \kappa$. Fix a bijection $\varphi : (\kappa^+)^{<\nu} \times \kappa \to \kappa^+$. φ will code all the possible strategies for II as subsets of κ^+. For each $\alpha < \kappa^+$ we will fix a partition $\mathcal{P}_\alpha = \langle P(\alpha, \gamma) : \gamma < \kappa \rangle$ of $\mathrm{Lev}(\alpha)$ into κ-many pieces and use these partitions to foil all of player II's strategies for $\mathcal{G}^{\nu}_{\kappa-1}(T)$.

Construction of $(T, \leq_T)$ and $\{\mathcal{P}_\alpha : \alpha < \kappa^+\}$. In the course of the construction we will be building for $t \in T_\beta$, $\beta \notin S$, a certain branch B^β_t (which we will call the canonical β-branch for t in T_β) which will help continue the construction.

Let $\mathrm{Lev}(0) = \{0\}$. Let $\mathrm{Lev}(1) = (\kappa \cdot 2) \setminus \{0\}$, and let $\mathcal{P}_1 = \langle P(1, \gamma) : \gamma < \kappa \rangle$ be some partition of $\mathrm{Lev}(1)$ into κ-many pieces. Suppose $\alpha \geq 1$ and $\mathrm{Lev}(\alpha)$ has been constructed. Put κ-many immediate successors above each element of $\mathrm{Lev}(\alpha)$ in such a way that $\mathrm{Lev}(\alpha+1) = \{\beta < \kappa^+ : \kappa \cdot (\alpha+1) \leq \beta < \kappa \cdot (\alpha+2)\}$. Let $\mathcal{P}_{\alpha+1} = \langle P(\alpha+1, \gamma) : \gamma < \kappa \rangle$ be a partition of $\mathrm{Lev}(\alpha+1)$ into κ-many pieces such that for each node t in $\mathrm{Lev}(\alpha)$ and each $\gamma < \kappa$, $\mathrm{Lev}(\alpha+1) \cap P(\alpha+1, \gamma)$ contains exactly one immediate successor of t.

Now suppose $\alpha < \kappa^+$ is a limit ordinal and T_α has been constructed. We have three cases.

Case 1. $\mathrm{cf}(\alpha) < \nu$. Then extend every α-branch of T_α to level α with exactly one extension in such a way that $\mathrm{Lev}(\alpha) = \{\beta < \kappa^+ : \kappa \cdot \alpha \leq \beta < \kappa \cdot (\alpha+1)\}$.

Case 2. $\mathrm{cf}(\alpha) \geq \nu$ and $\alpha \notin S$. We assume that $\forall \beta \in \lim(\alpha) \setminus S$, $\forall t \in T_\beta$, the canonical β-branch B_t^β has been extended to $\mathrm{Lev}(\beta)$ with exactly one extension, $\tilde{p}_t^\beta$. Let $t \in T_\alpha$. The canonical α-branch B_t^α is defined as follows. Let $\varepsilon = \mathrm{o.t.}(\{\gamma \in D_\alpha : \gamma > \mathrm{ht}(t)\})$ and let $\langle \beta_i : i < \varepsilon \rangle$ be the strictly increasing enumeration of $\{\gamma \in D_\alpha : \gamma > \mathrm{ht}(t)\}$. Let $x_0 \in \mathrm{Lev}(\beta_0)$ be the least ordinal such that $t <_T x_0$. For $i = j+1 < \varepsilon$, let $x_i \in \mathrm{Lev}(\beta_i)$ be the least ordinal such that $x_j <_T x_i$. For $i < \varepsilon$ a limit ordinal, $\beta_i \in \lim(D_\alpha)$, so $\beta_i \notin S$. Hence, $B_t^{\beta_i}$ was already constructed at stage β_i to be $\{s \in T_{\beta_i} : \exists j < i (s \leq_T x_j)\}$. $B_t^{\beta_i}$ was given the extension $\tilde{p}_t^{\beta_i}$ at $\mathrm{Lev}(\beta_i)$. Let $x_i = \tilde{p}_t^{\beta_i}$. Let $B_t^\alpha = \{s \in T_\alpha : \exists i < \varepsilon (s \leq_T x_i)\}$. B_t^α is called *the canonical α-branch for t in T_α*. For each $t \in T_\alpha$, extend the canonical α-branch B_t^α to $\mathrm{Lev}(\alpha)$ with exactly one extension so that $\mathrm{Lev}(\alpha) = \{\beta < \kappa^+ : \kappa \cdot \alpha \leq \beta < \kappa \cdot (\alpha+1)\}$.

Case 3. $\alpha \in S$. Let (C) be the statement, "$T_\alpha = \alpha$, $\{\mathcal{P}_\beta : \beta < \alpha\} \subseteq \{\mathcal{W}_\beta : \beta < \alpha\}$, and $\varphi''((\alpha)^{<\nu} \times \kappa) = \alpha$." Let (M) be the statement, "A_α is a maximal antichain in T_α." Let (F) be the statement, "A_α codes a partial strategy for II on T_α." If either (C) fails, or both (M) and (F) fail, then extend all the canonical α-branches of T_α to $\mathrm{Lev}(\alpha)$.

Suppose (C) holds. If (M) holds and (F) fails, then for each $t \in T_\alpha$, choose one r in T_α such that $r \geq_T t$ and $r \geq_T u$ for some $u \in A_\alpha$. Extend the canonical α-branch B_r^α to $\mathrm{Lev}(\alpha)$.

Now suppose (C) and (F) both hold. Let $f = \varphi^{-1}(A_\alpha)$. Then $f : (\alpha)^{<\nu} \to \kappa$ is a partial strategy for II on T_α. Fix a strictly increasing sequence $\langle \alpha_i : i < \nu \rangle$ such that each α_i is a successor ordinal and $\sup_{i<\nu} \alpha_i = \alpha$. This is possible since $\alpha \in S$ implies $\mathrm{cf}(\alpha) = \nu$. We let I play $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$ and II play according to f. For each $\beta < \alpha$, let $g(\beta)$ be the ordinal below α such that $\mathcal{P}_\beta = \mathcal{W}_{g(\beta)}$. Then each $\mathcal{P}_{\alpha_i} = \mathcal{W}_{g(\alpha_i)}$. For each $i < \nu$, let

$$\gamma_i = f(\langle g(\alpha_j) : j \leq i \rangle). \tag{14}$$

Then when I plays the sequence $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$ and II plays by f, II chooses to leave out $P(\alpha_i, \gamma_i)$ on round i.

Let $t \in T_\alpha$. If (M) holds, let $r \in T_\alpha$ be such that $r \geq_T t$ and $r \geq_T u$ for some $u \in A_\alpha$. Otherwise, let $r = t$. Let $i(t) < \nu$ be the least ordinal such that $\alpha_{i(t)} > \mathrm{ht}(r)$. We will construct a chain $\langle x_i : i(t) \leq i < \nu \rangle$ in T_α such that $x_{i(t)} \geq_T r$; for each $i(t) \leq i < j < \nu$, $x_i <_T x_j$; and for each $i(t) \leq i < \nu$, $x_i \in P(\alpha_i, \gamma_i)$. Let $p_{i(t)}$ be the least ordinal in $\mathrm{Lev}(\alpha_{i(t)} - 1)$ such that $p_{i(t)} >_T r$. Let $x_{i(t)}$ be the immediate successor of $p_{i(t)}$ in $P(\alpha_{i(t)}, \gamma_{i(t)})$. For $i(t) < i = j + 1 < \nu$, let p_i be the least ordinal in $\mathrm{Lev}(\alpha_i - 1)$ such that $p_i \geq_T x_j$. Let x_i be the immediate successor of p_i in $P(\alpha_i, \gamma_i)$. For i a limit ordinal with $i(t) < i < \nu$, let $\lambda_i = \sup_{j<i} \alpha_j$. Then λ_i is a limit ordinal with $\mathrm{cf}(\lambda_i) < \nu$, so every branch in T_{λ_i} has an extension in $\mathrm{Lev}(\lambda_i)$, by Case 1. Let p'_i be the least ordinal in $\mathrm{Lev}(\lambda_i)$ such that $p'_i > x_j$ for all $i(t) \leq j < i$. α_i is a successor ordinal greater than λ_i, so let p_i be the least ordinal in $\mathrm{Lev}(\alpha_i - 1)$ such that $p_i \geq_T p'_i$. Let x_i be the immediate successor of p_i in $P(\alpha_i, \gamma_i)$.

Let $b^\alpha_t = \{s \in T_\alpha : \text{for some } i \text{ with } i(t) \leq i < \nu,\ s <_T x_i\}$. b^α_t is an α-branch in T_α which passes through the pieces of the partitions which II chooses to leave out on each level α_i according to f. For each $t \in T_\alpha$, extend b^α_t to $\mathrm{Lev}(\alpha)$.

For each of the subcases of Case 3, for each $t \in T_\alpha$ we have chosen one α-branch containing t to extend to $\mathrm{Lev}(\alpha)$. Extend each of these branches uniquely to $\mathrm{Lev}(\alpha)$ in such a way that $\mathrm{Lev}(\alpha) = \{\beta < \kappa^+ : \kappa \cdot \alpha \leq \beta < \kappa \cdot (\alpha + 1)\}$. Now let $\mathcal{P}_\alpha = \langle P(\alpha, \gamma) : \gamma < \kappa \rangle$ be some partition of $\mathrm{Lev}(\alpha)$ into κ many pieces.

Let $T = \bigcup_{\alpha<\kappa} \mathrm{Lev}(\alpha)$. This concludes the construction of $(T, \leq_T)$. By the usual argument, $(T, \leq_T)$ is a κ^+-Suslin tree.

Let $C_T = \{\alpha < \kappa^+ : T_\alpha = \alpha\}$, $C_P = \{\alpha < \kappa^+ : \{\mathcal{P}_\beta : \beta < \alpha\} \subseteq \{\mathcal{W}_\beta : \beta < \alpha\}\}$, and $C_\varphi = \{\alpha < \kappa^+ : \varphi''((\alpha)^{<\nu} \times \kappa) = \alpha\}$. C_T and C_P are club subsets of κ^+. C_φ is a $\geq \nu$-club subset of κ^+.

Claim 39 *II does not have a winning strategy for $\mathcal{G}^\nu_{\kappa-1}(T)$.*

Fix a strategy $h : (\kappa^+)^{<\nu} \to \kappa$ for II in $\mathcal{G}^\nu_{\kappa-1}(T)$. $\{\beta \in S : A_\beta = \varphi''(h) \cap \beta\}$ is a stationary subset of $\mathrm{cof}(\nu)$. Hence, it is $\geq \nu$-stationary, by Proposition 36. Since C_φ is $\geq \nu$-club, there exists an $\alpha \in \{\beta \in S : A_\beta = \varphi''(h) \cap \beta\} \cap C_\varphi \cap C_T \cap C_P$. Let $f = \varphi^{-1}(A_\alpha)$. Then $f = h \restriction (\alpha)^{<\nu}$. $\alpha \in \{\beta \in S : A_\beta = \varphi''(h) \cap \beta\} \cap C_\varphi \cap C_T \cap C_P$ implies $\mathrm{Lev}(\alpha)$ was constructed according to Case 3 with statements (C) and (F) holding for f. Let $\langle \alpha_i : i < \nu \rangle$ be the strictly increasing sequence we picked such that each α_i is a successor ordinal and $\sup_{i<\nu} \alpha_i = \alpha$. Let I play the sequence $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$. Let $q \in \mathrm{Lev}(\alpha)$. Then there is some $t \in T_\alpha$ such that q is the unique extension of the α-branch b^α_t. By our construction, the unique element $x_{i(t)} \in \mathrm{Lev}(\alpha_{i(t)}) \cap b^\alpha_t$ is in $P(\alpha_{i(t)}, \gamma_{i(t)})$, the piece of the partition $\mathcal{P}_{\alpha_{i(t)}}$ which f chooses to leave out on round $i(t)$ when

I has played the sequence $\langle \mathcal{P}_{\alpha_j} : j \leq i(t) \rangle$. Since q was an arbitrary member of $\mathrm{Lev}(\alpha)$, there is no $(\alpha+1)$-branch in T which misses all of the pieces of the partitions $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$ which f chose to leave out. Hence, if II plays according to the strategy h, II loses when I plays the sequence $\langle \mathcal{P}_{\alpha_i} : i < \nu \rangle$. Thus, h is not a winning strategy for II. Since h was arbitrary, II does not have a winning strategy for $\mathcal{G}^{\nu}_{\kappa-1}(T)$; hence, II does not have a winning strategy for $\mathcal{G}^{\nu}_{\kappa-1}$ in r.o.(T^*).

$\square$

Remark 40 (1) In Case 3 when (C) and (F) hold for α, for each $t \in T_\alpha$ we only needed to extend some α-branch containing the element $x_i \in P(\alpha_{i(t)}, \gamma_{i(t)})$ to ensure that II does not have a winning strategy for $\mathcal{G}^{\nu}_{\kappa-1}$ in r.o.(T^*). Our construction shows that II does not have a winning strategy in r.o.(T^*) in the following game, which is even weaker than $\mathcal{G}^{\nu}_{\kappa-1}$: Players I and II play the game $\mathcal{G}^{\nu}_{\kappa-1}$ constructing a sequence $\langle a, W_\alpha, b_\alpha : \alpha < \nu \rangle$. After the play is over, II gets to choose some set $X \subseteq \nu$ of cardinality ν. I wins the play iff $\bigwedge_{\alpha \in X} \bigvee (W_\alpha \setminus \{b_\alpha\}) = \mathbf{0}$.

(2) In the case when $\kappa^{<\kappa} = \kappa$, the previous construction can be slightly modified so that using only $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$ we can construct a κ^+-Suslin algebra $\mathbb{B}$ which contains a $< \kappa$-closed dense subset, and in which the game $\mathcal{G}^{\kappa}_{\lambda-1}$ is undetermined for each $\omega \leq \lambda \leq \kappa$. Hence, also for each λ, μ with $2 \leq \mu \leq \min(\kappa, \lambda)$, the game $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ is undetermined in $\mathbb{B}$.

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