ON FAMILIES OF SUBSETS OF NATURAL NUMBERS DECIDING THE NORM CONVERGENCE IN ℓ_1

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Abstract. The classical Schur theorem asserts that the weak convergence and the norm convergence in the Banach space ℓ_1 coincide. In this paper we study complexity and cardinality of subfamilies $\mathcal{F}$ of $\wp(\omega)$ such that a sequence $\langle x_n\colon\ n \in \omega\rangle \subseteq \ell_1$ is norm convergent whenever $\lim_{n\to\infty}\sum_{j\in A} x_n(j) = 0$ for every $A \in \mathcal{F}$. We call such families *Schur* and prove that they cannot have cardinality less than the pseudo-intersection number $\mathfrak{p}$. On the other hand, we also show that every non-meager subset of the Cantor space 2^ω is a Schur family when thought as a subset of $\wp(\omega)$, implying that the minimal size of a Schur family is bounded from above by $\operatorname{non}(\mathcal{M})$, the uniformity number of the ideal of meager subsets of 2^ω.

1. Introduction

We start with the following motivation. By ℓ_1 and ℓ_∞ we denote the Banach spaces of all summable complex-valued sequences and all bounded complex-valued sequences, respectively. Recall that the dual space ℓ_∞^* of ℓ_∞ is isometrically isomorphic to the space $\operatorname{ba}(\wp(\omega))$ of all bounded complex-valued finitely additive measures on $\wp(\omega)$, the power set of ω. As a predual space of ℓ_∞, ℓ_1 embeds isometrically into $\operatorname{ba}(\wp(\omega))$. Hence, every element $x = \langle x(n)\colon\ n \in \omega\rangle \in \ell_1$ may be considered as a measure $\mu_x \in \operatorname{ba}(\wp(\omega))$ given for every $A \in \wp(\omega)$ by the formula:

$$\mu_x(A) = \sum_{j\in A} x(j).$$

Since the natural embedding $\ell_1 \hookrightarrow \operatorname{ba}(\wp(\omega))$ is an isometry, we have

$$\|x\|_1 = \sum_{j\in\omega} \big|x(j)\big| = \sum_{j\in\omega} \big|\mu_x\big(\{j\}\big)\big| = \|\mu_x\|.$$

Motivated by this, for every $\mu \in \operatorname{ba}(\wp(\omega))$ we define the element $\mu \restriction \omega$ of ℓ_1 by the formula:

$$\mu \restriction \omega = \big\langle \mu\big(\{j\}\big)\colon\ j \in \omega\big\rangle.$$

Notice that $\|\mu \restriction \omega\|_1 \le \|\mu\| < \infty$ and the equality occurs only in the case when $\mu = \mu_x$ for some $x \in \ell_1$. Note also that $x = \mu_x \restriction \omega$ for every $x \in \ell_1$.

Phillips [16, Lemma 3.3] proved an important result (*Phillips's lemma*) stating that every sequence $\langle \mu_n \in \operatorname{ba}(\wp(\omega))\colon\ n \in \omega\rangle$ such that $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \wp(\omega)$ satisfies the equality $\lim_{n\to\infty} \|\mu_n \restriction \omega\|_1 = 0$. Note that this lemma concerns sequences of measures from *entire* $\operatorname{ba}(\wp(\omega))$, not only from ℓ_1 embedded into $\operatorname{ba}(\wp(\omega))$. However, if we restrict attention only to measures from ℓ_1, then by the identification of every $A \in \wp(\omega)$ with its characteristic function $\chi_A \in \ell_\infty$, we immediately obtain an important result due to Schur [18] asserting that for every sequence $\langle x_n \in \ell_1\colon\ n \in \omega\rangle$ such that $\lim_{n\to\infty}\langle \chi_A, x_n\rangle = 0$ for every $A \in \wp(\omega)$, we have $\lim_{n\to\infty} \|x_n\|_1 = 0$. (The symbol $\langle y, x\rangle$ for $x \in \ell_1$ and $y \in \ell_\infty$ denotes

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the sum $\sum_{j\in\omega} y(j)x(j)$). Thus, Schur's theorem asserts that to decide the norm convergence of a sequence $\langle x_n \in \ell_1\colon\ n \in \omega\rangle$ it is sufficient to check for every $A \in \wp(\omega)$ the convergence of the sequence $\langle\langle\chi_A, x_n\rangle\colon\ n\in\omega\rangle$. In this paper we are interested in obtaining smaller than $\wp(\omega)$ families of subsets of ω which still decide the norm convergence of sequences in ℓ_1.

Definition 1.1. A family $\mathcal{F} \subseteq \wp(\omega)$ is *a Phillips family* if $\lim_{n\to\infty}\|\mu_n \restriction \omega\|_1 = 0$ for every sequence $\langle \mu_n \in \mathrm{ba}(\wp(\omega))\colon\ n \in \omega\rangle$ such that $\lim_{n\to\infty}\mu_n(A) = 0$ for every $A \in \mathcal{F}$.

Definition 1.2. A family $\mathcal{F} \subseteq \wp(\omega)$ is *a Schur family* if $\lim_{n\to\infty}\|x_n\|_1 = 0$ for every sequence $\langle x_n \in \ell_1\colon\ n\in\omega\rangle$ such that $\lim_{n\to\infty}\langle\chi_A, x_n\rangle = 0$ for every $A \in \mathcal{F}$.

It is immediate that every Phillips family is Schur. In Section 2 we show that every Schur family (and hence Phillips) is of cardinality at least equal to the pseudo-intersection number $\mathfrak{p}$. On the other hand, in Theorem 3.2 we strengthen Schur's classical theorem and show in ZFC that every non-meager subset of the Cantor space 2^ω is a Phillips family (and hence Schur) when thought as a subfamily of $\wp(\omega)$.

In Section 4.1 we provide some remarks concerning cardinal invariants of the continuum related to Phillips and Schur families. The final section of the paper, Section 4.2, is devoted to the applications of Schur families to the classical Banach–Steinhaus theorem (also known as the Uniform Boundedness Principle).

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1.2. **Notation and terminology.** Our notation and terminology are standard. In particular, we refer the reader to the books of Diestel [5] and Bartoszyński and Judah [2] for all the necessary information concerning Banach space theory and set theory, respectively.

2. The pseudo-intesection number $\mathfrak{p}$ and Schur families

In this section we present the proof that every Schur family (and hence every Phillips family) has cardinality not smaller than the pseudo-intersection number $\mathfrak{p}$. In the following, $\mathcal{Q}$ denotes the subset of c_{00} consisting of all rational-valued sequences.

Lemma 2.1. *For every $N \in \omega$, finite subset $F \subseteq S_{\ell_\infty}$ and $\varepsilon \in (0,1)$ there exists $x \in \mathcal{Q}$ such that $\|x\|_1 \geq N$ and $|\langle y, x\rangle| < \varepsilon$ for every $y \in F$.*

Proof. There exists $x' \in \bigcap_{y\in F}\ker y$ such that $\|x'\|_1 \geq N+1$. Since every $y \in F$ is uniformly continuous, there exists $\delta \in (0,1)$ such that $|y(x)| < \varepsilon$ whenever $y \in F$, $x \in \ell_1$ and $\|x - x'\|_1 < \delta$. Choose $x \in \mathcal{Q}$ such that $\|x - x'\|_1 < \delta$; then:

$$\|x\|_1 = \|x' + (x - x')\|_1 \geq \|x'\|_1 - \|x - x'\|_1 > N + 1 - \delta > N.$$

□

Theorem 2.2. *Let $\mathcal{F} \subseteq \wp(\omega)$ be such that $|\mathcal{F}| < \mathfrak{p}$. Then, there exists $\langle x_n \in \mathcal{Q}\colon\ n\in\omega\rangle$ such that $\sup_{n\in\omega}\|x_n\|_1 = \infty$ and $\lim_{n\to\infty}\langle\chi_A, x_n\rangle = 0$.*

Proof. For every $N, k \in \omega$ and $A \in \mathcal{F}$ put:

$$\mathcal{Q}(N) = \big\{x \in \mathcal{Q}\colon\ \|x\| \geq N\big\} \quad \text{and} \quad \mathcal{Q}(A,k) = \big\{x \in \mathcal{Q}\colon\ \big|\langle \chi_A, x\rangle\big| < 1/k\big\}.$$

By Lemma 2.1, the following family has the strong finite intersection property:

$$\mathcal{G} = \big\{\mathcal{Q}(N)\colon\ N \in \omega\big\} \cup \big\{\mathcal{Q}(A,k)\colon\ A \in \mathcal{F}, k \in \omega\big\}.$$

Since $|\mathcal{G}| = |\mathcal{F}| \cdot \omega < \mathfrak{p}$, there exists infinite $\mathcal{X} \subseteq \mathcal{Q}$ almost contained in every element of $\mathcal{G}$. The elements of $\mathcal{X}$ form the required sequence. □

Corollary 2.3. *Every Schur family is of cardinality at least* $\mathfrak{p}$. □

Recall that MA_κ(*σ-centered*) denotes Martin's axiom for σ-centered posets and at most κ many dense subsets (cf. Bartoszyński and Judah [2, Section 1.4.B]) and the result of Bell [3] stating that MA_κ(*σ-centered*) holds if and only if $\mathfrak{p} > \kappa$.

Corollary 2.4. *Let κ be a cardinal number and assume* MA_κ(*σ-centered*). *If $\mathcal{F} \subseteq \wp(\omega)$ is a Schur family, then $|\mathcal{F}| > \kappa$.*

In particular, if Martin's axiom holds, then every Schur family is of cardinality $\mathfrak{c}$. □

However, the next proposition shows that consistently all Schur families may have cardinality much bigger than $\mathfrak{p}$.

Proposition 2.5. *It is consistent that all Schur families have cardinality $\mathfrak{c}$ but $\mathfrak{p} = \omega_1 < \mathfrak{c}$.*

Proof. Start with a model M of ZFC in which $\mathfrak{p} = \mathfrak{c}$ and a Suslin tree $\mathbb{T}$ exists (see e.g. Kunen and Tall [10, Corollary 10] and Fremlin [9, Section 11]). Let N be a $\mathbb{T}$-generic extension of M. Then, $\mathfrak{p} = \omega_1 < \mathfrak{c}$ holds in N (Džamonja, Hrušák and Moore [6, Theorem 6.16]). However, we claim that every Schur family in N has cardinality $\mathfrak{c}$. To see that, assume that there exists a Schur family $\mathcal{F} \in N$ such that $\big(|\mathcal{F}| < \mathfrak{c}\big)^N$. Since the forcing $\mathbb{T}$ is ω_1-Baire, i.e. intersections of countably many open dense subsets are open dense, $\big(\wp(\omega)\big)^M = \big(\wp(\omega)\big)^N$. Let κ be a cardinal in N such that $|\mathcal{F}| = \kappa$. $\mathbb{T}$ satisfies the countable chain condition, so $\mathbb{T}$ preserves cardinals and there is $\mathcal{G} \subseteq \wp(\omega)$ in M such that $\mathcal{F} \subseteq \mathcal{G}$ and $(|\mathcal{G}| = \kappa)^M$ (cf. Kunen [11, Lemma VII.5.5]). But then, in M, $\mathcal{G}$ is a Schur family and $|\mathcal{G}| < \mathfrak{p}$, a contradiction with Proposition 2.3. □

3. Non-meager subsets of 2^ω and Phillips families

The aim of this section is to prove in ZFC that every non-meager subset of the Cantor space 2^ω containing all finite subsets of ω is a Phillips family and hence there exists a Phillips family of cardinality $\mathrm{non}(\mathcal{M})$, the minimal cardinality of a non-meager subset of the Cantor set 2^ω. All measures considered in the sequel are bounded and finitely additive.

Lemma 3.1. *Let $\delta > 0$ and $\big\langle \mu_n\colon\ n \in \omega\big\rangle$ be a sequence of real-valued measures such that $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in [\omega]^{<\omega}$, but $\|\mu_n \restriction \omega\|_1 \geq 6\delta$ for every $n \in \omega$. Then, there exist increasing sequences $\big\langle N_k \in \omega\colon\ k \in \omega\big\rangle$ and $\big\langle n_k\colon\ k \in \omega\big\rangle$ and an antichain $\big\langle F_k \in [\omega]^{<\omega}\colon\ k \in \omega\big\rangle$ in $[\omega]^{<\omega}$ such that for every $k \in \omega$ the following conditions are satisfied:*

- $F_k \subseteq \big[N_k, N_{k+1}\big)$,
- $\big|\mu_{n_k}\big(F_k\big)\big| > \delta$,
- $\sum_{j<N_k} \big|\mu_{n_k}\big(\{j\}\big)\big| < \delta/4$ *and* $\sum_{j\geq N_{k+1}} \big|\mu_{n_k}\big(\{j\}\big)\big| < \delta/4$.

Proof. The proof goes by induction. Assume that for some $K \in \omega$ we have constructed sequences $N_0, \ldots, N_{K-1}$, $n_0, \ldots, n_{K-1}$ and $F_0, \ldots, F_{K-1}$ as required. To obtain N_K, n_K and F_K we proceed as follows.

Since $\|\mu_{n_{K-1}} \restriction \omega\|_1 < \infty$, there is $N_K > \max F_{K-1}$ (if $K = 0$, just put $N_0 = 0$) such that

$$\sum_{j \geq N_2} \left|\mu_{n_{K-1}}(\{j\})\right| < \delta/4.$$

Now, there exists $n_K > n_{K-1}$ (again, if $K = 0$, put $n_0 = 0$) such that

$$\sum_{j < N_K} \left|\mu_{n_K}(\{j\})\right| < \delta/4.$$

Finally, there exists $F_K \subseteq [\omega \setminus \{0, \ldots, N_K - 1\}]^{<\omega}$ such that

$$\left|\mu_{n_K}(F_K)\right| \geq \frac{1}{5} \sum_{j \geq N_K} \left|\mu_{n_K}(\{j\})\right| = \frac{1}{5}\Big(\sum_{j \in \omega} \left|\mu_{n_K}(\{j\})\right| - \sum_{j < N_K} \left|\mu_{n_K}(\{j\})\right|\Big) >$$

$$\frac{1}{5}(6\delta - \delta) = \delta.$$

$\square$

We think of subsets of 2^ω as subfamilies of $\wp(\omega)$, so we say that $\mathcal{F} \subseteq 2^\omega$ *contains all finite sets* if $\mathcal{F}$ contains all points $x \in 2^\omega$ such that $x(i) = 0$ for all but finitely many $i \in \omega$.

Theorem 3.2. *Let $\mathcal{F} \subseteq 2^\omega$ be a non-meager set containing all finite sets. Then, $\mathcal{F}$ is a Phillips family.*

Proof. We first prove that $\mathcal{F}$ is a Phillips family for sequences of real-valued measures. So assuming $\mathcal{F}$ is not, there exists a sequence $\langle \mu_n\colon n \in \omega \rangle$ of real-valued measures such that $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in [\omega]^{<\omega}$, but there exists $\delta > 0$ for which $\|\mu_n \restriction \omega\|_1 \geq 6\delta$ for every $n \in \omega$. Since $[\omega]^{<\omega} \subseteq \mathcal{F}$, there exist increasing sequences $\langle N_k \in \omega\colon k \in \omega \rangle$ and $\langle n_k\colon k \in \omega \rangle$ and an antichain $\langle F_k \in [\omega]^{<\omega}\colon k \in \omega \rangle$ as in Lemma 3.1.

For every $n \in \omega$ and $G \in [\omega]^{<\omega}$ there exists $k(G, n) > n$ such that $N_{k(G,n)} > \max G$ (and hence $\min F_{k(G,n)} > \max G$). Define the function

$$h^n_G\colon G \cup \big[N_{k(G,n)}, N_{k(G,n)+1}\big) \to \{0, 1\}$$

as follows:

$$h^n_G(x) = \begin{cases} 1 & \text{if } x \in G \cup F_{k(G,n)}, \\ 0 & \text{if } x \in \big[N_{k(G,n)}, N_{k(G,n)+1}\big) \setminus F_{k(G,n)}. \end{cases}$$

For every $n \in \omega$ and $G \subseteq [\omega]^{<\omega}$ define the following open subset of 2^ω:

$$O^n_G = \big\{g \in 2^\omega\colon\ h^n_G \subseteq g\big\}.$$

For every $n \in \omega$ put:

$$X_n = \bigcup_{G \in [\omega]^{<\omega}} O^n_G.$$

The set X_n is open dense in 2^ω for every $n \in \omega$. Since $\mathcal{F}$ is non-meager as a subset of 2^ω, there exists $f \in \mathcal{F} \cap \bigcap_{n\in\omega} X_n$. For every $n \in \omega$ there exists $G \in [\omega]^{<\omega}$ such that $h^n_G \subseteq f$ and hence:

$$\left|\mu_{n_{k(G,n)}}(f)\right| \geq$$

$$\left|\mu_{n_{k(G,n)}}\big(F_{k(G,n)}\big)\right| - \sum_{j < N_{k(G,n)}} \left|\mu_{n_{k(G,n)}}(\{j\})\right| - \sum_{j \geq N_{k(G,n)+1}} \left|\mu_{n_{k(G,n)}}(\{j\})\right| >$$

$$\delta - \delta/4 - \delta/4 = \delta/2.$$

Hence, $\limsup_{n\in\omega} \left|\mu_n(f)\right| \geq \delta/2 > 0$ and $f \in \mathcal{F}$, which is a contradiction.

Finally, we show that $\mathcal{F}$ is a Phillips family for sequences of complex-valued measures. Let $\left\langle\mu_n\colon\ n \in \omega\right\rangle$ be a sequence of complex-valued measures on $\wp(\omega)$ such that $\lim_{n\to\infty} \mu_n(A) = 0$ for every $A \in \mathcal{F}$. Then,

$$\lim_{n\to\infty} \operatorname{Re}\left(\mu_n(A)\right) = \lim_{n\to\infty} \operatorname{Im}\left(\mu_n(A)\right) = 0$$

for every $A \in \mathcal{F}$, and hence

$$\lim_{n\to\infty} \left\|\left(\operatorname{Re}(\mu_n)\right) \restriction \omega\right\|_1 = \lim_{n\to\infty} \left\|\left(\operatorname{Im}(\mu_n)\right) \restriction \omega\right\|_1 = 0,$$

since $\mathcal{F}$ is a Phillips family for sequences of real-valued measures. By the triangle inequality we obtain that $\lim_{n\to\infty} \left\|\mu_n \restriction \omega\right\|_1 = 0$. □

Corollary 3.3. *There exists a Phillips family of cardinality* $\operatorname{non}(\mathcal{M})$*.*

Since $\operatorname{non}(\mathcal{M}) = \omega_1 < \mathfrak{c}$ holds e.g. in the Cohen, Sacks or Miller models (see Blass [4, Chapter 11]), we obtain the following corollary.

Corollary 3.4. *It is consistent that $\omega_1 < \mathfrak{c}$ and there exists a Phillips family of cardinality ω_1. In particular, the existence of a Phillips family (or a Schur family) of cardinality strictly less than $\mathfrak{c}$ is undecidable in ZFC+¬CH.* □

4. Final remarks and consequences

In this final section of the paper we do several remarks and show some consequences of the results proved in the previous sections. We also ask some questions.

4.1. **Cardinal invariants of the continuum.** In Section 2 we proved that there is no countable Schur family and in Section 3 we showed in ZFC that there is a Phillips family of cardinality $\operatorname{non}(\mathcal{M})$. Hence, it seems reasonable to introduce the following cardinal invariants of the continuum related to Phillips and Schur families.

Definition 4.1. *The Phillips number* $\mathfrak{phil}$ is the smallest cardinality of a Phillips family:

$$\mathfrak{phil} = \min\left\{|\mathcal{F}|\colon\ \mathcal{F} \subseteq \wp(\omega) \text{ is a Phillips family}\right\}.$$

The Schur number $\mathfrak{schur}$ is the smallest cardinality of a Schur family:

$$\mathfrak{schur} = \min\left\{|\mathcal{F}|\colon\ \mathcal{F} \subseteq \wp(\omega) \text{ is a Schur family}\right\}.$$

Since every Phillips family is Schur, we immediately have that $\mathfrak{schur} \leq \mathfrak{phil}$. The following corollary expresses all the results from the previous sections in terms of the numbers $\mathfrak{phil}$ and $\mathfrak{schur}$.

Corollary 4.2.

(1) *Under Martin's axiom,* $\mathfrak{phil} = \mathfrak{schur} = \mathfrak{c}$.
(2) $\mathfrak{p} \leq \mathfrak{schur} \leq \mathfrak{phil} \leq \operatorname{non}(\mathcal{M})$.
(3) *It is consistent that* $\omega_1 = \mathfrak{p} < \mathfrak{schur} = \mathfrak{phil} = \operatorname{non}(\mathcal{M}) = \mathfrak{c}$.
(4) *It is consistent that* $\omega_1 = \mathfrak{p} = \mathfrak{phil} = \mathfrak{schur} = \operatorname{non}(\mathcal{M}) < \mathfrak{c}$. □

We would like to know whether it is possible to express $\mathfrak{phil}$ and $\mathfrak{schur}$ exactly in terms of the classical cardinal invariants of the continuum studied e.g. in Bartoszyński and Judah [2] or Blass [4].

Problem 4.3.

(1) *Give (better) bounds for* $\mathfrak{phil}$ *and* $\mathfrak{schur}$*.*
(2) *Determine the exact values of* $\mathfrak{phil}$ *and* $\mathfrak{schur}$ *in terms of classical cardinal invariants.*

It seems also interesting to distinguish the class of Phillips families and the class of Schur families.

Question 4.4.

(1) Is every Schur family Phillips?
(2) If no, is it consistent that $\mathfrak{schur} < \mathfrak{phil}$?

4.2. **Weak* Banach–Steinhaus sets in ℓ_∞.** Let X be an infinite-dimensional Banach space with a predual space X_* and the dual space X^*. Then, X_* isometrically embeds into X^* and the Banach–Steinhaus theorem states in particular that if a sequence $\langle x_n \in X_*\colon\ n \in \omega\rangle$ is pointwise bounded on X, i.e. $\sup_{n\in\omega}\left|\langle x_n, y\rangle\right| < \infty$ for every $y \in X$, then it is uniformly bounded, i.e. $\sup_{n\in\omega}\left\|x_n\right\| < \infty$. In this section, we are interested whether we can consider a small subset D of S_X to decide the uniform boundedness of the sequence $\langle x_n\colon\ n \in \omega\rangle$.

Definition 4.5. A subset D of the unit sphere S_X is *weak* Banach–Steinhaus in* X if every sequence $\langle x_n \in X_*\colon\ n \in \omega\rangle$ which is *pointwise bounded on* D, i.e. $\sup_{n\in\omega}\left|\langle x_n, y\rangle\right| < \infty$ for every $y \in D$, is uniformly bounded.

Nygaard and Põldvere [15] provide several characterizations of weak* Banach–Steinhaus sets (under the name of *weak*-thick sets*). We prove that those sets cannot be too small.

Proposition 4.6. *Every weak* Banach–Steinhaus set $D \subseteq S_X$ in an infinite-dimensional Banach space X with a predual X_* is uncountable and linearly weak* dense (i.e. $\overline{\mathrm{span}}^{weak^*}(D) = X$).*

Proof. We only show that D is uncountable, since the proof of linear weak* density of D is similar.

Assume D is countable, i.e. $D = \{y_n\colon\ n \in \omega\}$. Put:

$$A_n = \mathrm{span}\,\{y_0, \ldots, y_{n-1}\}.$$

Recall that $(X, weak^*)^*$ denotes the space of all weak* continuous functionals on X. Since $(X, weak^*)^* = X_*$ (cf. Rudin [17, Theorem 3.10]), by the Hahn–Banach theorem, there exists $x_n \in X_*$ such that $x_n(z) = 0$ for every $z \in A_n$ and $\langle x_n, y_n\rangle = n$. For every $m \in \omega$ we have:

$$\sup_{n\in\omega}|\langle x_n, y_m\rangle| = \sup_{n\in\{0,\ldots,m\}}|\langle x_n, y_m\rangle| < \infty,$$

but $\sup_{n\in\omega}\left\|x_n\right\| = \infty$. Thus, D is not weak* Banach–Steinhaus. □

We are now going to the following setting: $X_* = \ell_1$, $X = \ell_\infty$ and $X^* = \mathrm{ba}(\wp(\omega))$. An immediate corollary of Theorem 2.2 is the following result — note that this also yields that there are no countable weak* Banach–Steinhaus sets in ℓ_∞.

Corollary 4.7. *Every weak* Banach–Steinhaus set in ℓ_∞ is of cardinality at least $\mathfrak{p}$. In particular, assuming Martin's axiom, every weak* Banach–Steinhaus set in ℓ_∞ is of cardinality $\mathfrak{c}$.* □

The next proposition reveals a relation between Schur families and weak* Banach–Steinhaus sets in ℓ_∞.

Proposition 4.8. *If $\mathcal{F} \subseteq \wp(\omega)$ is a Schur family, then the set $D = \{\chi_A\colon\ A \in \mathcal{F}\}$ is weak* Banach–Steinhaus in ℓ_∞.*

Proof. Assume $\langle x_n \in \ell_1\colon\ n \in \omega\rangle$ is pointwise bounded on D, but $\lim_{n\to\infty}\left\|x_n\right\|_1 = \infty$. For every $n \in \omega$ define $x_n' \in \ell_1$ as follows:

$$x_n' = x_n/\sqrt{\left\|x_n\right\|_1}.$$

Then, for every $A \in \mathcal{F}$ we have $\lim_{n\to\infty} \langle \chi_A, x'_n \rangle = 0$. Since $\mathcal{F}$ is Schur, it follows that $\lim_{n\to\infty} \|x'_n\|_1 = 0$. But $\|x'_n\|_1 = \sqrt{\|x_n\|_1}$ for every $n \in \omega$, a contradiction. $\square$

Corollary 4.9. *There exists a weak* Banach–Steinhaus set in ℓ_∞ of cardinality* $\mathrm{non}(\mathcal{M})$. *Hence, it is consistent that $\omega_1 < \mathfrak{c}$ and there exists a weak* Banach–Steinhaus set in ℓ_∞ of cardinality ω_1. In particular, the existence of weak* Banach–Steinhaus sets in ℓ_∞ of cardinality strictly smaller than $\mathfrak{c}$ is undecidable in ZFC+$\neg$CH.* $\square$

Note that for every Banach space $(X, \|\cdot\|)$ the density character of $(X^*, weak^*)$ is not greater than the density character of $(X, \|\cdot\|)$ (cf. Fabian et al. [7, page 576]). Hence, the weak* density character of ℓ_∞ is ω. Corollary 4.7 implies that not every weak* dense subset of ℓ_∞ is a weak* Banach–Steinhaus set (i.e. those of cardinality less than $\mathfrak{p}$ are not). However, if $D \subseteq S_{\ell_\infty}$, then the same proposition states that linear weak* density of D is a necessary condition for D to be weak* Banach–Steinhaus. On the other hand, since the density character of ℓ_∞ is $\mathfrak{c}$, Corollary 4.9 yields that consistently there exists a weak* Banach–Steinhaus set which is not linearly dense in ℓ_∞.

Question 4.10. *Assume Martin's axiom. Is every weak* Banach–Steinhaus set in ℓ_∞ linearly dense?*

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