

# GLOBAL CO-STATIONARITY OF THE GROUND MODEL FROM A NEW COUNTABLE LENGTH SEQUENCE

NATASHA DOBRINEN

ABSTRACT. Suppose  $V \subseteq W$  are models of ZFC with the same ordinals, and that for all regular cardinals  $\kappa$  in  $W$ ,  $V$  satisfies  $\square_\kappa$ . If  $W \setminus V$  contains a sequence  $r : \omega \rightarrow \gamma$  for some ordinal  $\gamma$ , then for all cardinals  $\kappa < \lambda$  in  $W$  with  $\kappa$  regular in  $W$  and  $\lambda \geq \gamma$ ,  $(\mathcal{P}_\kappa(\lambda))^W \setminus V$  is stationary in  $(\mathcal{P}_\kappa(\lambda))^W$ . That is, a new  $\omega$ -sequence achieves global co-stationarity of the ground model.

## 1. INTRODUCTION

Suppose  $V \subseteq W$  are models of ZFC with the same ordinals,  $\kappa$  is regular and uncountable in  $W$ , and  $\lambda \geq (\kappa^+)^W$ . We say that *the ground model is co-stationary in  $W$*  if  $(\mathcal{P}_\kappa(\lambda))^W \setminus (\mathcal{P}_\kappa(\lambda))^V$  is stationary in  $(\mathcal{P}_\kappa(\lambda))^W$ . Note that  $(\mathcal{P}_\kappa(\lambda))^V = (\mathcal{P}_\kappa(\lambda))^W \cap V$ ; hence,  $(\mathcal{P}_\kappa(\lambda))^W \setminus (\mathcal{P}_\kappa(\lambda))^V = (\mathcal{P}_\kappa(\lambda))^W \setminus V$ . We shall subsequently drop the superscript for the larger model  $W$  with the convention that  $\mathcal{P}_\kappa(\lambda)$  denotes  $(\mathcal{P}_\kappa(\lambda))^W$ .

Abraham in [1] showed that if  $\mathbb{P}$  is a c.c.c. forcing which adds a new real, then  $\mathcal{P}_{\aleph_1}(\lambda) \setminus V$  is stationary in  $V^\mathbb{P}$  for all  $\lambda \geq \aleph_2$ . Answering a question of Abraham, Gitik showed in [5] that a new real in the larger model is enough to obtain co-stationarity of the ground model. In fact, he showed more.

**Theorem 1.1** (Gitik [5]). *Let  $V \subseteq W$  be models of ZFC with the same ordinals,  $\kappa$  a regular uncountable cardinal in  $W$ , and  $\lambda \geq (\kappa^+)^W$ . Suppose that there is a real in  $W \setminus V$ . Then  $\mathcal{P}_\kappa(\lambda) \setminus V$  is stationary in  $W$ .*

We shall say that the ground model is *globally co-stationary* if for some  $\kappa_0$ ,  $\mathcal{P}_\kappa(\lambda) \setminus V$  is stationary for all cardinals  $\lambda > \kappa \geq \kappa_0$  in  $W$ , with  $\kappa$  regular in  $W$ . Theorem 1.1 shows that any new real in the larger model achieves global co-stationarity of the ground model.

In this paper, we are interested in how the ground model is affected when the larger model contains a new sequence of countable length.

**Question 1.2.** *Is a new sequence of length  $\omega$  enough to ensure global co-stationarity of the ground model?*

An upper bound for this was obtained by Dobrinen and Friedman in [4].

**Theorem 1.3** (Dobrinen/Friedman [4]). *Suppose  $\nu \geq \omega_1$  and there is a proper class of  $\nu$ -Erdős cardinals in  $V'$ . Then there is a class generic extension  $V$  of  $V'$  in which the following holds: Suppose  $\kappa' > \nu$  is regular, and  $\mathbb{P}$  adds a new function  $r : \omega_1 \rightarrow \nu$  and is  $\kappa'$ -c.c. (or just satisfies the  $(\rho^+, \rho^+, < \rho)$ -distributive law for all successor cardinals  $\rho \geq \kappa'$ , and is  $\theta$ -c.c. for the least regular limit cardinal  $\theta \geq \kappa'$ ). Then  $\mathcal{P}_\kappa(\lambda) \setminus V$  is stationary in  $V^\mathbb{P}$  for each regular  $\kappa \geq \kappa'$  and all  $\lambda \geq \kappa^+$  in  $V^\mathbb{P}$ .*

As shown in [4], when  $\nu = \omega_1$ , the proper class of  $\nu$ -Erdős cardinals is necessary, as in that case, a covering theorem of Magidor from [8] applies. In fact, it was

---

2000 *Mathematics Subject Classification.* 03E05, 03E35, 03E65, 05C05.

*Key words and phrases.*  $\mathcal{P}_\kappa\lambda$ , co-stationarity, non-reflecting stationary set.

This work was supported by FWF grant P 16334-N05. The author wishes to thank Justin Moore for invaluable help and Paul Larson for direction.

proved in [4] that global co-stationarity of the ground model in a forcing extension obtained by an  $\aleph_2$ -c.c. forcing which adds a new subset of  $\aleph_1$  is equiconsistent with a proper class of  $\omega_1$ -Erdős cardinals. However, Theorem 1.3 left open the following questions.

**Question 1.4.** *Suppose  $\mathbb{P}$  adds no new reals but does add a new  $\omega$ -sequence. Let  $\nu$  be the least such that  $\mathbb{P}$  adds a new function  $r : \omega \rightarrow \nu$ . Does it follow that  $(\mathcal{P}_\kappa(\lambda))^V$  is co-stationary in  $V^\mathbb{P}$  for all cardinals  $\aleph_1 < \kappa < \lambda$  in  $V^\mathbb{P}$  with  $\kappa$  regular in  $V^\mathbb{P}$  and  $\lambda \geq \nu$ ? (This is Question 1.3 from [4].)*

**Question 1.5.** *If  $\nu > \omega_1$ , is a proper class of  $\nu$ -Erdős cardinals necessary to achieve global co-stationarity of the ground model from a new sequence  $r : \omega_1 \rightarrow \nu$ ?*

**Question 1.6.** *Are  $\nu$ -Erdős cardinals necessary to achieve global co-stationarity of the ground model if there is a new sequence  $r : \omega \rightarrow \nu$ ?*

The Main Theorem gives a negative answer to Question 1.6 and a positive answer to Question 1.4 in  $L$ . Moreover, the larger model need not be a forcing extension of the ground model.

**Main Theorem.** *Let  $V \subseteq W$  be models of ZFC with the same ordinals. Suppose  $\nu$  is regular in  $V$  and is the least cardinal in  $V$  such that there is a new sequence  $r : \omega \rightarrow \nu$  in  $W \setminus V$ . Suppose that for each regular  $\kappa \geq \nu$  in  $W$  there is a non-reflecting stationary subset of  $\{\alpha < \kappa^+ : cf(\alpha) = \nu\}$  in  $V$ . Then for all cardinals  $\aleph_1 \leq \kappa < \lambda$  in  $W$  with  $\kappa$  regular in  $W$  and  $\lambda \geq \nu$ ,  $\mathcal{P}_\kappa(\lambda) \setminus V$  is stationary in  $W$ .*

The reader is referred to [4] and [3] for more results on co-stationarity of the ground model for forcings which add new subsets of uncountable cardinals. Question 1.5 is still open.

## 2. DEFINITIONS AND BASIC FACTS

Throughout this paper, standard set-theoretic notation is used.  $\alpha, \beta, \gamma, \delta, \varepsilon$  are used to denote ordinals, while  $\kappa, \lambda, \mu, \nu, \rho, \theta$  are used to denote cardinals.  $\mathcal{P}_\kappa(X) = \{x \subseteq X : |x| < \kappa\}$ . Usually we use  $[X]^{<\omega}$  instead of  $\mathcal{P}_\omega(X)$  to denote the collection of finite subsets of  $X$ .  $(X)^{<\omega}$  denotes the tree of finite sequences of elements of  $X$  ordered by end-extension. Given an ordinal  $\delta$  and some  $X \subseteq \delta$ ,  $\lim(X)$  denotes the set of limit points of  $X$ .

Scott and Solovay asked for which cardinals  $\nu \geq \omega$  is there a complete Boolean algebra which adds a new sequence  $r : \omega \rightarrow \nu$  without adding any sequences  $g : \omega \rightarrow \theta$  for any  $\theta < \nu$ . Namba showed in [10] that for such  $\nu$ , the following must hold.

- (1) Either  $cf(\nu) = \omega$  or  $\nu$  is regular; and
- (2) For all  $\theta < \nu$ ,  $|\theta^\omega| < \nu$ .

These properties apply to all extension universes with the same ordinals, not just those obtained by forcing.

**Fact 2.1.** *Suppose  $V \subseteq W$  are models of ZFC with the same ordinals,  $\nu$  is the least ordinal such that there is a new sequence  $r : \omega \rightarrow \nu$  in  $W \setminus V$ .*

- (1) *If  $(cf(\nu))^V > \omega$ , then  $\nu$  is a regular cardinal in  $V$ .*
- (2) *For all  $\theta < \nu$ ,  $|\theta^\omega| < \nu$  in  $V$ .*

*Proof.* (1) Since  $\nu$  is the least ordinal such that there is a new sequence  $r : \omega \rightarrow \nu$  in  $W \setminus V$ ,  $\nu$  must be a cardinal in  $V$  and  $r$  must be cofinal in  $\nu$ . Suppose  $\omega < (cf(\nu))^V < \nu$ . Let  $\langle \nu_\alpha : \alpha < cf(\nu) \rangle$  be a cofinal sequence in  $\nu$ . Define  $g : \omega \rightarrow cf(\nu)$  by  $g(n) =$  the least  $\alpha$  such that  $\nu_\alpha > r(n)$ .  $g$  is unbounded in  $cf(\nu)$ , since  $r$  is cofinal in  $\nu$ . But this implies  $g \in W \setminus V$ , since  $(cf(\nu))^V > \omega$ . Contradiction to  $\nu$  being the least range of a new  $\omega$ -sequence.

(2) Suppose not. Then there is some  $\theta < \nu$  such that  $|\theta^\omega| \geq \nu$  in  $V$ . Let  $h : \nu \rightarrow \theta^\omega$  be a 1-1 function in  $V$ . Define  $g : \omega \times \omega \rightarrow \theta$  by  $g(i, j) = (h(r(i)))(j)$ .  $g \in V$ . For each  $i < \omega$ , define  $f_i : \omega \rightarrow \theta$  by  $f_i(j) = g(i, j)$ . Then the sequence  $\langle f_i : i < \omega \rangle$  is in  $V$ . But  $r(i) = h^{-1}(h(r(i))) = h^{-1}(f_i)$ , contradiction to  $r \notin V$ .  $\square$

*Remark 2.2.* (2) implies that if  $\nu$  is the least cardinal such that there is a new sequence  $r : \omega \rightarrow \nu$  in  $W \setminus V$ , then either  $\nu \geq \aleph_2$  or  $\nu = 2$ .

Next, we review some facts about co-stationarity of the ground model. Co-stationarity of the ground model in  $\mathcal{P}_\kappa(\lambda)$  implies co-stationarity of the ground model for certain other cardinals.

**Theorem 2.3** (Menas [9]). *Let  $A \subseteq B$  with  $|A| \geq \kappa$ . For  $Y \subseteq \mathcal{P}_\kappa(B)$ , let  $Y \upharpoonright A = \{y \cap A : y \in Y\}$ . If  $C \subseteq \mathcal{P}_\kappa(B)$  is club, then  $C \upharpoonright A$  contains a club set in  $\mathcal{P}_\kappa(A)$ .*

The next fact follows easily from Theorem 2.3.

**Fact 2.4** ([4]). *Let  $V \subseteq W$  be models of ZFC with the same ordinals and  $\kappa$  be regular and  $\lambda > \kappa$  in  $W$ . If  $\mathcal{P}_\kappa(\lambda) \setminus V$  is stationary in  $W$ , then for all  $\mu \geq \lambda$ ,  $\mathcal{P}_\kappa(\mu) \setminus V$  is also stationary in  $W$ .*

**Fact 2.5** ([4]). *Let  $V \subseteq W$  be models of ZFC with the same ordinals. If  $\kappa$  is a cardinal in  $W$  and  $\nu > \kappa$  is the least cardinal in  $V$  such that  $W \setminus V$  has a new function from  $\kappa$  into  $\nu$ , then  $\forall \lambda \geq \nu$ ,  $\mathcal{P}_{\kappa^+}(\lambda) \setminus V$  contains a cone. Moreover, for all cardinals  $\rho, \lambda$  in  $W$  with  $\rho$  regular in  $W$ ,  $\kappa < \rho \leq \nu \leq \lambda$ , and  $\text{cf}(\nu) \geq \rho$  in  $V$ , then  $\mathcal{P}_\rho(\lambda) \setminus V$  contains a cone.*

Note, however, that this tells us nothing about whether  $\mathcal{P}_\rho(\lambda) \setminus V$  is stationary in  $W$  for  $\lambda > \rho > \nu$ .

Finally, we give the necessary definitions concerning non-reflecting stationary sets.

**Definition 2.6** ([6]). A stationary set  $S \subseteq \kappa$  is *non-reflecting* if for all limit ordinals  $\alpha < \kappa$ ,  $S \cap \alpha$  is not stationary in  $\alpha$ .

**Definition 2.7** ([2]). Let  $\kappa$  be a cardinal and  $S \subseteq \kappa^+$ .  $\square_\kappa(S)$  holds if there is a sequence  $\langle C_\alpha : \alpha \in \text{lim}(\kappa^+) \rangle$  such that for each  $\alpha \in \text{lim}(\kappa^+)$ ,

- (1)  $C_\alpha \subseteq \alpha$  is club in  $\alpha$ ;
- (2)  $\text{cf}(\alpha) < \kappa \rightarrow \text{o.t.}(C_\alpha) < \kappa$ ; and
- (3) If  $\beta < \alpha$  and  $\beta \in \text{lim}(C_\alpha)$ , then  $C_\beta = C_\alpha \cap \beta$  and  $\beta \notin S$ .

$\square_\kappa$  denotes  $\square_\kappa(\emptyset)$ .

It is well-known that the existence of non-reflecting stationary subsets of  $\kappa^+$  follows from  $\square_\kappa$  and hence holds in  $L$ . In fact, more is true.

**Fact 2.8.** *Let  $\omega \leq \nu < \kappa$  with  $\nu$  regular and assume  $\square_\kappa$  holds. Then there exists a non-reflecting stationary  $S \subseteq \{\alpha < \kappa^+ : \text{cf}(\alpha) = \nu\}$  for which  $\square_\kappa(S)$  holds.*

### 3. MAIN THEOREM

Let  $V \subseteq W$  be models of ZFC with the same ordinals. When  $r : \omega \rightarrow \nu$  is in  $W \setminus V$ , we say that  $r$  *achieves global co-stationarity of the ground model* if for all cardinals  $\omega < \kappa < \lambda$  in  $W$  with  $\kappa$  regular in  $W$  and  $\lambda \geq \nu$ ,  $\mathcal{P}_\kappa(\lambda) \setminus V$  is stationary in  $W$ . In this section we show, assuming certain combinatorial principals, that if  $\nu$  is regular and is the least ordinal such that there exists a new sequence  $r : \omega \rightarrow \nu$ , then  $r$  achieves global co-stationarity of the ground model.

Our proof of the Main Theorem will use certain trees constructed by Gitik to prove Theorem 1.1. For each regular  $\kappa > \omega$ , Gitik constructs a certain subtree  $T \subseteq (\kappa^+)^{<\omega}$  and constructs three branches through  $T$  to code the new real. In Theorem 3.3, given an  $r : \omega \rightarrow \nu$  in  $W \setminus V$ , we will construct two branches through

Gitik's tree which, along with a particular function  $f$ , suffice to code a new cofinal sequence from  $\omega$  into  $\nu$ , which thus cannot be in  $V$ . The next lemma gives conditions under which the function we use exists and can be naturally extracted from work of Todorćević in [12]. We thank Justin Moore for pointing it out and include his proof. Given a pairset  $a$ , we let  $a^0 = \min(a)$  and  $a^1 = \max(a)$ .

**Lemma 3.1.** *Suppose  $\aleph_0 \leq \nu < \kappa$  are regular cardinals and that there exists a non-reflecting stationary subset of  $\{\alpha < \kappa : \text{cf}(\alpha) = \nu\}$ . Then there is a function  $f : [\kappa]^2 \rightarrow \nu$  satisfying the following:*

$$\begin{aligned} & \text{Suppose } A, B \subseteq [\kappa]^2, |A| = |B| = \kappa, \\ (*f) \quad & \forall a_0, a_1 \in A \ a_0 \cap a_1 = \emptyset, \text{ and } \forall b_0, b_1 \in B \ b_0 \cap b_1 = \emptyset. \\ & \text{Then } \forall \varepsilon < \nu \ \exists a \in A \ \exists b \in B \ \forall i, j < 2 \ f(a(i), b(j)) > \varepsilon. \end{aligned}$$

*Proof.* Let  $S$  be a non-reflecting stationary subset of  $\{\alpha < \kappa : \text{cf}(\alpha) = \nu\}$ . For each limit ordinal  $\alpha < \kappa$ , let  $C_\alpha \subseteq \alpha$  be a club in  $\alpha$  such that  $C_\alpha \cap S = \emptyset$ . Let  $C_0 = \emptyset$ , and for each ordinal  $0 < \alpha < \kappa$ , let  $C_{\alpha+1} = \{\alpha\}$ . For  $\alpha < \beta < \kappa$ , the *trace from  $\alpha$  to  $\beta$*  is

$$(3.1) \quad \text{tr}(\alpha, \beta) = \{\beta_i : \beta_0 = \beta, \beta_{i+1} = \min(C_{\beta_i} \setminus \alpha) \text{ if } \beta_i > \alpha\}.$$

Define  $f : [\kappa]^2 \rightarrow \nu$  by

$$(3.2) \quad f(\alpha, \beta) = \max\{\text{o.t.}(C_\xi \cap \alpha) : \xi \in \text{tr}(\alpha, \beta) \cap S\}.$$

Let  $A, B$  be as in  $(*f)$ , and  $\varepsilon < \nu$ . Choose a sequence  $\langle a_\alpha : \alpha < \kappa \rangle$  of elements of  $A$  such that  $\alpha < \beta < \kappa \rightarrow a_\alpha^1 < a_\beta^0$ . Let  $A_1 = \{a_\alpha^1 : \alpha < \kappa\}$ .  $\text{Lim}(A_1)$  is club in  $\kappa$ , so let  $\delta \in \text{lim}(A_1) \cap S$ . There is a subsequence  $\langle a_{\alpha_i} : i < \nu \rangle$  such that  $\sup_{i < \nu} a_{\alpha_i}^1 = \delta$ , and for each  $i < \nu$ ,  $\text{o.t.}(C_\delta \cap a_{\alpha_i}^0) > \varepsilon$ .

Fix  $b \in B$  such that  $b^0 > \delta$ . Fix  $k < 2$  and let  $\beta = b^k$ . Note that for  $\zeta \in \text{tr}(\delta, \beta)$ ,  $\delta \in C_\zeta$  iff  $\zeta = \delta + 1$ . Let  $\delta_0 = \sup\{\sup(C_\zeta \cap \delta) : \zeta \in \text{tr}(\delta, \beta) \text{ and } \zeta > \delta\}$ . Then  $\delta_0 < \delta$ , since for all  $\zeta \in \text{tr}(\delta, \beta)$  such that  $\zeta > \delta$ ,  $C_\zeta \cap \delta$  is bounded in  $\delta$ .

Claim: For each  $\xi$  such that  $\delta_0 < \xi < \delta$ ,  $\delta \in \text{tr}(\xi, \beta)$ . Let  $\beta = \beta_0 > \beta_1 > \dots > \beta_n = \delta$  enumerate  $\text{tr}(\delta, \beta)$ . Let  $\xi$  be such that  $\delta_0 < \xi < \delta$ . Calculate  $\text{tr}(\xi, \beta)$ : Let  $\zeta_0 = \beta = \beta_0$ . Assume  $l < n$  and  $\zeta_l = \beta_l$ . Let  $\zeta_{l+1} = \min(C_{\zeta_l} \setminus \xi)$ .  $\beta_l \in \text{tr}(\delta, \beta)$  and  $\beta_l > \delta$ , so  $\sup(C_{\zeta_l} \cap \delta) = \sup(C_{\beta_l} \cap \delta) < \delta_0$ . Hence,  $C_{\zeta_l} \cap \delta = C_{\zeta_l} \cap \xi$ , which implies  $C_{\zeta_l} \setminus \xi = C_{\zeta_l} \setminus \delta$ . Therefore,  $\zeta_{l+1} = \min(C_{\zeta_l} \setminus \xi) = \min(C_{\beta_l} \setminus \delta) = \beta_{l+1}$ . By induction,  $\zeta_n = \beta_n = \delta$ .

Now choose  $i < \nu$  for which  $\delta_0 < a_{\alpha_i}^0$ .  $\delta \in \text{tr}(a_{\alpha_i}^m, \beta)$  for each  $m < 2$ , so  $f(a_{\alpha_i}^m, \beta) \geq \text{o.t.}(C_\delta \cap a_{\alpha_i}^m) > \varepsilon$ .  $\square$

*Remark 3.2.* J. Moore has pointed out that the “2” in Lemma 3.1 can be replaced by “finite”.

The next theorem contains most of the work towards proving the Main Theorem.

**Theorem 3.3.** *Let  $V \subseteq W$  be models of ZFC with the same ordinals. Suppose  $\nu \geq \aleph_2$  is regular in  $V$  and is the least cardinal in  $V$  such that there is a new sequence  $r : \omega \rightarrow \nu$  in  $W \setminus V$ . Suppose  $\kappa \geq \nu$  and  $\kappa$  is regular in  $W$ . If in  $V$  there is a non-reflecting stationary subset of  $\{\alpha < \kappa^+ : \text{cf}(\alpha) = \nu\}$ , then  $\mathcal{P}_\kappa(\kappa^+) \setminus V$  is stationary in  $W$ .*

*Proof.* Let  $\nu \geq \aleph_2$  be a regular cardinal in  $W$ , and assume that  $\nu$  is least such that there is a sequence  $r : \omega \rightarrow \nu$  in  $W \setminus V$ . In  $W$ , let  $\kappa > \nu$  be a regular cardinal, and let  $C \subseteq \mathcal{P}_\kappa(\kappa^+)$  be club. By a theorem due to Kueker in [7], there is a function  $g : [\kappa^+]^{<\omega} \rightarrow \mathcal{P}_\kappa(\kappa^+)$  such that  $C_g$  is a club subset of  $C$ , where  $C_g = \{x \in \mathcal{P}_\kappa(\kappa^+) : (\forall y \in [x]^{<\omega}) \ g(y) \subseteq x\}$ . Define  $\text{cl}^0(x) = x$ ,  $\text{cl}^1(x) = \bigcup\{g(y) : y \in [x]^{<\omega}\} \cup x$ , and

$\text{cl}^{n+1}(x) = \text{cl}^1(\text{cl}^n(x))$ . Define  $\text{cl}_g(x) = \bigcup_{n < \omega} \text{cl}^n(x)$ . Note: For each  $x \in \mathcal{P}_\kappa(\kappa^+)$ ,  $\text{cl}_g(x) \in C_g$ .

The following paragraph is extracted from Gitik's proof of Theorem 1.1 in [5]. We take the liberty of revising a bit of the notation.

Let  $\langle T_0, \prec \rangle$  be the tree of all finite increasing sequences from  $\{\alpha < \kappa^+ : \text{cf}(\alpha) = \kappa\}$ . For every  $\langle \gamma \rangle$  from  $\text{Lev}_{T_0}(1)$ , the first level of  $T_0$ , and for every  $\bar{\eta} \succ \langle \gamma \rangle$  in  $T_0$ , define the ordinal  $\alpha(\bar{\eta})$  to be the supremum of  $\gamma \cap \text{cl}_g(\bar{\eta})$ . Then  $\alpha(\bar{\eta}) < \gamma$  since  $\text{cf}(\gamma) = \kappa$ . By XI Lemma 3.7 of Shelah in [11], we can shrink the tree above  $\langle \gamma \rangle$  to a tree  $T_0^{\langle \gamma \rangle}$  such that every splitting in  $T_0^{\langle \gamma \rangle}$  is still a stationary subset of  $\kappa^+$ , and there exists a  $\gamma_* < \gamma$  such that for every  $\bar{\eta} \in T_0^{\langle \gamma \rangle}$ ,  $\alpha(\bar{\eta}) < \gamma_*$ . Let  $T_1 = \bigcup \{T_0^{\langle \gamma \rangle} : \langle \gamma \rangle \in \text{Lev}_{T_0}(1)\}$ . Repeat the process for each node in  $\text{Lev}_{T_1}(2)$ , and so forth. In this manner, we obtain a tree  $T'$  such that  $\forall \bar{\eta} = \langle \alpha_0, \dots, \alpha_{n-1} \rangle \in T'$ ,

- (i)  $\text{Succ}_{T'}(\bar{\eta})$  is a stationary subset of  $\{\alpha < \kappa^+ : \text{cf}(\alpha) = \kappa\}$ ;
- (ii)  $\exists \alpha(\bar{\eta}) < \alpha_{n-1}$  such that  $\forall \bar{\zeta} \succ_{T'} \bar{\eta}$  in  $T'$ ,  $\alpha_{n-2} \leq \bigcup(\alpha_{n-1} \cap \text{cl}_g(\bar{\zeta})) < \alpha(\bar{\eta})$ .

Using (i), shrink  $T'$  level by level to obtain a  $T$  which satisfies, in addition to (i) and (ii), also

- (iii)  $\exists \beta(\bar{\eta})$  such that  $\forall \gamma \in \text{Succ}_T(\bar{\eta})$ ,  $\beta(\bar{\eta}) = \alpha(\bar{\eta} \frown \langle \gamma \rangle) < \gamma$ .

Note: If  $\bar{\eta} = \langle \alpha_0, \dots, \alpha_{n-1} \rangle$ , then  $\forall \alpha_n \in \text{Succ}_T(\bar{\eta})$ ,

$$(3.3) \quad \alpha_{n-1} \leq \bigcup(\alpha_n \cap \text{cl}_g(\bar{\eta} \frown \langle \alpha_n \rangle)) < \alpha(\bar{\eta} \frown \langle \alpha_n \rangle) = \beta(\bar{\eta}) < \alpha_n.$$

We point out the following useful property of the tree  $T$  and the chosen  $\beta(\bar{\eta})$ 's.

**Fact 3.4.** *If  $b$  is a branch through  $T$  and  $x = \text{cl}_g(b)$ , then for each  $k < \omega$ ,  $b(k) = \min(x \setminus \beta(b \upharpoonright k))$ .*

Firstly,  $b(k) \in b \subseteq x$ . Secondly, for each  $k+1 < n < \omega$ ,  $\bigcup(b(k) \cap \text{cl}_g(b \upharpoonright n)) < \beta(b \upharpoonright k) < b(k)$ , by (ii) and (iii). Therefore,  $[\beta(b \upharpoonright k), b(k)) \cap x = \emptyset$ .

By Lemma 3.1, let  $f : [\kappa^+]^2 \rightarrow \nu$  satisfy  $(*f)$  in  $V$ . We construct two branches  $b, c$  through  $T$  which we will close under  $g$ . The resulting two elements of  $C$  will code an  $\omega$ -sequence unbounded in  $\nu$ , hence not in  $V$ .

Construct a strictly increasing sequence  $\langle b^\delta(0), c^\delta(0) : \delta < \kappa^+ \rangle$  such that

- (a)  $\forall \delta < \kappa^+$ ,  $b^\delta(0), c^\delta(0) \in \text{Succ}_T(\langle \rangle)$ ;
- (b)  $\forall \delta < \varepsilon < \kappa^+$ ,  $b^\delta(0) < c^\delta(0) < b^\varepsilon(0) < c^\varepsilon(0) < \kappa^+$ ;
- (c)  $\forall \delta < \kappa^+$   $b^\delta(0) < \beta(\langle b^\delta(0) \rangle) < c^\delta(0)$ .

Let  $A(0) = \{\{b^\delta(0), c^\delta(0)\} : \delta < \kappa^+\}$ . Break up  $A(0)$  into two disjoint sets each of cardinality  $\kappa^+$ . Since  $(*f)$  holds, there exist  $\delta_0 < \varepsilon_0 < \kappa^+$  such that  $f(b^{\delta_0}(0), c^{\varepsilon_0}(0)) > r(0)$ . Let  $b(0) = b^{\delta_0}(0)$  and  $c(0) = c^{\varepsilon_0}(0)$ . For  $0 < n < \omega$ , choose  $b(n)$  and  $c(n)$  as follows: Given  $b \upharpoonright n$  and  $c \upharpoonright n$ , construct a strictly increasing sequence  $\langle b^\delta(n), c^\delta(n) : \delta < \kappa^+ \rangle$  such that

- (a)  $\beta(c \upharpoonright n) < b^0(n)$ ;
- (b)  $\forall \delta < \kappa^+$ ,  $b^\delta(n) \in \text{Succ}_T(b \upharpoonright n)$ ,  $c^\delta(n) \in \text{Succ}_T(c \upharpoonright n)$ ;
- (c)  $\forall \delta < \varepsilon < \kappa^+$ ,  $b^\delta(n) < c^\delta(n) < b^\varepsilon(n) < c^\varepsilon(n) < \kappa^+$ ;
- (d)  $\forall \delta < \kappa^+$ ,  $b^\delta(n) < \beta((b \upharpoonright n) \frown \langle b^\delta(n) \rangle) < c^\delta(n)$ .

Let  $A(n) = \{\{b^\delta(n), c^\delta(n)\} : \delta < \kappa^+\}$ . Split  $A(n)$  into two disjoint pieces each of cardinality  $\kappa^+$  and apply  $(*f)$  to obtain  $\delta_n < \varepsilon_n < \kappa^+$  such that  $f(b^{\delta_n}(n), c^{\varepsilon_n}(n)) > r(n)$ . Let  $b(n) = b^{\delta_n}(n)$  and  $c(n) = c^{\varepsilon_n}(n)$ .

Now we have branches  $b, c$  through  $T$  such that for each  $n < \omega$ ,  $f(b(n), c(n)) > r(n)$ . Therefore,  $b, c$  code an unbounded function from  $\omega$  into  $\nu$ . Since  $(\text{cf}(\nu))^V > \omega$ , this function must be in  $W \setminus V$ . Since  $f \in V$ , at least one of  $b, c$  must be in  $W \setminus V$ . Let  $x = \text{cl}_g(b)$  and  $y = \text{cl}_g(c)$ .

**Claim.**  *$b$  and  $c$  can be recovered from  $V$ , using  $x$  and  $y$  as oracles.*

*Proof.*  $b(0) = \min(x \setminus \beta(\langle \rangle))$  and  $c(0) = \min(y \setminus \beta(\langle \rangle))$ , by Fact 3.4. Suppose we have  $b(n)$  and  $c(n)$ .  $b(n+1) = \min(x \setminus \beta(b \upharpoonright (n+1)))$ , by Fact 3.4. The  $\beta$  function is not necessarily in  $V$ , since it is constructed using  $g$ . However, we know that  $\beta(b \upharpoonright (n+1)) = \beta((b \upharpoonright n) \frown \langle b^{\delta_n}(n) \rangle) < c^{\delta_n}(n) < c^{\varepsilon_n}(n) = c(n) < b^0(n+1) \leq b(n+1)$ . Therefore,  $b(n+1) = \min(x \setminus c(n))$ .  $c(n+1) = \min(y \setminus \beta(c \upharpoonright (n+1)))$ , by Fact 3.4.  $\beta(c \upharpoonright (n+1)) < b^0(n+1) \leq b(n+1) < c(n+1)$ . Therefore,  $c(n+1) = \min(y \setminus b(n+1))$ .  $\square$

Hence, at least one of  $x, y$  is not in  $V$ . Therefore,  $C \cap (\mathcal{P}_\kappa(\kappa^+) \setminus V) \neq \emptyset$ .  $\square$

*Remark 3.5.* In the proof of Theorem 3.3, it is not necessary for the function  $f$  to satisfy  $(*f)$ , but rather the following weak version:  $f : [\kappa^+]^2 \rightarrow \nu$  is such that for each  $A \subseteq [\kappa^+]^2$  satisfying  $(|A| = \kappa^+ \text{ and } \forall a, b \in A \ a \cap b = \emptyset)$ , for each  $\varepsilon < \nu$  there are  $a, b \in A$  such that  $f(a^0, b^1) > \varepsilon$ . It is open whether this is strictly weaker.

**Main Theorem.** *Let  $V \subseteq W$  be models of ZFC with the same ordinals. Suppose  $\nu$  is regular in  $V$  and is the least cardinal in  $V$  such that there is a new  $\omega$ -sequence  $r : \omega \rightarrow \nu$  in  $W \setminus V$ . Suppose that for each regular  $\kappa \geq \nu$  in  $W$  there is a non-reflecting stationary subset of  $\{\alpha < \kappa^+ : cf(\alpha) = \nu\}$  in  $V$ . Then for all cardinals  $\aleph_1 \leq \kappa < \lambda$  in  $W$  with  $\kappa$  regular in  $W$  and  $\lambda \geq \nu$ ,  $\mathcal{P}_\kappa(\lambda) \setminus V$  is stationary in  $W$ .*

*Proof.* If  $\nu < \aleph_2$ , then  $\nu = 2$  by Remark 2.2. Gitik's Theorem 1.1 then gives the result. So assume  $\nu \geq \aleph_2$ . By Theorem 3.3 and Fact 2.4, for each  $\nu \leq \kappa < \lambda$  in  $W$  with  $\kappa$  regular in  $W$ ,  $\mathcal{P}_\kappa(\lambda) \setminus V$  is stationary in  $W$ . For  $\aleph_1 \leq \kappa < \nu \leq \lambda$ ,  $\mathcal{P}_\kappa(\lambda) \setminus V$  is stationary in  $W$ , by Fact 2.5 along with the fact that  $\nu$  is regular in  $V$ .  $\square$

**Corollary 3.6.** *Suppose  $V \models \text{ZFC}$  and satisfies  $\square_\kappa$  whenever  $\kappa$  is a regular cardinal in  $V$ . Let  $W \supseteq V$  be any model of ZFC with the same ordinals as  $V$ . Suppose  $\nu$  is regular in  $V$  and is the least cardinal in  $V$  such that there is a sequence  $r : \omega \rightarrow \nu$  in  $W \setminus V$ . Then  $r$  achieves global co-stationarity of  $V$  in  $W$ .*

**Example 3.7** (Namba forcing over  $L$ ). Let  $\nu \geq \aleph_2$  be regular, and let  $\mathbb{P}_\nu$  denote Namba forcing on  $\nu$ .  $\mathbb{P}_\nu$  is the collection of subtrees  $p \subseteq (\nu)^{<\omega}$  such that for each node  $t \in p$  above the stem of  $p$ ,  $t$  has  $\nu$ -many immediate successors.  $q \leq p$  iff  $q \supseteq p$ . Namba proved that if  $\nu^\omega = \nu$ , then  $\mathbb{P}_\nu$  adds a new sequence  $r : \omega \rightarrow \nu$ , but for all  $\theta < \nu$ ,  $\mathbb{P}_\nu$  adds no new sequences from  $\omega$  into  $\theta$  [10].

In  $L$ ,  $\square_\kappa$  holds for all cardinals  $\kappa$ . Moreover,  $\nu^\omega = \nu$ , since  $\nu \geq \aleph_2$  is regular and  $L$  satisfies GCH. Hence, by the Main Theorem, for all cardinals  $\aleph_1 \leq \kappa < \lambda$  in  $L^{\mathbb{P}_\nu}$  with  $\kappa$  regular in  $L^{\mathbb{P}_\nu}$  and  $\lambda \geq \nu$ ,  $\mathcal{P}_\kappa(\lambda) \setminus L$  is stationary in  $L^{\mathbb{P}_\nu}$ .

## REFERENCES

1. Uri Abraham and Saharon Shelah, *Forcing closed and unbounded sets*, The Journal of Symbolic Logic **48** (1983), no. 3, 643–657.
2. Keith J. Devlin, *Constructibility*, Springer-Verlag, 1984.
3. Natasha Dobrinen and Sy-David Friedman, *Internal consistency and global co-stationarity of the ground model*, Submitted.
4. \_\_\_\_\_, *Co-stationarity of the ground model*, The Journal of Symbolic Logic **71** (2006), no. 3, 1029–1043.
5. Moti Gitik, *Nonsplitting subsets of  $\mathcal{P}_\kappa(\kappa^+)$* , The Journal of Symbolic Logic **50** (1985), no. 4, 881–894.
6. Thomas Jech, *Set theory*, the 3rd millennium ed., Springer, 2003.
7. David W. Kueker, *Löwenheim-Skolem and interpolation theorems in infinitary languages*, Bulletin of the American Mathematical Society **78** (1972), 211–215.
8. Menachem Magidor, *Representing sets of ordinals as countable unions of sets in the core model*, Transactions of the American Mathematical Society **317** (1990), no. 1, 91–126.
9. Telis K. Menas, *On strong compactness and supercompactness*, Annals of Mathematical Logic **7** (1974/75), 327–359.
10. Kanji Namba, *Independence proof of  $(\omega, \omega_\alpha)$ -distributive law in complete Boolean algebras*, Commentarii Mathematici Universitatis Sancti Pauli **19** (1971), 1–12.
11. Saharon Shelah, *Proper and Improper Forcing*, second ed., Springer-Verlag, 1998.

12. Stevo Todorćević, *Coherent sequences*, Handbook of Set Theory (Matthew Foreman, Akihiro Kanamori, and Menachem Magidor, eds.), Kluwer, To appear.

KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC, WÄHRINGERSTRASSE 25, 1090  
WIEN, AUSTRIA

*E-mail address:* `dobrinen@logic.univie.ac.at`

*URL:* `http://www.logic.univie.ac.at/~dobrinen/`