Fibers of generic maps on surfaces

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Abstract

Using special triangulations of a compact 2-dimensional topological manifold without boundary S, for every closed subset $F \subseteq S$ we construct a dense in the mapping space $C(F,[0,1])$ family of piecewise linear mappings whose fibers consist of components homeomorphic to subcontinua of the figure eight. The number of fibers with a figure-eight component is evaluated for each such map in the case $F = S$. We then prove that every fiber of a generic map in $C(F,[0,1])$ consists only of components being either a singleton or a figure-eight-like hereditarily indecomposable continuum. This extends a result of Z. Buczolich and U.B. Darji.

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1. Introduction

Given any two metric spaces X and Y, by $C(X,Y)$ we denote the space of all continuous mappings from X into Y with the supremum norm. By saying that *a generic continuous function* $f \in C(X,Y)$ *has property* $\mathcal{P}$ we mean the existence of a dense $\mathbb{G}_\delta$ subset $G \subseteq C(X,Y)$ such that every $f \in G$ has the property $\mathcal{P}$. For compact metric spaces and the unit interval $I = [0,1]$ with the natural topology, M. Levin proved the following theorem:

Theorem 1.1 ([1])**.** *Let X be a compact metric space. Then every component of every fiber of a generic $f \in C(X,I)$ is a hereditarily indecomposable continuum.*

A *continuum* is a connected compact metric space. We say that a continuum is *indecomposable* if it cannot be represented as a union of its two proper subcontinua and it is *hereditarily indecomposable* if its every subcontinuum is indecomposable.

Independently of Levin, J. Krasinkiewicz obtained a stronger result:

Theorem 1.2 ([2])**.** *Let X be a compact metric space and M be a manifold of positive dimension. Then every component of every fiber of a generic $f \in C(X,M)$ is a hereditarily indecomposable continuum.*

J. Song and E.D. Tymchatyn generalized the Krasinkiewicz theorem to polygons:

Theorem 1.3 ([3])**.** *Let X be a compact metric space and P be a locally finite polygon. Then every component of every fiber of a generic $f \in C(X,P)$ is a hereditarily indecomposable continuum.*

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These three theorems characterize fibers of a generic map between two given spaces in a very general way – they state only that components of these fibers are hereditarily indecomposable continua. A major step toward a more precise description is the paper by Z. Buczolich and U.B. Darji ([4]), where the characterization is given of components of fibers of a generic map from the 2-dimensional sphere S^2 into the unit interval I in terms of ε-mappings onto *the figure eight* $\mathcal{8}$ – a space homeomorphic to the wedge of two circles $S^1 \vee S^1$. A function $f \in C(X,Y)$ is an *ε-mapping* if the preimage of every point has diameter less than ε. For a finite graph P a continuum X is *P-like* if for every $\varepsilon > 0$ there exists an ε-mapping from X onto P.

Theorem 1.4 (Buczolich, Darji, [4]). *Every component of every fiber of a generic $f \in C(S^2, I)$ is either a singleton or an $\mathcal{8}$-like hereditarily indecomposable continuum.*

This paper generalizes the theorem of Buczolich and Darji to any closed subset of a compact surface (Theorem 3.4). The idea of the proof is based on the technique of Buczolich and Darji, i.e. by constructing appropriate triangulations of a surface we prove the existence of a dense family of continuous mappings with components of fibers being points, circles or eights. The existence of this family is crucial for the proof of the main theorem (Lemma 3.3). A very simple method of triangulating S^2 presented in [4] cannot be directly carried over an arbitrary compact surface, so in this paper we propose a new way of constructing triangulations of compact surfaces. We call those triangulations eight-like (cf. Definition 2.1). They have a series of features common with the triangulation presented in [4], hence the proofs of Lemmas 2.3-2.8 and 2.15 are strongly based on the proofs of corresponding lemmas from [4].

2. Triangulations of surfaces

By a *surface* we always mean a 2-dimensional topological manifold, i.e. a non-empty locally Euclidean, second countable, Hausdorff space. By a triangulation $\mathcal{T}$ of a surface S we mean a pair $(\mathcal{K}, \varphi)$ where $\mathcal{K}$ is an Euclidean simplicial complex and $\varphi : |\mathcal{K}| \to S$ is a homeomorphism from the polygon $|\mathcal{K}|$ induced by $\mathcal{K}$ onto S. For simplicity, we identify the complex $\mathcal{K}$ with its polygon $|\mathcal{K}|$. If $v = \varphi(x)$ for some 0-simplex $x \in \mathcal{K}$ ($\approx |\mathcal{K}|$), then v is a *vertex* of $\mathcal{T}$. We define similarly *edges* and *triangles* of $\mathcal{T}$ as images by φ of respectively 1- and 2-simplices from $\mathcal{K}$. If vertices u and v of $\mathcal{T}$ are distinct end points of an edge e of $\mathcal{T}$, then we denote e simply by uv. The sets of vertices and edges of $\mathcal{T}$ are denoted by $\mathcal{V}$ and $\mathcal{E}$, respectively. The set of triangles of $\mathcal{T}$ is identified with $\mathcal{T}$ itself, i.e. $t \in \mathcal{T}$ means that t is a triangle of $\mathcal{T}$. Given a vertex $v \in \mathcal{V}$, we denote by $N(v)$ the set of all neighbours of v in $\mathcal{T}$, i.e. $N(v) = \{w \in \mathcal{V} : \ vw \in \mathcal{E}\}$. $V(e)$ and $V(t)$ denote, respectively, the 2-element set of end points of an edge $e \in \mathcal{E}$ and the 3-element set of vertices of a triangle $t \in \mathcal{T}$. Given a graph G, the *degree* of a vertex v of G, denoted by $\deg_G v$, is the cardinality of $N(v)$ in G.

Definition 2.1. *A triangulation $\mathcal{T}$ is eight-like if every $v \in \mathcal{V}$ has degree greater than* 3 *and there exists a* 3*-colouring $c : \mathcal{V} \to \{-, 0, +\}$ of vertices of $\mathcal{T}$ satisfying the following conditions:*

1. *for every $v \in \mathcal{V}$:*
   - *if $\deg_{\mathcal{T}} v \neq 4, 6$, then $c(v) \neq 0$;*
   - *if $\deg_{\mathcal{T}} v = 4$ and $c(v) = 0$, then there exist distinct $x, y \in N(v)$ such that $c(x), c(y) \in \{-, +\}$;*
   - *if $\deg_{\mathcal{T}} v = 6$ and $c(v) = 0$, then there exist distinct $x, y, z \in N(v)$ such that $c(\{x, y, z\}) = \{-, +\}$;*

2. *every edge* $e \in \mathcal{E}$ *has at least one vertex coloured by* 0*, i.e.* $0 \in c(V(e))$*;*
3. *every triangle* $t \in \mathcal{T}$ *has a vertex coloured by either* $-$ *or* $+$*, i.e.* $c(V(t)) \cap \{-,+\} \neq \varnothing$ *(note that by the second condition:* $|c(V(t))| = 2$*).*

We say that c *is associated with* $\mathcal{T}$*.*

Let S be a *closed* surface (i.e. compact and without boundary). If $\mathcal{T}$ is a triangulation of S, then every triangle $t \in \mathcal{T}$ is homeomorphic to the triangle $T \subset \mathbb{R}^2$ spanned on the vertices $(0,0)$, $(1,0)$ and $(0,1)$. Fix a homeomorphism $\varphi_t : T \to t$.

Definition 2.2. *A mapping* $f \in C(S,I)$ *is* $\mathcal{T}$*-triangular for a triangulation* $\mathcal{T}$ *of* S *if* $f|_{\mathcal{V}} : \mathcal{V} \to I$ *is one-to-one and for every* $t \in \mathcal{T}$ *the mapping* $f \circ \varphi_t : T \to I$ *is linear.*

A simple consequence of this definition is the following

Lemma 2.3. *Let* $f \in C(S,I)$ *be* $\mathcal{T}$*-triangular for a triangulation* $\mathcal{T}$ *of* S*. Let* $y \in I$ *and* $t \in \mathcal{T}$*. Assume* $f^{-1}(y) \cap t \neq \varnothing$*. Then exactly one of the following holds:*

- $f^{-1}(y) \cap t = \{v\}$ *for some vertex* $v \in \mathcal{V}$*;*
- $f^{-1}(y) \cap t$ *is an arc containing exactly one vertex* $v \in \mathcal{V}$ *and joining* v *with the side of* t *opposite to* v*;*
- $f^{-1}(y) \cap t$ *is an arc intersecting two sides of* t *and containing no vertex of* $\mathcal{T}$*.*

Since we consider only finite triangulations, from Lemma 2.3 we immediately get

Corollary 2.4. *Let* $f \in C(S,I)$ *be* $\mathcal{T}$*-triangular for a triangulation* $\mathcal{T}$ *of* S*. For every* $y \in I$*, the fiber* $f^{-1}(y)$ *has finitely many components, each of which is a graph.*

We now introduce the notion of an extremal function, which is crucial for this paper:

Definition 2.5. *Let* $\mathcal{T}$ *be a triangulation of* S *and* $c : \mathcal{V} \to \{-,0,+\}$ *a 3-colouring of its vertices. A* $\mathcal{T}$*-triangular function* $f : S \to I$ *is c-extremal if for every* $v \in \mathcal{V}$ *the condition* $c(v) = +$ *(resp.* $c(v) = -$*) holds if and only if there is a local maximum (resp. minimum) of* f *at* v*.*

Condition 1 from Definition 2.1 implies the following

Lemma 2.6. *Let* $\mathcal{T}$ *be an eight-like triangulation of* S *and* $c : \mathcal{V} \to \{-,0,+\}$ *be a colouring associated with* $\mathcal{T}$*. Let a* $\mathcal{T}$*-triangular function* $f : S \to I$ *be c-extremal. Then, if* $f(v) = 0$ *for* $v \in \mathcal{V}$*, then there exist distinct* $v_1, v_2 \in N(v)$ *and distinct* $t_1, t_2 \in \mathcal{T}$ *such that* $v_i \in \mathcal{V}(t_i)$ *and* f *has a local extremum at* v_i *(*$i = 1,2$*).*

The proof of the following important lemma strictly follows the proof of Lemma 5.4 from [4].

Lemma 2.7. *Let a triangulation* $\mathcal{T}$ *of* S *be eight-like and* $c : \mathcal{V} \to \{-,0,+\}$ *be a colouring associated with it. Let* $f \in C(S,I)$ *be a* $\mathcal{T}$*-triangular c-extremal function. For every* $y \in f(S)$*, every component* M *of the fiber* $f^{-1}(y)$ *is homeomorphic to one of the following three spaces: a point, the circle* S^1 *and the figure eight* $\mathcal{8}$*. Moreover, every fiber* $f^{-1}(y)$ *has at most one component which is non-homeomorphic to* S^1*.*

Proof. Let M be a component of $f^{-1}(y)$ for some $y \in f(S)$. Let us consider the following cases:

1. If $M \cap \mathcal{V} = \emptyset$, then by Lemma 2.3, for every $t \in \mathcal{T}$, the intersection $M \cap t$ is empty or is an arc. Thus every point x of the graph M has degree 2. Theorem 9.6 [5], stating that a continuum N is homeomorphic to the circle S^1 if and only if every point of N has degree 2, implies that M is homeomorphic to S^1.

2. If $M \cap \mathcal{V} = \{v\}$ and $c(v) \neq 0$, then f has a local extremum at v. Hence $M = \{v\}$.

3. Let $M \cap \mathcal{V} = \{v\}$ and $c(v) = 0$. By the injectivity of $f|_{\mathcal{V}}$ and Lemma 2.3, it follows that every point $x \in M \setminus \{v\}$ has degree 2 in M. Let us compute $\deg_M v$ with respect to $\deg_{\mathcal{T}} v$:

a) Let $\deg_{\mathcal{T}} v = 4$. By Lemma 2.6, there exist distinct $v_1, v_3 \in N(v)$ such that $c(v_1), c(v_3) \in \{-,+\}$. The second condition of Definition 2.1 guarantees that v_1 and v_3 are not neighbours. Let v_2 and v_4 be distinct from v_1 and v_3 neighbours of v (Figure 1a).
Let us first assume that $c(v_1) = c(v_3) = +$. If $f(v) > f(v_2)$, then $f^{-1}(y)$ intersects the edges v_1v_2 and v_2v_3. If $f(v) < f(v_2)$, then $f^{-1}(y)$ intersects neither v_1v_2 nor v_2v_3. Similarly, if $f(v) > f(v_4)$, then $f^{-1}(y)$ intersects the edges v_1v_4 and v_3v_4, but intersects none of them if $f(v) < f(v_4)$. Inequalities $f(v) < f(v_2)$ and $f(v) < f(v_4)$ cannot hold simultaneously, since $c(v) = 0$ and f is c-extremal. Thus $\deg_M v \in \{2,4\}$.
We prove analogously that $\deg_M v \in \{2,4\}$ when $c(v_1) = c(v_3) = -$ and that $\deg_M v = 2$ when $c(v_1) \neq c(v_3)$.

b) Let $\deg_{\mathcal{T}} v = 6$. By Condition 1 of Definition 2.1 there exist distinct vertices $v_1, v_3, v_5 \in N(v)$ such that $c(v_1) = c(v_3) \neq c(v_5) \in \{-,+\}$. Without loss of generality we assume that $c(v_5) = -$ (other cases are symmetric). Denote the remaining neighbours of v by v_2, v_4, v_6 (Figure 1b).
If $f(v) > f(v_2)$, then $f^{-1}(y)$ intersects the edges v_1v_2 and v_2v_3. If $f(v) < f(v_2)$, then $f^{-1}(y)$ intersects none of them. If $f(v) > f(v_4)$, then $f^{-1}(y)$ intersects the edge v_3v_4 but not the edge v_4v_5. Similarly, if $f(v) < f(v_4)$, then $f^{-1}(y)$ intersects the edge v_4v_5 but not the edge v_3v_4. Analogously, we analyse the value $f(v_6)$. Thus $\deg_M v \in \{2,4\}$.

If $\deg_M v = 2$, then M is homeomorphic to the circle S^1. If $\deg_M v = 4$, then M is homeomorphic to the figure eight $\wp$ (cf. [4, p. 240]). □

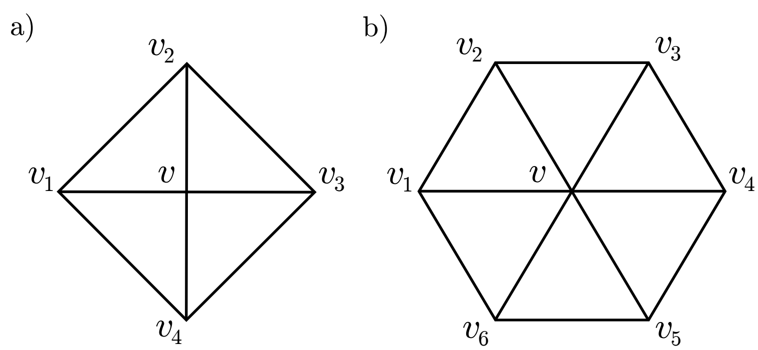

Figure 1: a) $\deg_{\mathcal{T}} v = 4$, b) $\deg_{\mathcal{T}} v = 6$

From the finiteness of considered triangulations and the proof of the above lemma we immediately get the following

Corollary 2.8. *Let $f \in C(S, I)$ be a $\mathcal{T}$-triangular c-extremal function for some eight-like triangulation $\mathcal{T}$ and for colouring $c : \mathcal{V} \to \{-, 0, +\}$ associated with $\mathcal{T}$. Then the set of all those points $y \in f(S)$ that $f^{-1}(y)$ has a component homeomorphic to a point or to the figure eight $\mathcal{E}$ is finite.*

2.1. Construction of eight-like triangulations

In the following section we prove that every closed surface S admits an arbitrary small eight-like triangulation, i.e. for every $\varepsilon > 0$ there is an eight-like triangulation $\mathcal{T}$ such that every $t \in \mathcal{T}$ has diameter less than ε.

Let us recall the following well-known classification of closed surfaces:

Theorem 2.9. *Every connected closed surface is homeomorphic to exactly one of the following ($m \geq 1$):*

1. *the sphere S^2;*
2. *the connected sum of m tori $\mathbb{T}^2$;*
3. *the connected sum of m real projective planes $\mathbb{R}P^2$.*

An elegant proof of Theorem 2.9 can be found e.g. in [6, Chapter 6]. A very important step in the proof is an observation that every closed surface S can be obtained by appropriate "gluing" edges of an even-sided regular polygon $P \subset \mathbb{R}^2$ called a *fundamental polygon of S* (cf. [6, Chapter 6]). This observation allows us to restrict our attention only to triangulations of planar regular polygons.

Let then S be a closed surface and P its fundamental n-gon where $n \geq 4$ is divisible by 4. Assume P is contained in $\mathbb{R}^2 \approx \mathbb{C}$ and its vertices are

$$v_l = \exp\left(i\pi \frac{2l+1}{n}\right) \quad \text{for} \quad l = 0, 1, \ldots, n-1,$$

i.e. v_l's are all the n-th complex roots of -1.

Let us fix $\varepsilon > 0$. The polygon P is compact, hence the quotient map gluing edges is uniformly continuous, so it is sufficient to construct an eight-like ε-triangulation of P. We proceed with the construction in several steps.

Step 1. Let $N \in \mathbb{N}$ be divisible by 4 and such that $1/N < \varepsilon/16$. For every $k = 1, 2, \ldots, N$ let P_k denote the boundary of the regular n-gon spanned by the vertices

$$v_l^k = \frac{k}{N} \exp\left(i\pi \frac{2l+1}{n}\right) \quad \text{for} \quad l = 0, 1, \ldots, n-1.$$

Notice that

$$P_N = \mathrm{bd}_{\mathbb{R}^2}(P) \quad \text{and} \quad v_l^N = v_l \quad \text{for} \quad l = 0, 1, \ldots, n-1.$$

The polygonal curves P_k divide P into $N-1$ n-gonal annuli and one n-gon containing the point $(0,0)$ (Figure 2). The length of a side of each P_k is $\frac{2k}{N} \sin \frac{\pi}{n}$.

Step 2. On every P_k we mark counterclockwisely, equidistantly, $(k+2)n$ points u_m^k, $m = 0, 1, \ldots, (k+2)n-1$, in such a way that $v_0^k = u_0^k$ (Figure 3). Notice that every side of every P_k is divided into $k+2$

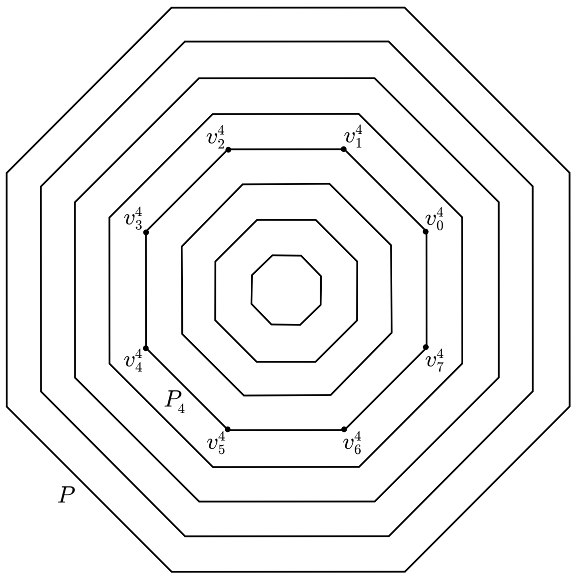

Figure 2: The n-gons P_k

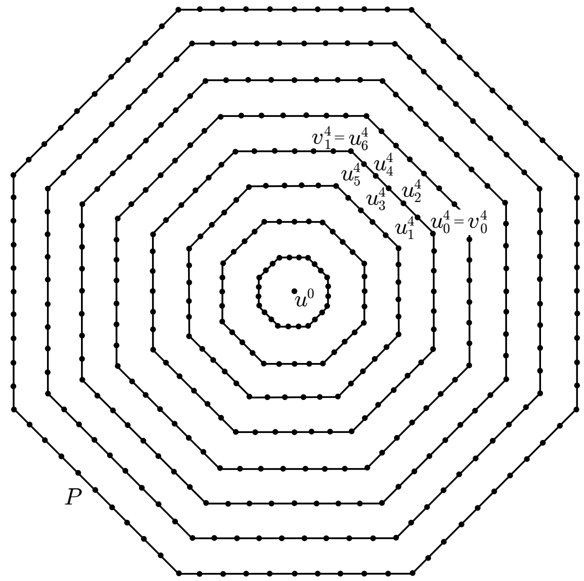

Figure 3: Vertices of the triangulation $\mathcal{T}$

equal segments and that $v_l^k = u_{l(k+2)}^k$ for $l = 0, 1, \ldots, n-1$. We denote

$$u_{(k+2)n}^k \coloneqq u_0^k, \quad u_{-1}^k \coloneqq u_{(k+2)n-1}^k \quad \text{and} \quad u^0 \coloneqq (0,0).$$

The distance beetwen two adjacent points u_m^k and u_{m+1}^k is $\frac{2k}{(k+2)N} \sin \frac{\pi}{n}$, which is less than $\varepsilon/2$.

We declare the points u_m^k and u^0 to be the vertices of the eight-like triangulation we are constructing, i.e. put

$$\mathcal{V} \coloneqq \left\{ u_m^k \;:\; k = 1, 2, \ldots, N;\; m = 0, 1, \ldots, (k+2)n - 1 \right\} \cup \left\{ u^0 \right\}.$$

An initial set of edges is defined as follows:

$$\tilde{\mathcal{E}} \coloneqq \{ \overline{u_m^k u_{m+1}^k} \;:\; k = 1, 2, \ldots, N;\; m = 0, 1, \ldots, (k+2)n - 1 \},$$

where $\overline{ab}$ denotes the shortest segment contained in P joining points a and b. Notice that every $\overline{u_m^k u_{m+1}^k}$ is contained in P_k. The set $\tilde{\mathcal{E}}$ will be extended in the next steps.

The graph $(\mathcal{V}, \tilde{\mathcal{E}})$ is a disconnected graph consisting of N disjoint cyclic graphs Q_k, $k = 1, 2, \ldots, N$ (inducing in fact triangulations of the polygonal curves P_k) and a one-vertex graph Q_0 corresponding to the point u^0.

Step 3. In this step we add to $\tilde{\mathcal{E}}$ edges joining vertices of the graphs Q_{k-1} and Q_k for $k = 1, 2, \ldots, N$. We obtain a connected graph $(\mathcal{V}, \mathcal{E})$ inducing an eight-like ε-triangulation $\mathcal{T}$.

First we add to $\tilde{\mathcal{E}}$ edges joining vertices $u_m^1 \in V(Q_1)$, for $m = 0, 1, \ldots, 3n-1$, and the vertex u^0, i.e.

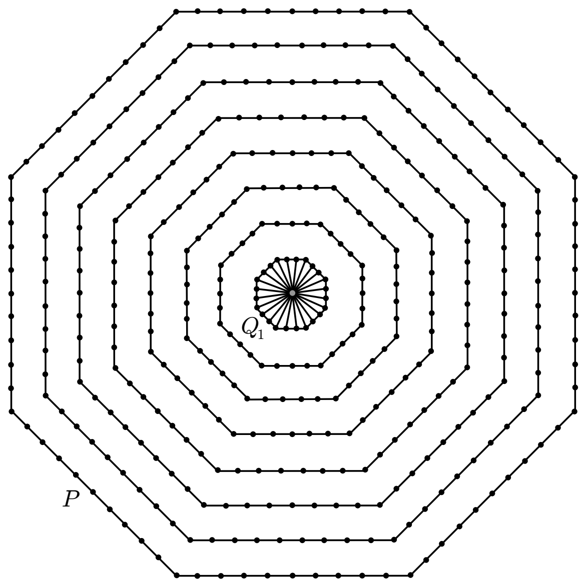

Figure 4: Edges of $\mathcal{E}_0$ joining vertices of Q_1 and the vertex u^0

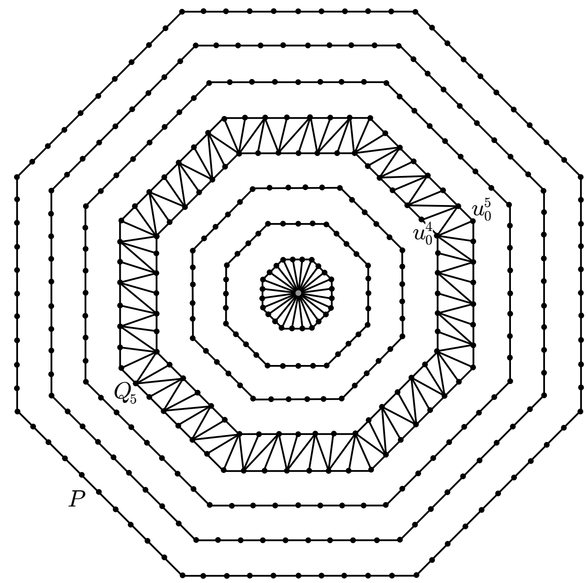

Figure 5: Edges of $\mathcal{E}_1$ joining vertices of Q_5 with vertices of Q_4

the segments $\overline{u_m^1 u^0}$ (Figure 4). Denote the obtained set of edges by $\mathcal{E}_0$. Since the method of adding edges joining vertices of Q_k and Q_{k-1}, for $k = 2, 3, \ldots, N$, depends on the remainder of division of k by 4, we proceed with the construction in the following four substeps (notice that the index l represents the number of a side of the polygon bounded by P_k and the index j enumerates the vertex lying on this side):

$k \equiv 1 \mod 4$ (Figure 5):

$$\begin{aligned}
\mathcal{E}_1 &:= \{\overline{u^k_{(k+2)l+j} u^{k-1}_{(k+1)l+j}} :\ l = 0, 1, \ldots, n-1;\ j = 0, 1, 2, \ldots, k+1\} \\
&\cup \{\overline{u^k_{(k+2)l+j} u^{k-1}_{(k+1)l+j-1}} :\ l = 0, 1, \ldots, n-1;\ j = 1, 3, 5, \ldots, k\} \\
&\cup \{\overline{u^k_{(k+2)l+j} u^{k-1}_{(k+1)l+j+1}} :\ l = 0, 1, \ldots, n-1;\ j = 1, 3, 5, \ldots, k\} \\
&\cup \mathcal{E}_0
\end{aligned}$$

$k \equiv 2 \mod 4$ (Figure 6):

$$\begin{aligned}
\mathcal{E}_2 &:= \{\overline{u^k_{(k+2)l+j} u^{k-1}_{(k+1)l+j-1}} :\ l = 0, 1, \ldots, n-1;\ j = 0, 1, 2, \ldots, k+1\} \\
&\cup \{\overline{u^k_{(k+2)l+j} u^{k-1}_{(k+1)l+j-2}} :\ l = 0, 1, \ldots, n-1;\ j = 2, 4, 6, \ldots, k\} \\
&\cup \{\overline{u^k_{(k+2)l+j} u^{k-1}_{(k+1)l+j}} :\ l = 0, 1, \ldots, n-1;\ j = 0, 2, 4, \ldots, k\} \\
&\cup \mathcal{E}_1
\end{aligned}$$

$k \equiv 3 \mod 4$ (Figure 7):

$$
\begin{aligned}
\mathcal{E}_3 &:= \{\overline{u^k_{(k+2)l+j}u^{k-1}_{(k+1)l+j-1}} :\ l = 0,1,\dots,n-1;\ j = 1,2,3,\dots,k+1\} \\
&\cup \{\overline{u^k_{(k+2)l+j}u^{k-1}_{(k+1)l+j-2}} :\ l = 0,1,\dots,n-1;\ j = 2,4,6,\dots,k+1\} \\
&\cup \{\overline{u^k_{(k+2)l+j}u^{k-1}_{(k+1)l+j}} :\ l = 0,1,\dots,n-1;\ j = 0,2,4,6,\dots,k+1\} \\
&\cup \mathcal{E}_2
\end{aligned}
$$

$k \equiv 0 \mod 4$ (Figure 8):

$$
\begin{aligned}
\mathcal{E}_4 &:= \{\overline{u^k_{(k+2)l+j}u^{k-1}_{(k+1)l+j-1}} :\ l = 0,1,\dots,n-1;\ j = 2,4,6,\dots,k\} \\
&\cup \{\overline{u^k_{(k+2)l+j}u^{k-1}_{(k+1)l+j}} :\ l = 0,1,\dots,n-1;\ j = 0,1,2,\dots,k+1\} \\
&\cup \{\overline{u^k_{(k+2)l+j}u^{k-1}_{(k+1)l+j+1}} :\ l = 0,1,\dots,n-1;\ j = 0,2,4,6,\dots,k\} \\
&\cup \mathcal{E}_3
\end{aligned}
$$

Put $\mathcal{E} := \mathcal{E}_4$.

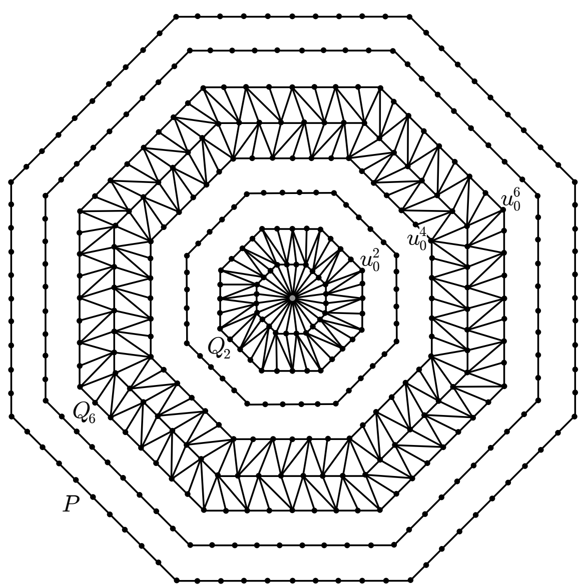

Figure 6: Edges of $\mathcal{E}_2$ joining Q_2 with Q_1 and Q_6 with Q_5

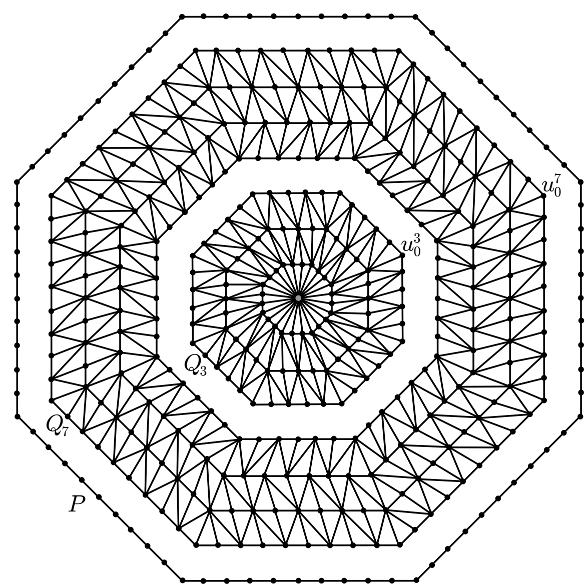

Figure 7: Edges of $\mathcal{E}_3$ joining Q_3 with Q_2 and Q_7 with Q_6

Lemma 2.10. *The graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ induces a triangulation $\mathcal{T}$ of S.*

Proof. Let us make the following three observations:

- the boundary of P is triangulated by Q_N, which is a subgraph of $\mathcal{T}$;
- vertices of P belong to $\mathcal{V}$;
- vertices of Q_N are in equal distance from each other and on each side of P there is the same number of vertices of Q_N.

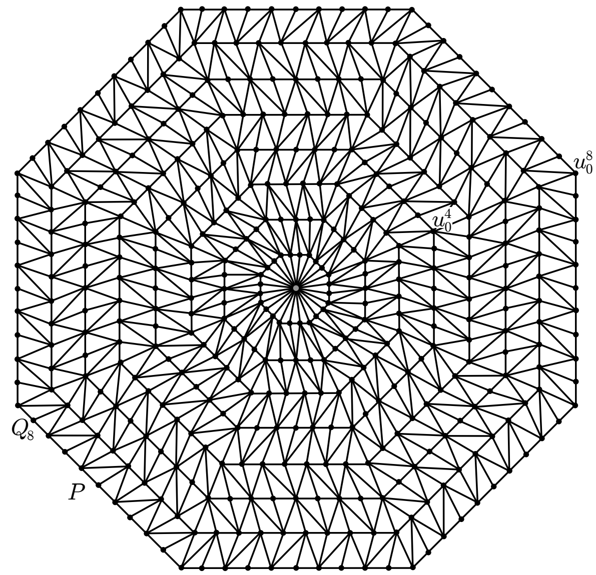

Figure 8: Edges of $\mathcal{E}_4$ joining Q_4 with Q_3 and Q_8 with Q_7

Since the quotient map $P \to S$ glues edges linearly, this means that if $\mathcal{T}$ is a triangulation of P, then it is a triangulation of S.

We need several more simple facts concerning the construction:

- the vertex u^0 of Q_0 is joined only with vertices of Q_1;
- vertices of Q_k are joined only with vertices of Q_{k-1}, Q_k and Q_{k+1} ($k = 1, 2, \dots, N-1$);
- vertices of Q_N are joined only with vertices of Q_N and Q_{N-1};
- every vertex of Q_k is joined with a vertex of Q_{k-1} ($k = 1, 2, \dots, N$);
- every vertex of Q_k is joined with a vertex of Q_{k+1} ($k = 0, 2, \dots, N-1$).

It follows from the above facts that in order to show that P is triangulated by $\mathcal{T}$, it is sufficient to observe that $\mathcal{G}$ induces triangulations of polygonal annuli bounded by P_k and P_{k-1}, triangulated themselves by Q_k and Q_{k-1} ($k = 1, 2, \dots, N$). The case of $k = 1$ is obvious. The case of $k > 1$ follows from the definition of $\mathcal{E}_i$ where $k \equiv i \mod 4$. Let $k > 1$ and let u_p^{k-1} be joined with a vertex u_q^k. Then the vertex u_{p+1}^{k-1} is joined with u_q^k or u_{q+1}^k. Indeed, if it is not joined with u_q^k, then u_p^{k-1} and u_{p+1}^{k-1} are both joined with u_{q+1}^k. Hence, since every vertex of Q_{k-1} is joined with a vertex of Q_k, for every two consecutive vertices u_p^{k-1} and u_{p+1}^{k-1}, there exists a triangle in $\mathcal{G}$ contained in the annulus bounded by P_{k-1} and P_k and containing those two vertices. Moreover, since the conjuction "u_p^{k-1} *is joined with* u_q^k and u_{p+1}^{k-1} *is joined with* u_{q-1}^k" never holds, the intersection of any two distinct edges $e, e' \in \mathcal{E}$ is contained in $\mathcal{V}$. This altogether implies that $\mathcal{T}$ triangulates the annulus bounded by P_k and P_{k-1}. □

Lemma 2.11. *$\mathcal{T}$ is an ε-triangulation.*

Proof. As we noticed in the second step of the construction of $\mathcal{T}$, the distance between two adjacent vertices in every Q_k is less than $\varepsilon/2$. Hence, by the triangle inequality, it is enough to show that every edge joining Q_k and Q_{k-1} has length less than $\varepsilon/2$.

The cases of edges joining vertices of Q_1 and u^0 and of edges $\overline{u^k_{(k+2)l}u^{k-1}_{(k+2)l-1}}$, for $k > 0$ such that $k \equiv 2 \mod 4$, are easy. For simplicity of the proof, we present only the case of edges joining vertices contained in the l-th sides of Q_k and Q_{k-1} for l such that $n = 4l$. These are sides parallel to the x-axis, contained in the upper half-plane and vertices of $\mathcal{T}$ laying on them have coordinates:

$$\left(\frac{k}{N}\cos\left(\pi\frac{2l+1}{n}\right) + \frac{2kj}{(k+2)N}\sin\frac{\pi}{n},\ \frac{k}{N}\sin\left(\pi\frac{2l+1}{n}\right)\right) \quad \text{for} \quad Q_k,$$
$$\left(\frac{k-1}{N}\cos\left(\pi\frac{2l+1}{n}\right) + \frac{2(k-1)j}{(k+1)N}\sin\frac{\pi}{n},\ \frac{k-1}{N}\sin\left(\pi\frac{2l+1}{n}\right)\right) \quad \text{for} \quad Q_{k-1}.$$

Let $1 < k \le N$ and $2 \le j \le k+1$. Accordingly to the construction of $\mathcal{T}$ it is enough to estimate the distance from the vertex $u^k_{(k+2)l+j}$ to the vertices $u^{k-1}_{(k+1)l+j-2}$, $u^{k-1}_{(k+1)l+j-1}$, $u^{k-1}_{(k+1)l+j}$ and $u^{k-1}_{(k+1)l+j+1}$. Estimating the square of the distance between $u^k_{(k+2)l+j}$ and $u^{k-1}_{(k+1)l+j-2}$ we get:

$$\left|u^k_{(k+2)l+j} - u^{k-1}_{(k+1)l+j-2}\right| \le \frac{1}{N} + \frac{\sqrt{16}}{N} + \frac{\sqrt{8}}{N} < \frac{1}{N} + \frac{4}{N} + \frac{3}{N} = \frac{8}{N} < \frac{\varepsilon}{2}.$$

Analogously, the distances from $u^k_{(k+2)l+j}$ to $u^{k-1}_{(k+1)l+j-1}$, $u^{k-1}_{(k+1)l+j}$ and $u^{k-1}_{(k+1)l+j+1}$, for $1 \le j \le k+1$, can be estimated. The case of $j = 0$ may be considered separately. □

Let us now define a 3-colouring $c : \mathcal{V} \to \{-, 0, +\}$ satisfying conditions of Definition 2.1. Let $k \in \{1, 2, \dots, N\}$ and $m \in \{0, 1, \dots, (k+2)n - 1\}$. Put:

$$c\left(u^0\right) := +,$$
$$c\left(u^k_m\right) := \begin{cases} 0\ , & k \text{ odd}, \\ +\ , & k \equiv 0 \mod 4,\ m \text{ even}, \\ -\ , & k \equiv 2 \mod 4,\ m \text{ even}, \\ 0\ , & k \text{ even},\ m \text{ odd}. \end{cases}$$

Checking that c defined as above satisfies conditions of Definition 2.1 is easy (Figure 9). Thus we get

Lemma 2.12. *The triangulation $\mathcal{T}$ is eight-like.*

Repeating the method from the proof of Lemma 2.7 and using the preceding constructions of the triangulation $\mathcal{T}$ and the 3-colouring c, we can easily compute the number of fibers of a c-extremal $\mathcal{T}$-triangular function having a component homeomorphic to the figure ∞. Recall that $v_0, v_1, \dots, v_{n-1}$ denote all the vertices of the polygonal P. Let $\pi : P \to S$ be the quotient map.

Lemma 2.13. *Every c-extremal $\mathcal{T}$-triangular function $f \in C(S, I)$ has exactly $n(N^2 + 4N - 4)/8$ fibers with a component homeomorphic to the figure ∞. Besides, f has exactly $nN(N+4)/16$ minima and exactly $n(N^2 + 4N - 16)/16 + 1 + \rho$ maxima, where $\rho := |\pi(\{v_0, v_1, \dots, v_{n-1}\})|$.*

Proof. If $c(v) = -$ for $v \in \mathcal{V}$, then $v = u^k_m$ for some $k \le N$ such that $k \equiv 2 \mod 4$, and m even. Thus the number of minima is equal to

$$\frac{1}{2}\sum_{k=1}^{N/4} n((4k-2)+2) = nN(N+4)/16.$$

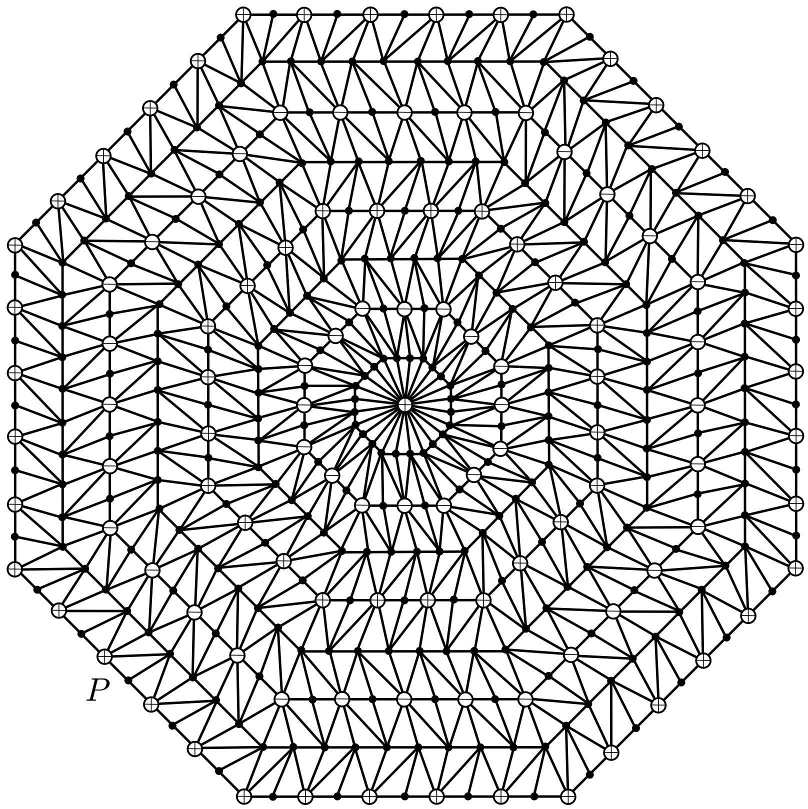

Figure 9: Vertices coloured by + are marked with ⊕ and by − with ⊖

Computing the number of maxima is a bit more complicated. If $c(v) = +$ for $v \in \mathcal{V}$, then either $v = u^0$, or $v = u_m^k$ for some $k \le N$ such that $k \equiv 0 \mod 4$, and m even. The number of maxima with $k < N$ is equal to

$$\frac{1}{2}\sum_{k=1}^{N/4-1} n(4k+2) = n(N^2-16)/16.$$

The quotient map π glues each edge of P with exactly one other edge, hence every $u_m^N \notin \{v_0, v_1, \ldots, v_{n-1}\}$ is glued with exactly one other $u_{m'}^N \notin \{v_0, v_1, \ldots, v_{n-1}\}$ – this contributes additional $nN/4$ vertices to the number of maxima. On the other hand, every vertex v_i of P may be glued with more than one other vertex of P, so there are only ρ vertices of P mapped to points of S at which f has maxima. Thus the entire number of maxima of f is equal to

$$n(N^2-16)/16 + \frac{1}{4}nN + 1 + \rho = n(N^2+4N-16)/16 + 1 + \rho.$$

Let us now compute the number of fibers with a component homeomorphic to $\mathscr{P}$. Let M be a component of the fiber $f^{-1}(y)$ for some $y \in f(S)$. M may be homeomorphic to $\mathscr{P}$ only if M contains a vertex $v \in \mathcal{V}$ coloured by 0. Let v be such a vertex. If $v \in V(Q_1)$ and $\deg_{\mathcal{T}} v = 4$, then $\deg_M v = 2$, hence M is homeomorphic to the circle S^1. If $v \notin V(Q_1)$ or $\deg_{\mathcal{T}} v \ne 4$, then there exists a unique path of distinct vertices u_0, u_1, u_2, u_3, u_4 in $\mathcal{T}$ such that $v \in \{u_1, u_2, u_3\}$, $c(\{u_1, u_2, u_3\}) = \{0\}$,

$c(u_0) = c(u_4) \in \{-, +\}$ and either

- there is $k \leq N-4$ such that $u_i \in V(Q_{k+i})$ for every $0 \leq i \leq 4$, or
- $u_0 \in V(Q_{N-2}), u_1 \in V(Q_{N-1}), u_2 \in V(Q_N), u_3 \in V(Q_{N-1}), u_4 \in V(Q_{N-2})$.

The vertex u_2 is of degree 4 in $\mathcal{T}$ and has two neighbours coloured by the same nonzero colour. On the other hand, for every two vertices u_m^k and u_{m+2}^k, where k and m are even (hence both vertices are coloured by the same nonzero colour), there is a unique path such as described above and containing u_{m+1}^k. Using the method presented in the proof of Lemma 2.7, one can easily show that there is exactly one vertex in $\{u_1, u_2, u_3\}$ such that a component of a fiber of f containing this vertex is homeomorphic to $\mathcal{S}$. Hence, the number of all fibers having a component homeomorphic to the figure $\mathcal{S}$ equals to the number of all vertices of $\mathcal{T}$ coloured by 0 and laying between two vertices of the same nonzero colour. This number is equal to

$$nN(N+4)/16 + n(N^2 - 16)/16 + n(N+2)/4 = n(N^2 + 4N - 4)/8. \qquad \square$$

Lemmas 2.10, 2.11 and 2.12 imply the following important theorem:

Theorem 2.14. *Let S be a closed surface and $\varepsilon > 0$. Then there exists an eight-like ε-triangulation of S.*

Let d be a metric on S. The existence of an arbitrarily small eight-like triangulation allows us to prove the following crucial lemma (cf. [4, Lemma 5.6]):

Lemma 2.15. *Let S be a closed surface. The set of all extremal functions is dense in $C(S, I)$. More precisely, given any $g \in C(S, I)$ and $\varepsilon, \gamma > 0$, there exists an extremal function $f \in C(S, I)$ such that $\|g - f\| < \varepsilon$ and for every $x \in S$ there is $x' \in S$ such that $d(x, x') < \gamma$ and $f^{-1}(f(x'))$ has a component M homeomorphic to the figure $\mathcal{S}$, containing x' and for which $\deg_M x' = 4$.*

Proof. Let $g \in C(S, I)$ and $1 > \varepsilon, \gamma > 0$. Since S is compact, g is uniformly continuous. Hence there exists $\delta > 0$ such that if $d(p, q) < \delta$, then $|g(p) - g(q)| < \varepsilon/8$. Set up $\eta := \min(\gamma, \delta)/3$. Let $\mathcal{T}$ be an eight-like η-triangulation of S and $c : \mathcal{V} \to \{-, 0, +\}$ be the 3-colouring associated with $\mathcal{T}$. We construct a $\mathcal{T}$-triangular c-extremal function $f \in C(S, I)$ such that $\|f - g\| < \varepsilon$. Notice that without loss of generality we can assume that $g(S) \subseteq [\varepsilon/2, 1 - \varepsilon/2]$. To finish the proof it is sufficient to define f as a one-to-one function on $\mathcal{V}$. Let $v \in \mathcal{V}$.
If $c(v) = +$, then we choose $f(v)$ so that:

$$g(v) + \frac{\varepsilon}{4} < f(v) < g(v) + \frac{\varepsilon}{2}.$$

If $c(v) = -$, then choose $f(v)$ so that:

$$g(v) - \frac{\varepsilon}{2} < f(v) < g(v) - \frac{\varepsilon}{4}.$$

If $c(v) = 0$, then let $f(v)$ be chosen in such a way that:

$$g(v) - \frac{\varepsilon}{8} < f(v) < g(v) + \frac{\varepsilon}{8}.$$

Now extend f linearly on every triangle of $\mathcal{T}$ so that $f : S \to I$ is $\mathcal{T}$-triangular. We have to show that f is c-extremal and that $\|f - g\| < \varepsilon$. Let $t \in \mathcal{T}$ be a triangle with vertices v_1, v_2, v_3 such that $c(v_1) = +$ and $c(v_2) = c(v_3) = 0$. We have for $i = 2, 3$:

$$f(v_i) < g(v_i) + \frac{\varepsilon}{8} < g(v_1) + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = g(v_1) + \frac{\varepsilon}{4} < f(v_1),$$

so f has a maximum at v_1. Moreover, for every $x \in t$ we have

$$f(x) \in \left(g(v_1) - \frac{\varepsilon}{4},\ g(v_1) + \frac{\varepsilon}{2}\right),$$

hence

$$|f(x) - g(x)| \le |f(x) - g(v_1)| + |g(v_1) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{8} < \varepsilon.$$

The case of $c(v_1) = -$ is similar.

The second part of the lemma follows from the observation made in the proof of Lemma 2.13: every point $x \in S$ belongs to some triangle of $\mathcal{T}$ and every triangle $t \in \mathcal{T}$ has a vertex u contained in a path u_1, u_2, u_3 of vertices coloured by 0 and such that there exists a unique index $i \in \{1, 2, 3\}$ for which $f^{-1}(f(u_i))$ has a component M such that $u_i \in M$, $\deg_M u_i = 4$ and M is homeomorphic to ∞. □

Lemmas 2.15 and 2.13 immediately imply the following

Corollary 2.16. *The set of all functions having a fiber with a component homeomorphic to ∞ is dense in $C(S, I)$.*

2.2. Closed subsets of surfaces

In the following section we generalize Lemma 2.15 to all closed subsets of closed surfaces. First, let us notice that the set $\mathcal{S}$ of all non-homeomorphic subcontinua of ∞ is finite: it consists of the point ·, the arc |, the circle ○, the triod ≺, the cross ×, the circle with one hair ⟜, the circle with two hairs ∝ and the figure eight ∞. Thus, by Corollary 2.4, Lemma 2.15 may be expressed in a bit more general form:

Lemma 2.17. *Let S be a closed surface. The family of all functions with all fibers consisting of components homeomorphic to a subcontinuum of ∞ and such that the number of components is finite is dense in $C(S, I)$.*

This lemma can be easily generalized to all closed subsets of a closed surface:

Lemma 2.18. *Let F be a closed subset of a closed surface S. The family of all functions with all fibers consisting of components homeomorphic to a subcontinuum of ∞ is dense in $C(F, I)$.*

Proof. Let $g \in C(F, I)$ and $\varepsilon > 0$. By the Tietze theorem there exists a continuous extension of g over the surface S, i.e. there exists $G \in C(S, I)$ such that $G|_F = g$. By Lemma 2.17 there exists a function $f \in C(S, I)$ which has all fibers consisting of subcontinua of the figure ∞ and is ε-close to G. The restriction of f to F satisfies $\|f|_F - g\| < \varepsilon$ and has the property demanded in the thesis of the lemma. □

Since the Euclidean plane $\mathbb{R}^2$ embeds into S^2, Lemma 2.18 works also for every compact subset of $\mathbb{R}^2$. Moreover, as every compact surface with boundary is homeomorphic to a closed surface with a finite number of open discs removed (cf. [6, Exercise 6.5]), we immediately obtain

Corollary 2.19. *Let S be a compact surface with boundary. The family of all functions with all fibers consisting of components homeomorphic to a subcontinuum of $\mathcal{P}$ is dense in $C(S,I)$.*

3. Fibers of a generic map $f \in C(F,I)$

Let F be a closed subset of a closed surface. In the following section we prove that a generic $f \in C(F,I)$ has the property that every component of every fiber is either a singleton or it is an $\mathcal{P}$-like hereditarily indecomposable continuum.

Recall that the Hausdorff distance between two closed subsets K and L of a compact metric space (X,d) is defined by the formula $d_H(K,L) = \max\left(\sup_{x\in K}\inf_{y\in L} d(x,y),\ \sup_{y\in L}\inf_{x\in K} d(x,y)\right)$. We will need the following two technical lemmas:

Lemma 3.1 ([4, Lemma 4.5])**.** *Let P be a graph and M be a continuum contained in a compact metric space (X,d). Let $\varepsilon > 0$. Suppose $f : M \to P$ is a continuous ε-surjection. Then, there exists $\eta > 0$ such that if N is a continuum in X with $d_H(M,N) < \eta$, then there exists a (2ε)-mapping from N onto P.*

Lemma 3.2 ([4, Lemma 4.11])**.** *Let P be a non-degenerate subcontinuum of $\mathcal{P}$. If M is a hereditarily indecomposable P-like continuum, then M is $\mathcal{P}$-like.*

Let $\mathcal{S}$ denote the set of all non-homeomorphic subcontinua of $\mathcal{P}$. The following lemma was proved as Lemma 5.16 in [4], however, the proof is a main place where we use Lemma 2.18, thus for the self-containment of the paper we include it here.

Lemma 3.3. *A generic map $f \in C(F,I)$ has the following property: if M is a non-degenerate component of a fiber of f, then there exists $P \in \mathcal{S}$ such that M is a P-like continuum.*

Proof. Let $\mathcal{F}_\varepsilon$ be a family of all functions $f \in C(F,I)$ such that there exists a fiber $f^{-1}(y)$ with a component M for which there is no ε-mapping onto any element of $\mathcal{S}$. We will show that the closure $\overline{\mathcal{F}_\varepsilon}$ is nowhere dense in $C(F,I)$.

Let $\{f_n\}$ be a sequence in $\mathcal{F}_\varepsilon$ convergent to some $f \in C(F,I)$ in the supremum norm. Let sequences $\{y_n\}$ and $\{M_n\}$ be such that M_n is a component of $f_n^{-1}(y_n)$ for which there is no ε-mapping onto any element of $\mathcal{S}$. Since F is compact, without loss of generality, we can assume that $y_n \to y$ and $M_n \to M$ in the Hausdorff metric. Thus, $f(M) = \{y\}$. Let N be a component of $f^{-1}(y)$ containing M.

There is no $(\varepsilon/2)$-mapping from N onto any element of $\mathcal{S}$. Indeed, assume that there exists such a mapping. Since M is a subcontinuum of N, there exists an $(\varepsilon/2)$-mapping from M onto an element of $\mathcal{S}$. By Lemma 3.1 for sufficiently large $n < \omega$ there exists an ε-mapping from M_n onto an element of $\mathcal{S}$, which contradicts a choice of M_n.

Thus we have shown that for every $f \in \overline{\mathcal{F}_\varepsilon}$ there exists $y \in f(F)$ such that there is a component M of $f^{-1}(y)$ for which there is no $(\varepsilon/2)$-mapping onto any element of $\mathcal{S}$. According to Lemma 2.18, $\overline{\mathcal{F}_\varepsilon}$ is nowhere dense.

Put $\mathcal{M} := C(F,I) \setminus \bigcup_{n<\omega} \overline{\mathcal{F}_{1/n}}$. $\mathcal{M}$ is a dense $\mathbb{G}_\delta$-subset of $C(F,I)$ consisting of all continuous functions having the property that every component of every fiber is a P-like continuum for some $P \in \mathcal{S}$. □

Theorem 3.4. *Every component of every fiber of a generic $f \in C(F, I)$ is either a singleton or is an $\mathcal{P}$-like hereditarily indecomposable continuum.*

Proof. The theorem immediately follows from Levin's theorem and Lemmas 3.3 and 3.2. □

Remark. It follows from the Dimension-Lowering Mapping Theorem ([7, Theorem 4.3.6]) that if the set F has dimension 2, then a generic $f \in C(F, I)$ in Theorem 3.4 must have a fiber with a non-degenerate component M. The component M is a 1-junctioned curve, so there is a point $p \in M$ such that each subcontinuum $C \subseteq M \setminus \{p\}$ is a pseudoarc (cf. [8, Theorem 7]). Moreover, one can easily deduce that the family of all pseudoarcs in M is a dense $\mathbb{G}_\delta$ subset of the hyperspace of all subcontinua of M.

4. Open problems

In the last section of this paper we state several problems concerning generalizations of Theorem 3.4.

Problem 4.1. *Generalize Theorem 3.4 to other 2-dimensional topological spaces, e.g. pseudo-manifolds, simplicial complexes or CW-complexes.*

Problem 4.2. *Characterize fibers of a generic map from a compact surface into a finite graph.*

Problem 4.3. *Characterize fibers of a generic map from the n-dimensional sphere S^n into the unit interval I for $n > 2$.*

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