FAMILIES OF SETS RELATED TO ROSENTHAL'S LEMMA

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Abstract. A family $\mathcal{F} \subseteq [\omega]^\omega$ is called *Rosenthal* if for every Boolean algebra $\mathcal{A}$, bounded sequence $\langle \mu_k\colon\ k \in \omega \rangle$ of measures on $\mathcal{A}$, antichain $\langle a_n\colon\ n \in \omega \rangle$ in $\mathcal{A}$, and $\varepsilon > 0$, there exists $A \in \mathcal{F}$ such that $\sum_{n \in A, n \neq k} \mu_k(a_n) < \varepsilon$ for every $k \in A$. Well-known and important Rosenthal's lemma states that $[\omega]^\omega$ is a Rosenthal family. In this paper we provide a necessary condition in terms of antichains in $\wp(\omega)$ for a family to be Rosenthal which leads us to a conclusion that no Rosenthal family has cardinality strictly less than $\mathrm{cov}(\mathcal{M})$, the covering of category. We also study ultrafilters on ω which are Rosenthal families — we show that the class of Rosenthal ultrafilters contains all selective ultrafilters (and consistently selective ultrafilters comprise a proper subclass).

1. Introduction

Rosenthal's lemma is one of the most fundamental results in vector measure theory with numerous applications to the theory of operators on Banach spaces and the study of weak topologies, cf. e.g. Diestel [9, Chapter VII], Diestel and Uhl [10, Section I.4], Haydon [13, Propositions 1B and 1C], Koszmider and Shelah [18, Lemma 2.2]. The lemma in its particular form reads as follows.

Rosenthal's lemma. *Given an antichain $\langle a_n\colon\ n \in \omega \rangle$ in a Boolean algebra $\mathcal{A}$, a sequence of non-negative finitely additive measures $\langle \mu_k\colon\ k \in \omega \rangle$ on $\mathcal{A}$ satisfying for every $k \in \omega$ the inequality $\sum_{n \in \omega} \mu_k(a_n) \leq 1$, and $\varepsilon > 0$, there exists an infinite set $A \in [\omega]^\omega$ such that for every $k \in A$ the following holds:*

$$\sum_{\substack{n \in A \\ n \neq k}} \mu_k(a_n) < \varepsilon.$$

In this paper we are interested in addressing the following question concerning possible choices of the set A.

Question 1.1. *Can the set A in the conclusion of Rosenthal's lemma be chosen from a previously fixed family $\mathcal{F} \subseteq [\omega]^\omega$?*

An easy analysis of common proofs of the lemma, e.g. of simple Kupka's proof ([20, Lemma 1]), shows that they only appeal to *the numbers* $\mu_k(a_n)$'s, not to *the measures* μ_k's or *elements of the Boolean algebra* $\mathcal{A}$ as such, hence to prove the lemma it is sufficent only to consider the infinite real-entried matrix $\langle m_n^k\colon\ n, k \in \omega \rangle$, where $m_n^k = \mu_k(a_n)$ for each $n, k \in \omega$.

Definition 1.2. An infinite matrix $\langle m_n^k\colon\ n, k \in \omega \rangle$ is called *Rosenthal* if $m_n^k \geq 0$ for every $n, k \in \omega$ and $\sum_{n \in \omega} m_n^k \leq 1$ for every $k \in \omega$.

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Definition 1.3. A non-empty family $\mathcal{F} \subseteq [\omega]^\omega$ is called *Rosenthal* if for every Rosenthal matrix $\langle m_n^k \colon n, k \in \omega \rangle$, and $\varepsilon > 0$, there exists $A \in \mathcal{F}$ such that for every $k \in A$ the following inequality holds:

$$\sum_{\substack{n \in A \\ n \neq k}} m_n^k < \varepsilon.$$

Thus, Question 1.1 asks whether a given family $\mathcal{F} \subseteq [\omega]^\omega$ is Rosenthal (and Rosenthal's lemma asserts that $[\omega]^\omega$ is).

In Section 2 we provide a necessary condition for a family to be Rosenthal in terms of antichains in $\wp(\omega)$. We then use it to prove that no Rosenthal family can be *simpler* than every family of meager subsets covering the real line $\mathbb{R}$, i.e. we prove that no family of cardinality strictly less than $\mathrm{cov}(\mathcal{M})$, the covering of category, is a Rosenthal family (Corollary 2.6).

On the other hand, in Theorem 3.6 of Section 3 we will answer Question 1.1 affirmatively for a family $\mathcal{F}$ being a base of a selective ultrafilter (assuming such an ultrafilter exists). Selective ultrafilters, as well as their weaker variants like P-points and Q-points, constitute an important tool of infinite Ramsey theory or transfinite combinatorics in general; see e.g. Blass [5, 6, 7], Comfort and Negrepontis [8], Grigorieff [12], Laflamme [21] or Laflamme and Leary [22]. However, their existence is independent of ZFC (cf. Section 3).

The converse to Theorem 3.6 does not hold — in Theorem 3.17 under the assumption of Martin's axiom for σ-centered partially ordered sets we construct an example of a P-point ultrafilter which is a Rosenthal family but not a Q-point.

Recalling the result of Baumgartner and Laver [4] stating that in the model obtained by iterating the Sacks forcing there exists a selective ultrafilter with a base of cardinality ω_1 while the continuum $\mathfrak{c}$ is equal to ω_2, we get that consistently there exists a Rosenthal family $\mathcal{F}$ of cardinality strictly less than $\mathfrak{c}$. Since under Martin's axiom every Rosenthal family is of cardinality $\mathfrak{c}$ (Corollary 2.7), we obtain that the existence of Rosenthal families of cardinality strictly less than $\mathfrak{c}$ is undecidable in ZFC+$\neg$CH (Corollary 3.8).

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2. Rosenthal families and $\mathrm{cov}(\mathcal{M})$

In this section we provide a simple necessary (but not sufficient) condition for a subfamily of $[\omega]^\omega$ to be Rosenthal. We start with the following auxiliary definition. Recall that a sequence $\langle a_n \colon n \in \omega \rangle$ of subsets of ω is *an antichain* if $a_n \cap a_m = \emptyset$ for every distinct $n, m \in \omega$.

Definition 2.1. A family $\mathcal{F} \subseteq [\omega]^\omega$ has *the antichain property* if there exists an antichain $\langle a_n \in \wp(\omega) \colon n \in \omega \rangle$ such that for every $A \in \mathcal{F}$ there exists $n \in \omega$ such that $|a_n| \geq 2$ and $a_n \subseteq A$.

Proposition 2.2. *If a family $\mathcal{F} \subseteq [\omega]^\omega$ has the antichain property, then it is not Rosenthal.*

Proof. Assume $\mathcal{F}$ has the antichain property and let $\langle a_n \colon n \in \omega \rangle$ be an antichain witnessing it. We may assume that $|a_n| = 2$ for every $n \in \omega$; denote $a_n = \{p_n, r_n\}$. Define an infinite matrix $\langle m_n^k \colon n, k \in \omega \rangle$ as follows:

$$m_n^k = \begin{cases} 1 & \text{if } \{k, n\} = \{p_l, r_l\} \text{ for some } l \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\langle a_n\colon\ n\in\omega\rangle$ is an antichain, $\langle m_n^k\colon\ n,k\in\omega\rangle$ is a Rosenthal matrix.

Let $A\in\mathcal{F}$ and $a_l=\{p_l,r_l\}\subseteq A$ for some $l\in\omega$. We have:

$$\sum_{\substack{n\in A\\ n\neq k}} m_n^{p_l}=m_{r_l}^{p_l}=1,$$

which proves that $\mathcal{F}$ cannot be Rosenthal. □

Proposition 2.3. *There exists a family $\mathcal{F}\subseteq[\omega]^\omega$ which is not Rosenthal and does not have the antichain property.*

Proof. Let $\langle\langle a_n^\alpha\colon\ n\in\omega\rangle\colon\ \alpha<\mathfrak{c}\rangle$ be an enumeration of all such antichains that $|a_n^\alpha|\geq 2$ for some $n\in\omega$. For every $\alpha<\mathfrak{c}$ let $A_\alpha\in[\omega]^\omega$ be such that $0\in A_\alpha$ and $|A_\alpha\cap a_n^\alpha|\leq 1$ for every $n\in\omega$. Put $\mathcal{F}=\{A_\alpha\colon\ \alpha<\mathfrak{c}\}$.

It is immediate that $\mathcal{F}$ does not have the antichain property. Also, since $0\in\bigcap\mathcal{F}$, $\mathcal{F}$ is not Rosenthal — the matrix $\langle m_n^k\colon\ n,k\in\omega\rangle$ defined as follows witnesses this fact:

$$m_n^k=\begin{cases}1 & \text{if } n=0,\\ 0 & \text{otherwise.}\end{cases}$$

□

Let us define the following cardinal characteristic of the continuum, which we have not encountered so far in the literature.

Definition 2.4. *The antichain number* is defined as follows:

$$\mathfrak{anti}=\min\{|\mathcal{F}|\colon\ \mathcal{F}\subseteq[\omega]^\omega \text{ does } \underline{\text{not}} \text{ have the antichain property}\}.$$

By Proposition 2.3, $\mathfrak{anti}\leq\mathfrak{c}$. The following proposition implies, in particular, that if $\mathcal{F}\subseteq[\omega]^\omega$ does not have the antichain property, then $\mathcal{F}$ is uncountable. Recall that MA_κ(countable) denotes Martin's axiom for countable posets and not more than κ many dense subsets of them.

Proposition 2.5. *Let κ be a cardinal number. Assuming MA_κ(countable), $\mathfrak{anti}>\kappa$.*

Proof. Define a poset $\mathbb{P}$ as follows:

$$\mathbb{P}=\{(a_1,\ldots,a_n)\colon\ n\in\omega,\ a_1,\ldots,a_n\in\wp(\omega)\text{ mutually disjoint pairs}\},$$

where $(a_1,\ldots,a_n)\leq(b_1,\ldots,b_m)$ if $n\geq m$ and $a_i=b_i$ for every $i\leq m$. Then, $\mathbb{P}$ is countable.

Let $\mathcal{F}\subseteq[\omega]^\omega$ be an arbitrary family such that $|\mathcal{F}|\leq\kappa$. We shall show that $\mathcal{F}$ has the pair property. For every $A\in\mathcal{F}$ and every $n\in\omega$ put:

$$D_A=\{(a_1,\ldots,a_m)\in\mathbb{P}\colon\ a_m\subseteq A\},$$

$$E_n=\{(a_1,\ldots,a_k)\in\mathbb{P}\colon\ k\geq n\}.$$

D_A's and E_n's are dense in $\mathbb{P}$. By MA_κ(*countable*), there exists a $\mathbb{P}$-generic ultrafilter G intersecting every D_A and every E_n. Put $g=\bigcup G$. By properties of G, the sequence g witnesses that $\mathcal{F}$ has the pair property. □

Keremedis [16, Theorem 1] (see also Bartoszyński and Judah [3, Theorem 2.4.5]) proved that given a cardinal number κ, MA_κ(countable) holds if and only if $\mathrm{cov}(\mathcal{M})>\kappa$, where $\mathrm{cov}(\mathcal{M})$ denotes the covering of category. Hence, we immediately get that $\mathrm{cov}(\mathcal{M})\leq\mathfrak{anti}$ as well as that no Rosenthal family is of cardinality strictly less than $\mathrm{cov}(\mathcal{M})$.

Corollary 2.6. *If $\mathcal{F}\subseteq[\omega]^\omega$ is a Rosenthal family, then $|\mathcal{F}|\geq\mathrm{cov}(\mathcal{M})$.*

Corollary 2.7. *Assuming* MA_κ*(countable), every Rosenthal family has cardinality* $\mathfrak{c}$*.*

It is easy to see that no base of an ultrafilter may have the antichain property and thus $\mathfrak{anti}$ is bounded from above by the ultrafilter number $\mathfrak{u}$. The following proposition provides a stronger upper bound for $\mathfrak{anti}$ — the reaping number $\mathfrak{r}$.

Proposition 2.8. $\mathfrak{anti} \leq \mathfrak{r}$*.*

Proof. Let $\mathcal{F} \subseteq [\omega]^\omega$ be unsplittable, i.e. for every $B \in [\omega]^\omega$ there exists $A \in \mathcal{F}$ such that one of the sets $A \cap B$ and $A \setminus B$ is finite. Without loss of generality we may assume that if $A \in \mathcal{F}$ and $n \in \omega$, then $A \setminus n \in \mathcal{F}$. Assume that $\mathcal{F}$ has the antichain property, i.e. there exists an antichain $\langle a_n\colon\ n \in \omega\rangle$ such that for every $A \in \mathcal{F}$ there exists $n \in \omega$ for which $a_n \subseteq A$ and $|a_n| \geq 2$. It is immediate that for every $A \in \mathcal{F}$ there exists a subantichain $\langle a_{n_k}\colon\ k \in \omega\rangle$ for which we have $\bigcup_{k\in\omega} a_{n_k} \subseteq A$. For each $n \in \omega$ pick $k_n \in a_n$ and put $B = \{k_n\colon n \in \omega\}$. Then, for every $A \in \mathcal{F}$ both sets $A \cap B$ and $A \setminus B$ are infinite — a contradiction. $\square$

It is also worth of noting that using measure-theoretic methods it can be shown that $\mathfrak{anti} \leq \mathfrak{d}$, where $\mathfrak{d}$ is the dominating number, however, the proof of this fact lies beyond the scope of this paper (see Sobota [24, Propositions 6.5.14 and 6.5.15]). Note that $\min(\mathfrak{r}, \mathfrak{d}) = \min(\mathfrak{u}, \mathfrak{d})$ due to Aubrey [2, Corollary 6.4].

3. Rosenthal families and ultrafilters

In the previous section we have found a necessary condition for a subfamily of $[\omega]^\omega$ to be Rosenthal, namely, such a family cannot have the antichain property. This led us to exclude from being Rosenthal those families which have too simple combinatorics, i.e. those with the cardinality strictly less than $\mathrm{cov}(\mathcal{M})$. In this section we will look for some Rosenthal families which are non-trivial, i.e. much different than $[\omega]^\omega$.

Let $\langle m_n^k\colon\ n, k \in \omega\rangle$ be a Rosenthal matrix and fix $\varepsilon > 0$. Let $\mathcal{F}$ for a moment be a family of all such $A \in [\omega]^\omega$ that:

$$\sum_{\substack{n\in A\\ n\neq k}} m_n^k < \varepsilon$$

for every $k \in A$. Note that if $A, B \in \mathcal{F}$, then $[A]^\omega \subseteq \mathcal{F}$ and $[A \cap B]^\omega \subseteq \mathcal{F}$ (the latter may be empty). Hence, it seems reasonable to look for a non-trivial Rosenthal family among such substructures of $[\omega]^\omega$ like ultrafilters or ideals. Also, the apparent similarity between Rosenthal's lemma and the infinite Ramsey theorem suggests that Ramsey (selective) ultrafilters may be good candidates and — as mentioned in the introductory section — they in fact are.

3.1. Selective ultrafilters.

Recall that an antichain $\mathcal{P} \subseteq \wp(\omega)$ is *a partition of* ω if $\omega = \bigcup \mathcal{P}$. By *an ultrafilter* we always mean a *non-principal* ultrafilter on ω, since principal ultrafilters are never Rosenthal families.

Definition 3.1. An ultrafilter $\mathcal{F}$ is *selective* (or *Ramsey*) if for every partition $\mathcal{P} \subseteq \wp(\omega) \setminus \mathcal{F}$ there is $A \in \mathcal{F}$ such that $|A \cap B| \leq 1$ for every $B \in \mathcal{P}$.

Selective ultrafilters are easy to construct under the Continuum Hypothesis or Martin's axiom, see e.g. Jech [14, Theorem 7.8] or Just and Weese [15, Section 19.3]. On the other hand, Kunen [19] proved that it is consistent that there are no selective ultrafilers.

There are many characterizations of selective ultrafilters, see e.g. Comfort and Negrepontis [8, Theorem 9.6], Argyros and Todorčević [1, Section B.I.1] or Grigorieff

[12, Corollary 16]. We will especially use Grigorieff's characterization in terms of trees.

Definition 3.2. Let $\mathcal{F}$ be an ultrafilter on ω. Let $A \subseteq \omega^{<\omega}$ be a tree. A is an *$\mathcal{F}$-tree* if for every $s \in A$ its *ramification* $\operatorname{ram}(s) = \{n\colon\ s^\frown n \in A\}$ is in $\mathcal{F}$. A branch $H \in \omega^\omega$ of A is an *$\mathcal{F}$-branch* if $\operatorname{ran} H \in \mathcal{F}$.

Definition 3.3. An ultrafilter $\mathcal{F}$ is a *T-ultrafilter* if every $\mathcal{F}$-tree $A \subseteq \omega^{<\omega}$ has an $\mathcal{F}$-branch.

Theorem 3.4 (Grigorieff [12, Corollary 1.15])**.** *Let $\mathcal{F}$ be an ultrafilter on ω. Then, $\mathcal{F}$ is selective if and only if $\mathcal{F}$ is a T-ultrafilter.* □

Before we go to the proof of the main theorem of this section, we prove the following auxiliary lemma.

Lemma 3.5. *Let $\langle m_n^k\colon\ n, k \in \omega\rangle$ be a Rosenthal matrix. Let $\mathcal{F}$ be an ultrafilter, $Y \in \mathcal{F}$ and $\delta > 0$. Then, there exists $Z \in \mathcal{F}$, $Z \subseteq Y$, such that for every $l \in Z$ there exists $X_l \in \mathcal{F}$, $X_l \subseteq Z$, satisfying the following two conditions:*

- *$l < \min(X_l)$, and*
- *$m_l^k < \delta$ for every $k \in X_l$.*

Proof. For every $l \in Y$ put:

$$A_l = \{k \in Y\colon\ k > l \ \&\ m_l^k \geq \delta\} \quad \text{and}$$

$$B_l = \{k \in Y\colon\ k > l \ \&\ m_l^k < \delta\}.$$

Since $Y \in \mathcal{F}$, either $A_l \in \mathcal{F}$ or $B_l \in \mathcal{F}$ (but not both!). Thus, define:

$$A = \{l \in Y\colon\ A_l \in \mathcal{F}\} \quad \text{and}$$

$$B = \{l \in Y\colon\ B_l \in \mathcal{F}\}.$$

Let K be the minimal natural number such that $K\delta > 1$. Then, $|A| < K$. Indeed, if there exist $l_1 < \ldots < l_K$ in A, then there exists $k \in A_{l_1} \cap \ldots \cap A_{l_K} \in \mathcal{F}$ and so $m_{l_1}^k, \ldots, m_{l_K}^k \geq \delta$, whence:

$$1 \geq \sum_{l \in \omega} m_l^k \geq \sum_{i=1}^{K} m_{l_i}^k \geq K\delta > 1,$$

a contradiction.

Let $N = \max(A) + 1$. Then put $Z = B \setminus N = Y \setminus N \in \mathcal{F}$ and $X_l = Z \cap B_l \in \mathcal{F}$ for every $l \in Z$. □

We now prove the main result of this chapter.

Theorem 3.6. *Let $\mathcal{F}$ be a selective ultrafilter and $\mathcal{U}$ its base. Then, $\mathcal{U}$ is a Rosenthal family.*

Proof. Let $\langle m_n^k\colon\ n, k \in \omega\rangle$ be a Rosenthal matrix and $\varepsilon > 0$.

We first construct an $\mathcal{F}$-tree $A \subseteq \omega^{<\omega}$ such that if $s \in A$ and $s^\frown k \in A$ for some $k \in \omega$, then the following conditions are satisfied:

(0) $k > \max(s)$,

(1) $\displaystyle\sum_{n<|s|} m_{s(n)}^k < \varepsilon/2$,

(2) if $s^\frown k \subsetneq t \in A$, then $\displaystyle\sum_{|s|<n<|t|} m_{t(n)}^k < \varepsilon/2$.

Note that if such a tree has been constructed, then its every branch $H \in \omega^\omega$ is increasing (due to the condition (0)) and for every $k \in \omega$ we have:

$$\sum_{\substack{n\in\omega \\ n\neq k}} m_{H(n)}^{H(k)} = \sum_{n<k} m_{H(n)}^{H(k)} + \sum_{n>k} m_{H(n)}^{H(k)} =$$

$$\sum_{n<k} m_{H(n)}^{H(k)} + \lim_{N\to\infty} \sum_{k<n<N} m_{H(n)}^{H(k)} < \varepsilon/2 + \lim_{N\to\infty} \varepsilon/2 = \varepsilon.$$

Let us now build the tree A. The construction will be conducted level by level. Let the 0-th level consist of the empty sequence $\emptyset$. We need to define the ramification $\mathrm{ram}(\emptyset)$, i.e. the 1-st level. Let $Y = \omega$. By Lemma 3.5, there exists a set $Z \in \mathcal{F}$, $Z \subseteq Y$, such that for every $l \in Z$ there is a set $X_{(l)} \in \mathcal{F}$, $X_{(l)} \subseteq Z$, satisfying the following two conditions:

- $l < \min\big(X_{(l)}\big)$, and
- $m_l^k < \varepsilon/2^2$ for every $k \in X_{(l)}$.

Put $\mathrm{ram}(\emptyset) = Z$, i.e. for every $l \in Z$ the 1-element sequence (l) belongs to A. Hence, the 1-st level has been constructed. Note that $\mathrm{ram}(\emptyset) \in \mathcal{F}$ and $X_{(l)} \subseteq \mathrm{ram}(\emptyset)$ for every $l \in \mathrm{ram}(\emptyset)$. The next levels of A will be built in such a way that if $l \in Z$ and $s \in A$ extends (l), then $s(1), \ldots, s(|s|-1) \in X_{(l)}$, whence $m_l^{s(i)} < \varepsilon/2^2$ for every $1 \leq i \leq |s|-1$.

Let $j \geq 1$ and assume we have built the j-th level of A in such a way that for every $s \in \omega^j$ there is a set $X_s \in \mathcal{F}$, $X_s \subseteq \mathrm{ram}\big(s \restriction j-1\big)$, such that the following two conditions are satisfied:

- $s(j-1) < \min\big(X_s\big)$, and
- $m_{s(j-1)}^k < \varepsilon/2^{j+1}$ for every $k \in X_s$,

(i.e. X_s was obtained with the aid of Lemma 3.5). Let thus $s \in \omega^j$ belong to the tree we have built so far; we want to choose $\mathrm{ram}(s) \in \mathcal{F}$. There exists $N \in \omega$ such that $\sum_{n>N} m_n^{s(j-1)} < \varepsilon/2$. Put $Y = X_s \setminus N \in \mathcal{F}$. By Lemma 3.5, there exists $Z \in \mathcal{F}$, $Z \subseteq Y$, such that for every $l \in Z$ there is a set $X_{s^\frown l} \in \mathcal{F}$, $X_{s^\frown l} \subseteq Z$, satisfying the following two conditions:

- $l < \min\big(X_{s^\frown l}\big)$, and
- $m_l^k < \varepsilon/2^{j+2}$ for every $k \in X_{s^\frown l}$.

Put $\mathrm{ram}(s) = Z$, i.e. for every $l \in Z$ the sequence $s^\frown l$ belongs to the being constructed tree A. Hence, the level $j+1$ has been constructed. Note that $\mathrm{ram}(s) \in \mathcal{F}$ and $X_{s^\frown l} \subseteq \mathrm{ram}(s) \subseteq X_s$ for every $l \in \mathrm{ram}(s)$. Also note that $s(j-1) < \min\big(X_s\big) \leq \min\big(\mathrm{ram}(s)\big)$.

Assume we have built the tree A in the way described above. Since $\mathrm{ram}(s) \in \mathcal{F}$ for every $s \in A$, A is an $\mathcal{F}$-tree. We need to check that the conditions (0)–(2) are satisfied. Let $s^\frown k \in A$.

- The condition (0) is satisfied due to the inequalities $s(|s|-1) < \min\big(X_s\big) \leq \min\big(\mathrm{ram}(s)\big)$.
- Since $k \in X_s \subseteq X_{(s(0),\ldots,s(|s|-2))} \subseteq \ldots \subseteq X_{(s(0))}$, we have that $m_{s(n)}^k < \varepsilon/2^{2+n}$ for every $0 \leq n \leq |s|-1$. Thus:

  $$\sum_{n<|s|} m_{s(n)}^k < \sum_{n<|s|} \varepsilon/2^{2+n} < \varepsilon/2,$$

  so the condition (1) is satisfied.

- If $s^\frown k \subsetneq t \in A$, then $t(|s|+1), \ldots, t(|t|-1) \in \operatorname{ram}(s^\frown k)$, so for $N = \min\big(\operatorname{ram}(s^\frown k)\big)$:

  $$\sum_{|s|<n<|t|} m^k_{t(n)} \le \sum_{n>N} m^k_n < \varepsilon/2,$$

  which shows that the condition (2) is satisfied.

Since A is an $\mathcal{F}$-tree and $\mathcal{F}$ is a T-ultrafilter (by Theorem 3.4), there exists an $\mathcal{F}$-branch $H \in \omega^\omega$. For every $k \in \omega$ we have:

$$\sum_{\substack{n\in\omega\\ n\neq k}} m^{H(k)}_{H(n)} < \varepsilon.$$

Let $U \in \mathcal{U}$ be contained in $\operatorname{ran} H$. Then, for every $k \in U$ it obviously holds:

$$\sum_{\substack{n\in U\\ n\neq k}} m^k_n < \varepsilon,$$

and the proof of the theorem is finished. □

Since there are models of ZFC where there exists a selective ultrafilter with a base of cardinality ω_1 and also the equality $\omega_2 = \mathfrak{c}$ holds (e.g. the Sacks model), the existence of "small" Rosenthal families is consistent.

Corollary 3.7. *It is consistent that there exists a Rosenthal family of cardinality ω_1 whereas $\mathfrak{c} = \omega_2$.* □

On the other hand, since under Martin's axiom every Rosenthal family has cardinality $\mathfrak{c}$, we have the following independence result.

Corollary 3.8. *The existence of a Rosenthal family of cardinality strictly less than $\mathfrak{c}$ is undecidable in* ZFC + ¬CH. □

Let us remark that the results from the previous and current sections imply that the minimal cardinality of a Rosenthal family is a cardinal invariant of the continuum. Let us thus introduce the following number.

Definition 3.9. *The Rosenthal number* $\mathfrak{ros}$ is defined as follows:

$$\mathfrak{ros} = \min\big\{|\mathcal{F}|\colon\ \mathcal{F} \subseteq [\omega]^\omega \text{ is a Rosenthal family}\big\}.$$

Let $\mathfrak{u}_s$ denote the minimal size of a base of a selective ultrafilter (or $\mathfrak{c}$ if no such ultrafilter exists).

Corollary 3.10.

(1) $\operatorname{cov}(\mathcal{M}) \le \mathfrak{ros} \le \mathfrak{u}_s$.
(2) Assuming Martin's axiom, $\mathfrak{ros} = \mathfrak{c}$. □

3.2. **P-points and Q-points.** In the previous section we have showed that every base of a selective ultrafilter is a Rosenthal family. Of course, every selective ultrafilter must have this property as well. Let us thus introduce the following class of ultrafilters.

Definition 3.11. An ultrafilter $\mathcal{F}$ is *Rosenthal* if it is a Rosenthal family.

In this section we will show that the class of Rosenthal ultrafilters is broader than the class of selective ones. More precisely, we will show that consistently there are Rosenthal P-points which are not selective. This shows that the converse to Theorem 3.6 does not hold and thus the Rosenthal property does not characterize selective ultrafilters.

Recall the following well-known classes of ultrafilters.

Definition 3.12. An ultrafilter $\mathcal{F}$ is:

- *a P-point* if for every partition $\mathcal{P} \subseteq \wp(\omega) \setminus \mathcal{F}$ of ω there is $A \in \mathcal{F}$ such that $A \cap B$ is finite for every $B \in \mathcal{P}$;
- *a Q-point* if for every partition $\mathcal{P} \subseteq [\omega]^{<\omega}$ of ω there is $A \in \mathcal{F}$ such that $|A \cap B| \leq 1$ for every $B \in \mathcal{P}$.

It is immediate that an ultrafilter is selective if and only if it is simultaneously a P-point and a Q-point. Extensive studies of the classes of ultrafilters may be found in e.g. Just and Weese [15, Section 19.3], Blass [5, 7], Comfort and Negrepontis [8], Laflamme [21] or Laflamme and Leary [22]. Note that the existence of P-points or Q-points is independent of ZFC — all those ultrafilters exist under the assumption of the Continuum Hypothesis or Martin's axiom (see Just and Weese [15, Section 19.3]), but Shelah [25] consistently showed that there are no P-points, and Miller [23] — no Q-points.

We start with lemmas. Recall that given an integer number $p > 0$ *the Ramsey number* $R(p)$ is the minimal number n such that for every 2-colouring $c\colon [n]^2 \to 2$ there exists $X \subseteq n$ such that $|X| = p$ and $c \restriction [X]^2$ is constant. The celebrated Ramsey theorem states that $R(p)$ exists for every $p > 0$ — see Graham, Rothschild and Spencer [11]. Let us call a partition $\langle a_n\colon\ n \in \omega\rangle$ of ω *uniform* if $|a_n| = n$ for every $n \in \omega$.

Lemma 3.13. *Let $\langle m_n^k\colon\ n, k \in \omega\rangle$ be a Rosenthal matrix and $\langle a_n\colon\ n \in \omega\rangle$ a uniform partition of ω. Let $\delta, \gamma \in (0, 1)$. For every integer $N > 1$ there exists integer $r_N > N$ such that for every $a \in [\omega]^{r_N}$ and $A \in [\omega \setminus a]^\omega$ such that $\{|a_n \cap A|\colon\ n \in \omega\}$ is infinite, there are $b \in [a]^N$ and $B \in [A]^\omega$ satisfying the following conditions:*

- $\sum_{n\in b} m_n^k < \delta$ *for every* $k \in B$,
- $\sum_{\substack{n\in b\\ n\neq k}} m_n^k < \delta$ *for every* $k \in b$,
- $\sum_{n\in B} m_n^k < \gamma$ *for every* $k \in b$,
- $\{|a_n \cap B|\colon\ n \in \omega\}$ *is infinite.*

Proof. Let $K \in \omega$ be such a number that $(K-1) \cdot \delta/N > 1$. Define the following numbers:

$$p_N = N \cdot K,$$
$$q_N = R\big(p_N\big),$$
$$r_N = R\big(q_N\big).$$

Clerly, $r_N > N$. We will now show that such defined r_N satisfies the thesis of the lemma. Let thus $a \in [\omega]^{r_N}$ and $A \in [\omega \setminus a]^\omega$ be such that $\{|a_n \cap A|\colon\ n \in \omega\}$ is infinite.

Define a colouring $c\colon [a]^2 \to 2$ in the following way:

$$c(i,j) = \begin{cases} 1 & \text{if } m_i^j < \delta/N, \\ 0 & \text{otherwise.} \end{cases}$$

for every $i < j \in a$. By the Ramsey theorem there exists $X \in [a]^{q_N}$ such that $c \restriction [X]^2$ is constant. If $c \restriction [X]^2 \equiv 0$, then for $j = \max(X)$ we have:

$$1 \geq \sum_{\substack{i\in X\\ i\neq j}} m_i^j \geq (q_N - 1)\cdot\delta/N \geq (p_N - 1)\cdot\delta/N > (K-1)\cdot\delta/N > 1,$$

a contradiction. So $c \restriction [X]^2 \equiv 1$.

Now, similarly as above define a colouring $d\colon [X]^2 \to 2$ (note the swap of the indices i and j with respect to the definition of c!):

$$d(i,j) = \begin{cases} 1 & \text{if } m_j^i < \delta/N, \\ 0 & \text{otherwise.} \end{cases}$$

for every $i < j \in X$. Again, by the Ramsey theorem and the argument as previously, there exists $Y \in [X]^{p_N}$ such that $d \restriction [Y]^2$ is constantly equal to 1.

We will now find the pair (b, B) — the set b will be an element of $[Y]^N$. Note that for every $b \in [Y]^N$ and every $k \in b$ we have:

$$\sum_{\substack{n\in b\\ n\neq k}} m_n^k < \delta.$$

Let $b_1, \ldots, b_N \in [Y]^K$ be pairwise disjoint; it follows that $Y = \bigcup_{i=1}^N b_i$.

Since $|b_1| = K < \infty$, there are $n_1 \in b_1$ and $B_1 \in [A]^\omega$ such that:

- $m_{n_1}^k < \delta/N$ for every $k \in B$, and
- $\{|a_n \cap B_1|\colon\ n \in \omega\}$ is infinite.

Indeed, for every $l \in b_1$ let $C_l \in \wp(A)$ be maximal such that $m_n^k < \delta/N$ for every $k \in C_l$. Since $A = \bigcup_{l\in b_1} C_l$, at least for one $l \in b_1$ the set $\{|a_n \cap C_l|\colon\ n \in \omega\}$ is infinite. Put $n_1 = l$ and $B_1 = C_l$.

Similarly, we can find $n_2 \in b_2, \ldots, n_N \in b_N$ and $B_2 \in [B_1]^\omega, \ldots, B_N \in [B_{N-1}]^\omega$ such that for every $i = 2, \ldots, N$:

- $m_{n_i}^k < \delta/N$ for every $k \in B_i$, and
- $\{|a_n \cap B_i|\colon\ n \in \omega\}$ is infinite.

Put $b = \{n_1, \ldots, n_N\}$. Note that for every $k \in B_N$ we have:

$$\sum_{n\in b} m_n^k < N \cdot \delta/N = \delta.$$

Let $M \in \omega$ be such that for every $i = 1, \ldots, N$ we have:

$$\sum_{n\in B_N\setminus M} m_n^{n_i} < \gamma.$$

Put $B = B_N \setminus M$. □

The following proposition is a generalization of Rosenthal's lemma.

Proposition 3.14. *Let $\langle m_n^k\colon\ n, k \in \omega\rangle$ be a Rosenthal matrix and $\langle a_n\colon\ n \in \omega\rangle$ a uniform partition of ω. Let $\varepsilon > 0$. For every $A \in [\omega]^\omega$ such that $\{|a_n \cap A|\colon\ n \in \omega\}$ is infinite there is $B \in [A]^\omega$ such that $\{|a_n \cap B|\colon\ n \in \omega\}$ is still infinite and for every $k \in B$ the following inequality holds:*

$$\sum_{\substack{n\in B\\ n\neq k}} m_n^k < \varepsilon.$$

Proof. To construct the set B we will inductively use Lemma 3.13. For $N = 2$, $\delta = \varepsilon/2^N$ and $\gamma = \varepsilon/2$, let r_N be as in Lemma 3.13. Since $\{|a_n \cap A|\colon\ n \in \omega\}$ is infinite, there is $M_N \in \omega$ ($M_N \geq r_N$) such that $|a_{M_N} \cap A| \geq r_N$. Let $K_N = \max(a_{M_N}) + 1$. Due the properites of r_M, there exists $b_N \in [a_{M_N} \cap A]^N$ and $B_N \in [A \setminus K_N]^\omega$ such that the following hold:

- $\sum_{n\in b_N} m_n^k < \delta$ for every $k \in B_N$,
- $\sum_{\substack{n\in b_N\\ n\neq k}} m_n^k < \delta$ for every $k \in b_N$,
- $\sum_{n\in B_N} m_n^k < \gamma$ for every $k \in b_N$,
- $\{|a_n \cap B_N|\colon\ n \in \omega\}$ is infinite.

Now, exactly as above, for any $N \geq 3$, $\delta = \varepsilon/2^N$ and $\gamma = \varepsilon/2$, $A = B_{N-1}$, use Lemma 3.13 to obtain r_N, M_N, a_{M_N}, K_N, $b_N \in [a_{M_N} \cap A]^N$ and $B_N \in [A \setminus K_N]^\omega$.

This way, we obtain an antichain $\langle b_N\colon\ N \geq 2\rangle$ in $[A]^{<\omega}$ such that:

(1) $\left|b_N \cap a_{R_N}\right| = N$ for every $N \geq 2$,
(2) for every $N \geq 2$ and $k \in b_N$ we have:

$$\sum_{\substack{n\in b_N\\ n\neq k}} m_n^k < \varepsilon/2^N \quad \text{and} \qquad \sum_{n\in \bigcup\limits_{M>N} b_M} m_n^k < \varepsilon/2,$$

(3) for every $N > M \geq 2$ and $k \in b_N$ we have:

$$\sum_{n\in b_M} m_n^k < \varepsilon/2^M.$$

Put: $B = \bigcup_{N\geq 2} b_N$. By (1) the set $\{|a_n \cap B|\colon\ \in\omega\}$ is infinite. Let $k \in B$ and let N be such that $k \in b_N$. We have:

$$\sum_{\substack{n\in B\\ n\neq k}} m_n^k = \sum_{n\in \bigcup\limits_{2\leq M<N} b_M} m_n^k + \sum_{\substack{n\in B_N\\ n\neq k}} m_n^k + \sum_{n\in \bigcup\limits_{M>N} b_M} m_n^k =$$

$$\sum_{2\leq M<N}\sum_{n\in b_M} m_n^k + \sum_{\substack{n\in B_N\\ n\neq k}} m_n^k + \sum_{n\in \bigcup\limits_{M>N} b_M} m_n^k <$$

$$\sum_{2\leq M<N} \varepsilon/2^M + \varepsilon/2^N + \varepsilon/2 < \varepsilon.$$

□

For a given partition $\mathcal{P}$ of ω, let us say that $C \in [\omega]^\omega$ is *a selector of* $\mathcal{P}$ if either $C \subseteq A$ for some $A \in \mathcal{P}$ or $C \cap A$ is finite for every $A \in \mathcal{P}$.

Lemma 3.15. *Let $\langle a_n\colon\ n\in\omega\rangle$ be a uniform partition of ω and $\mathcal{P} = \langle P_k\colon\ k\in\omega\rangle$ a partition of ω. Assume that for a set $B \in [\omega]^\omega$ the set $\{|a_n \cap B|\colon\ n\in\omega\}$ is infinite. Then, there exists a selector $C \in [B]^\omega$ of $\mathcal{P}$ such that $\{|a_n \cap C|\colon\ n\in\omega\}$ is infinite.*

Proof. If P_k is finite for every $k \in \omega$, then let $C = B$. Otherwise, there exists $k \in \omega$ such that P_k is infinite. Let $b_n = a_n \cap B$. Without loss of generality we may assume that the sequence $\langle |b_n|\colon\ n\in\omega\rangle$ is strictly increasing and $b_0 \neq \emptyset$; it follows that $|b_n| > n$ for every $n \in \omega$.

If there exists a sequence $\langle n_k\colon\ k\in\omega\rangle$ such that for some $l \in \omega$ the set $\{|b_{n_k} \cap P_l|\colon\ k\in\omega\}$ is infinite, then put $C = P_l \cap B$ and we are done.

Otherwise, for every $l \in \omega$ the set $\{|b_n \cap P_l|\colon\ n\in\omega\}$ is finite. We construct the set C inductively. Let $n_0 = 0$ and $c_0 = b_{n_0}$. Assume that for some $l \in \omega$ we have constructed the sequences $c_0, \ldots, c_l$ of finite sets and $n_0, \ldots, n_l \in \omega$ such that:

- $c_i \subseteq b_{n_i}$ for every $0 \leq i \leq l$,
- $|c_i| < |c_j|$ and $K_i \cap K_j = \emptyset$ for every $0 \leq i < j \leq l$, where $K_r = \{k \in \omega : P_k \cap c_r \neq \emptyset\}$.

Let:

$$m = \max_{k\in \bigcup\limits_{0\leq i\leq l} K_i} \left(\max_{n\in\omega} \left|b_n \cap P_k\right|\right)$$

and let $n_{l+1} \in \omega$ be such that:

$$n_{l+1} > |c_l| + m \cdot \sum_{0\leq i\leq l} |K_i|.$$

Define $c_{l+1} \subseteq b_{n_{l+1}}$ as follows:

$$c_{l+1} = b_{n_{l+1}} \setminus \bigcup_{0\leq i\leq l}\bigcup_{k\in K_i} P_k.$$

Then, $\left|c_{l+1}\right| > |c_l|$ and $K_{l+1} \cap \bigcup_{0 \leq i \leq l} K_i = \emptyset$.

Put $C = \bigcup_{n\in\omega} c_n$. Since for every $k \in \omega$ there is at most one $n \in \omega$ such that $P_k \cap c_n \neq \emptyset$, C is a selector of P. The sequence $\left\langle |c_l| \colon\ l \in \omega\right\rangle$ is strictly increasing and for every $l \in \omega$ we have $c_l \subseteq b_{n_l} \subseteq a_{n_l}$, hence the set $\left\{|a_n \cap C| \colon\ n \in \omega\right\}$ is infinite. □

The proof of the following lemma can be found in Just and Weese [15, Lemma 19.32]. Recall that MA(σ-centered) denotes Martin's axiom for σ-centered posets and strictly less than $\mathfrak{c}$ many dense subsets of them.

Lemma 3.16. *Assume* MA(σ*-centered*). *Let* $\langle a_n \colon\ n \in \omega\rangle$ *be a uniform partition of* ω. *Let* $\mathcal{B} \subseteq [\omega]^\omega$ *be such that* $|\mathcal{B}| < \mathfrak{c}$ *and for every finite* $H \subseteq \mathcal{B}$ *the set* $\left\{|\bigcap H \cap a_n| \colon\ n \in \omega\right\}$ *is infinite. Then, there exists a pseudo-intersection* $P \in [\omega]^\omega$ *of* $\mathcal{B}$ *such that* $\left\{|a_n \cap P| \colon\ n \in \omega\right\}$ *is infinite.*

We are in the position to construct a non-selective Rosenthal ultrafilter.

Theorem 3.17. *Assume* MA(σ*-centered*). *Then, there exists a Rosenthal P-point which is not a Q-point.*

Proof. Fix a uniform partition $\langle a_n \colon\ n \in \omega\rangle$ of ω. Denote the following sequences:

- $\langle C_\alpha \colon\ \alpha < \mathfrak{c}\rangle$ — an enumeration of all subsets of ω,
- $\langle \mathcal{P}_\alpha \colon\ \alpha < \mathfrak{c}\rangle$ — an enumeration of all infinite partitions of ω,
- $\langle M_\alpha \colon\ \alpha < \mathfrak{c}\rangle$ — an enumeration of all pairs $\left(\langle m_n^k \colon\ n, k \in \omega\rangle, \varepsilon\right)$, where the first coordinate is a Rosenthal matrix and the second one is a positive real number.

We will construct inductively a sequence $\langle B_\alpha \colon\ \alpha < \mathfrak{c}\rangle$ of infinite subsets of ω such that for every $\alpha < \beta < \mathfrak{c}$ the following hold:

(1) $B_\beta \setminus B_\alpha$ is finite,
(2) either $B_\alpha \subseteq C_\alpha$ or $B_\alpha \cap C_\alpha = \emptyset$,
(3) the set $\left\{|a_n \cap B_\alpha| \colon\ n \in \omega\right\}$ is infinite,
(4) B_α is a selector of $\mathcal{P}_\alpha$.
(5) if $M_\alpha = \left(\langle m_n^k \colon\ n, k \in \omega\rangle, \varepsilon\right)$, then for every $k \in B_\alpha$ we have:

$$\sum_{\substack{n\in B_\alpha\\ n\neq k}} m_n^k < \varepsilon.$$

Having this done, we put:

$$\mathcal{F} = \left\{A \in [\omega]^\omega \colon\ B_\alpha \setminus A \text{ is finite for some } \alpha < \mathfrak{c}\right\}.$$

$\mathcal{F}$ is an ultrafilter by (1) and (2), not a Q-point by (3), a P-point by (4), and a Rosenthal family by (5).

We start as follows. There exists $A \in \{C_0, \omega\setminus C_0\}$ such that $\left\{|a_n \cap A| \colon\ n \in \omega\right\}$ is infinite. By Proposition 3.14, for $M_0 = \left(\langle m_n^k \colon\ n, k \in \omega\rangle, \varepsilon\right)$ there exists $B \in [A]^\omega$ such that $\left\{|a_n \cap B| \colon\ n \in \omega\right\}$ is infinite and for every $k \in B$ we have:

$$\sum_{\substack{n\in B\\ n\neq k}} m_n^k < \varepsilon.$$

Finally, use Lemma 3.15 with $\mathcal{P} = \mathcal{P}_0$ to obtain a selector $C \in [B]^\omega$ of $\mathcal{P}_0$ such that $\left\{|a_n \cap C| \colon\ n \in \omega\right\}$ is infinite. Put: $B_0 = C$.

Let $0 < \beta < \mathfrak{c}$ and assume we have constructed a family $\mathcal{B} = \left\{B_\alpha \colon\ \alpha < \beta\right\}$ such that for every finite $H \subseteq \mathcal{B}$ the set $\left\{|a_n \cap \bigcap H| \colon\ n \in \omega\right\}$ is infinite. By Lemma 3.16, there exists a pseudo-intersection P of $\mathcal{B}$ such that $\left\{|a_n \cap P| \colon\ n \in \omega\right\}$ is infinite.

We now act similarly as in the 0-th step. There is $A \in \{P \cap C_\beta, P \setminus C_\beta\}$ such that $\{|a_n \cap A|\colon\ n \in \omega\}$ is infinite. By Proposition 3.14, for $M_\beta = \big(\langle m_n^k\colon\ n,k \in \omega\rangle, \varepsilon\big)$ there exists $B \in [A]^\omega$ such that $\{|a_n \cap B|\colon\ n \in \omega\}$ is infinite and for every $k \in B$ we have:

$$\sum_{\substack{n \in B\\ n \neq k}} m_n^k < \varepsilon.$$

Finally, use Lemma 3.15 with $\mathcal{P} = \mathcal{P}_\beta$ to obtain a selector $C \in [B]^\omega$ of $\mathcal{P}_\beta$ such that $\{|a_n \cap C|\colon\ n \in \omega\}$ is infinite. Put: $B_\beta = C$. □

Let us finish with the following important issue. We have just proved that the class of selective ultrafilters is consistently a proper subclass of Rosenthal ultrafilters. However, we have been so far unable to obtain an example of an ultrafilter which is not Rosenthal. As this issue is fundamental for the theory of Rosenthal ultrafilters (and Rosenthal families in general), we pose the following question.

Question 3.18. *Is every ultrafilter Rosenthal?*

Let $\mathbb{M}$ be a family of some Rosenthal matrices. We say that an ultrafilter $\mathcal{F}$ is *Rosenthal for* $\mathbb{M}$ if for every $\langle m_n^k\colon\ n,k \in \omega\rangle \in \mathbb{M}$ and $\varepsilon > 0$ there is $A \in \mathcal{F}$ such that:

$$\sum_{\substack{n \in A\\ n \neq k}} m_n^k < \varepsilon.$$

We have several remarks concerning the question.

Remark 3.19. If $\mathcal{F}$ is an ultrafilter which is Rosenthal for the family of all *finitely supported* Rosenthal matrices $\langle m_n^k\colon\ n,k \in \omega\rangle$, i.e. such that the set $\{n\colon m_n^k \neq 0\}$ is finite for every $k \in \omega$, then $\mathcal{F}$ is Rosenthal. Indeed, let $\langle m_n^k\colon\ n,k \in \omega\rangle$ be a Rosenthal matrix and $\varepsilon > 0$. For every $k \in \omega$ there exists N_k such that $\sum_{n > N_k} m_n^k < \varepsilon/2$. Define a new finitely supported Rosenthal matrix $\langle \widehat{m}_n^k\colon\ n,k \in \omega\rangle$ as follows:

$$\widehat{m}_n^k = \begin{cases} m_n^k & \text{if } n \leq N_k,\\ 0 & \text{otherwise.}\end{cases}$$

By the assumption, there is $A \in \mathcal{F}$ such that for every $k \in A$ we have:

$$\sum_{\substack{n \in A\\ n \neq k}} \widehat{m}_n^k < \varepsilon/2,$$

and hence:

$$\sum_{\substack{n \in A\\ n \neq k}} m_n^k = \sum_{\substack{n \in A\\ n \neq k}} \widehat{m}_n^k + \sum_{\substack{n \in A\\ n > N_k}} m_n^k < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Remark 3.20. Every ultrafilter is Rosenthal for the class of all *uniformly finitely supported* Rosenthal matrices $\langle m_n^k\colon\ n,k \in \omega\rangle$, i.e. such that there exists $M \in \omega$ for which $\big|\{n\colon m_n^k \neq 0\}\big| < M$ for all $k \in \omega$. Indeed, let $\mathcal{F}$ be an ultrafilter, $\langle m_n^k\colon\ n,k \in \omega\rangle$ uniformly finitely supported Rosenthal matrix with $M \in \omega$ witnessing that and $\varepsilon > 0$. Define a function $f\colon \omega \to [\omega]^M$ as follows:

$$f(k) = \{n \in \omega\colon m_n^k \neq 0 \text{ and } n \neq k\}.$$

Then, by Hajnal's Free Set Theorem (see e.g. Komjáth and Totik [17, Exercise 26.9]) there exist sets $A_1, \ldots, A_N \in \wp(\omega)$ for some $N \leq 2M+1$ such that $\omega =$

$\bigcup_{i=1}^{N} A_i$ and for every $i \leq N$ and $k \in A_i$ we have $f(k) \cap A_i = \emptyset$, and thus:

$$\sum_{\substack{n \in A_i \\ n \neq k}} m_n^k = 0.$$

Since $\mathcal{F}$ is an ultrafilter, there is $i \leq N$ such that $A_i \in \mathcal{F}$.

Remark 3.21. By the previous two remarks it follows that to answer Question 3.18 it is sufficent to check whether every ultrafilter is Rosenthal for the family of all finitely supported Rosenthal matrices $\langle m_n^k\colon\ n,k \in \omega\rangle$ such that:

$$\sup_{k\in\omega} \big|\{n\colon m_n^k \neq 0\}\big| = \infty.$$

Remark 3.22. Note that there are ZFC examples of non-P-points and non-Q-points. E.g. let $\mathcal{H}$ be a so-called *Fubini product* of two ultrafilters, i.e. given two ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on ω define $\mathcal{H}$ on $\omega \times \omega$ as follows:

$$\mathcal{H} = \Big\{X \in \wp(\omega\times\omega)\colon\ \big\{n \in \omega\colon\ \{m \in \omega\colon\ (n,m) \in X\} \in \mathcal{G}\big\} \in \mathcal{F}\Big\}.$$

Fix a bijection $\omega\times\omega \to \omega$ and identify $\mathcal{H}$ with an ultrafilter on ω. It is a folklore fact that $\mathcal{H}$ is neither a P-point nor a Q-point as well as that $\mathcal{F}$ and $\mathcal{G}$ are both below $\mathcal{H}$ in the sense of Rudin-Keisler order (see Blass [5, page 146]). Is $\mathcal{H}$ a Rosenthal ultrafilter?

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