THE NIKODYM PROPERTY IN THE SACKS MODEL

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ABSTRACT. We prove that if $\mathcal{A}$ is a σ-complete Boolean algebra in a ground model V of set theory, then $\mathcal{A}$ has the Nikodym property in every side-by-side Sacks forcing extension $V[G]$, i.e. every pointwise bounded sequence of measures on $\mathcal{A}$ in $V[G]$ is uniformly bounded. This gives a consistent example of a class of infinite Boolean algebras with the Nikodym property and of cardinality strictly less than the continuum.

1. INTRODUCTION

Let $\mathcal{A}$ be a Boolean algebra. A sequence of measures $\langle\mu_n\colon n\in\omega\rangle$ on $\mathcal{A}$ is *pointwise bounded* if $\sup_{n\in\omega}\left|\mu_n(A)\right|<\infty$ for every $A\in\mathcal{A}$ and it is *uniformly bounded* if $\sup_{n\in\omega}\left\|\mu_n\right\|<\infty$. The Nikodym Boundedness Theorem states that if $\mathcal{A}$ is σ-complete, then every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded. This principle, due to its numerous applications, is one of the most important results in the theory of vector measures, see Diestel and Uhl [7, Section I.3].

Since σ-completeness is rather a strong property of Boolean algebras, Schachermayer [12] made a detailed study of the Nikodym theorem and introduced the Nikodym property for general Boolean algebras.

Definition 1.1. A Boolean algebra $\mathcal{A}$ has the *Nikodym property* if every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.

The property has been studied by many authors, e.g. Darst [5], Seever [13], Haydon [10], Moltó [11], Freniche [8], Aizpuru [1, 2] or Valdivia [15].

Let us pose the following question. Let V be a model of ZFC+CH and $\mathcal{A}\in V$ be a σ-complete Boolean algebra of cardinality equal to the continuum $\mathfrak{c}$. Let $\mathbb{P}$ be a notion of forcing preserving ω_1 and G its generic filter over V. Assume that in the extension $V[G]$ the CH does not hold. Then, $\mathcal{A}$ will have cardinality ω_1 in $V[G]$, and hence it will no longer be σ-complete. However, will $\mathcal{A}$ still have the Nikodym property?

Brech [4, Theorem 3.1] proved that if $\mathbb{P}$ is the side-by-side Sacks forcing $\mathbb{S}^\kappa$ for some regular cardinal number κ, then $\mathcal{A}$ will have the *Grothendieck property* in $V[G]$, i.e. every sequence of measures in $V[G]$ which is weak* convergent on $\mathcal{A}$ is also weakly convergent. The Nikodym and Grothendieck properties are closely related to each other, see e.g. Schachermayer [12]. Thus, motivated by Brech's result, we studied the preservation of the Nikodym property by the Sacks forcing $\mathbb{S}^\kappa$ and proved that if $\mathcal{A}$ is a σ-complete Boolean algebra in V, then $\mathcal{A}$ has the Nikodym property in the $\mathbb{S}^\kappa$-generic extension $V[G]$ (Theorem 3.3).

Complementing the result of Brech was not the only reason we dealt with the side-by-side Sacks forcing $\mathbb{S}^\kappa$ instead of iterations. The other one was the fact the size of the continuum can be arbitrary large when forcing with $\mathbb{S}^\kappa$ ($\mathfrak{c}=\kappa$ holds in

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$V[G]$), while iterations give only models where the continuum is at most ω_2 (see Geschke and Quickert [9, Section 7]). This has an important consequence for us. In Sobota [14], the first author studied the relation between the Nikodym property and cardinal characteristics of the continuum. In particular, a ZFC construction of a Boolean algebra with the Nikodym property and of cardinality equal to $\operatorname{cof}(\mathcal{N})$, the cofinality of the σ-ideal $\mathcal{N}$ of subsets of the real line with zero Lebesgue measure, was presented. Since the construction was rather intricate, the natural question about the consistent existence of a *simple* example of a Boolean algebra with the Nikodym property and cardinality strictly smaller than arbitrarily large $\mathfrak{c}$ was posed. This paper answers this question.

1.1. **Terminology and notation.** Throughout the paper $\mathcal{A}$ will always denote a Boolean algebra. The Stone space of $\mathcal{A}$ is denoted by $K_{\mathcal{A}}$. Recall that by the Stone duality theorem $\mathcal{A}$ is isomorphic with the algebra of clopen subsets of $K_{\mathcal{A}}$; if $A \in \mathcal{A}$, then $[A]$ denotes the corresponding clopen subset of $K_{\mathcal{A}}$.

A subset X of a Boolean algebra $\mathcal{A}$ is an *antichain* if $x \wedge y = \mathbf{0}_{\mathcal{A}}$ for every distinct $x, y \in X$, i.e. every two distinct elements of X are *disjoint*. On the other hand, a subset X of a poset $\mathbb{P}$ is an *antichain* if no distinct $x, y \in X$ are compatible.

A *measure* $\mu\colon \mathcal{A} \to \mathbb{C}$ on $\mathcal{A}$ is always a finitely additive complex-valued function with finite variation. The measure μ has a unique Borel extension (denoted also by μ) onto the space $K_{\mathcal{A}}$, preserving the variation of μ. By the Riesz representation theorem the dual space $C\big(K_{\mathcal{A}}\big)^*$ of the Banach space of continuous complex-valued functions on $K_{\mathcal{A}}$ is isometrically isomorphic with the space of all measures on $\mathcal{A}$. For more information concerning measure theory and Banach spaces, see the book of Diestel [6].

V always denotes the set-theoretic universum. By $\mathbb{S}^\kappa$ we denote the side-by-side product of κ many Sacks forcings $\mathbb{S}$ for some uncountable regular cardinal number κ. Regarding all other notions related to the Sacks forcing, we follow the paper of Baumgartner [3]. If $s \in \mathbb{S}$ and $p \in s$, then $s|p = \big\{q \in s\colon\ q \subseteq p \text{ or } p \subseteq q\big\} \in \mathbb{S}$. If $n \in \omega$, then $l(n, s)$ denotes the *n-th forking level* of s.

Let $s, s' \in \mathbb{S}^\kappa, F \in [\operatorname{dom}(s)]^{<\omega}$ and $n \in \omega$. We put $l(F, n, s) = \big\{\sigma\colon\ \operatorname{dom}(\sigma) = F \ \&\ \forall \alpha \in F\colon\ \sigma(\alpha) \in l(n, s(\alpha))\big\}$. Note that $|l(F, n, s)| = 2^{n|F|}$. We write $s' \leq_{F,n} s$ if $s' \leq s$ and $l(F, n, s') = l(F, n, s)$. If $\sigma\colon F \to 2^{<\omega}$ is such that $\sigma(\alpha) \in s(\alpha)$ for every $\alpha \in F$, then we write $s|\sigma$ for a condition defined as $(s|\sigma)(\alpha) = s(\alpha)$ for $\alpha \in \operatorname{dom}(s) \setminus F$ and $(s|\sigma)(\alpha) = s(\alpha)|\sigma(\alpha)$.

2. Anti-Nikodym sequences in the Sacks model

In this section, assuming in a forcing extension the existence of sequences of measures on a ground model Boolean algebra $\mathcal{A}$ which are pointwise bounded but not uniformly bounded, we build (Proposition 2.9) in the ground model a special antichain in $\mathcal{A}$ which will be crucial in proving the main theorem of the paper — Theorem 3.3.

Definition 2.1. A sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on a Boolean algebra $\mathcal{A}$ is called *anti-Nikodym* if it is pointwise bounded but not uniformly bounded.

Lemma 2.2. *If a sequence $\langle \mu_n\colon\ n \in \omega\rangle$ of measures on a Boolean algebra $\mathcal{A}$ is anti-Nikodym, then there exists a point $t \in K_{\mathcal{A}}$ such that for every clopen neighborhood $U \in \mathcal{A}$ of t we have $\sup_{n\in\omega} \big\|\mu_n \restriction U\big\| = \infty$.*

The point t will be called *a Nikodym concentration point of the sequence* $\langle \mu_n\colon\ n \in \omega\rangle$.

Proof. Assume that for every point $t \in K_{\mathcal{A}}$ there exists $A_t \in \mathcal{A}$ such that $t \in [A_t]$ and $\langle \mu_n \upharpoonright A_t\colon\ n \in \omega \rangle$ is uniformly bounded. Then, by compactness of $K_{\mathcal{A}}$ there exist $t_1, \ldots, t_n \in K_{\mathcal{A}}$ such that $A_{t_1} \vee \ldots \vee A_{t_m} = \mathbf{1}_{\mathcal{A}}$. This in turn implies that

$$\sup_{n\in\omega} \|\mu_n\| = \sup_{n\in\omega} |\mu_n|(\mathbf{1}_{\mathcal{A}}) \leq \sup_{n\in\omega} |\mu_n|(A_{t_1}) + \ldots + \sup_{n\in\omega} |\mu_n|(A_{t_m}) =$$

$$\sup_{n\in\omega} \|\mu_n \upharpoonright A_{t_1}\| + \ldots + \sup_{n\in\omega} \|\mu_n \upharpoonright A_{t_m}\| < \infty,$$

which is a contradiction, since $\langle \mu_n\colon\ n \in \omega \rangle$ is not uniformly bounded. □

(Note that in the above proof we did not use the pointwise boundedness of $\langle \mu_n\colon\ n \in \omega \rangle$.)

Lemma 2.3. *Let $\langle \mu_n\colon\ n \in \omega \rangle$ be an anti-Nikodym sequence on $\mathcal{A}$ and let $t \in K_{\mathcal{A}}$ be its Nikodym concentration point. Assume that $t \in [A]$ for some $A \in \mathcal{A}$. Then, for every positive real number ρ and natural number M there exist an element $B \in \mathcal{A}$ and a natural number $n > M$ such that:*

- *$B \leq A$ and $t \in [A \setminus B]$,*
- *$|\mu_n(B)| > \rho$.*

Proof. Since $\langle \mu_n\colon\ n \in \omega \rangle$ is anti-Nikodym and $t \in [A]$, there exist $C \leq A$ and $n > M$ such that

$$|\mu_n(C)| > \sup_{m\in\omega} |\mu_m(A)| + \rho$$

and hence

$$|\mu_n(A \setminus C)| = |\mu_n(C) - \mu_n(A)| \geq |\mu_n(C)| - |\mu_n(A)| > \rho.$$

If $t \in [C]$, then put $B = A \setminus C$, otherwise put $B = C$. □

To the end of this section let $\mathcal{A}$ be a ground model infinite Boolean algebra.

Lemma 2.4. *Let $A_0, \ldots, A_k \in \mathcal{A}$, K, $M \in \omega$. Let $\langle \dot{\mu}_n\colon\ n \in \omega \rangle$ be a sequence of names for measures on $\mathcal{A}$, $\dot{t}$ a name for a point in $K_{\mathcal{A}}$ and $\dot{\rho}$ a name for a positive real number. Let $s \in \mathbb{S}^\kappa$ force that $\langle \dot{\mu}_n\colon\ n \in \omega \rangle$ is anti-Nikodym, $\dot{t}$ is its Nikodym concentration point and $\dot{t} \notin \bigcup_{j=0}^k [\check{A}_j]$.*

Then, there exist a sequence $B_1, \ldots, B_K$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\bigvee_{j=0}^k A_j$, a sequence $n_K > \ldots > n_1 > M$ of natural numbers and a condition $s^* \leq s$ forcing for every $1 \leq i \leq K$ that $\dot{t} \notin [\check{B}_i]$ and $|\dot{\mu}_{n_i}(\check{B}_i)| > \dot{\rho}$.

Proof. Use Lemma 2.3 inductively K times to obtain sequences $B_1, \ldots, B_K \in \mathcal{A}$, $n_K > \ldots > n_1 > M$ and $s_K \leq \ldots \leq s_1 \leq s$ such that for every $1 \leq i \leq K$ the element B_i is disjoint with $\bigvee_{j=0}^k A_j \vee \bigvee_{l=1}^{i-1} B_l$ and the condition s_i forces that $\dot{t} \notin [\check{B}_i]$ and $|\dot{\mu}_{n_i}(\check{B}_i)| > \dot{\rho}$. Let $s^* = s_K$. □

Using Lemma 2.4, we will usually assume that $s \Vdash \dot{\rho} = \sum_{j=0}^k \sup_{m\in\omega} |\dot{\mu}_m(\check{A}_j)| + \check{N} + 2$ for some given $N \in \omega$ (or something alike).

Lemma 2.5. *Let $K, P \in \omega$. Let $\mu_1, \ldots, \mu_K$ be a sequence of K measures on $\mathcal{A}$. Assume that $K \cdot \|\mu_j\| < P$ for every $1 \leq j \leq K$. Then, for every $Q > K \cdot P$ and every pairwise disjoint elements $C_1, \ldots, C_Q$ of $\mathcal{A}$ there exist natural numbers $k_1 < \ldots < k_{Q-K\cdot P}$ such that*

$$|\mu_j|(C_{k_l}) < 1/K$$

for every $1 \leq j \leq K$ and $1 \leq l \leq Q - K \cdot P$.

Proof. Let $Q > K \cdot P$ and $C_1, \ldots, C_Q$ be an antichain in $\mathcal{A}$. Notice that if there exist $k_1 < \ldots < k_P$ such that

$$\left|\mu_j\right|\left(C_{k_l}\right) \geq 1/K$$

for some $1 \leq j \leq K$ and every $1 \leq l \leq P$, then we have:

$$\|\mu_j\| \geq \sum_{l=1}^{P} \left|\mu_j\right|\left(C_{k_l}\right) \geq P \cdot 1/K > K \cdot \|\mu_j\| \cdot 1/K = \|\mu_j\|,$$

a contradiction, so for every $1 \leq j \leq K$ there must exist at most $P-1$ elements C_{k_l}'s such that

$$\left|\mu_j\right|\left(C_{k_l}\right) \geq 1/K.$$

Hence, the thesis of the lemma holds for some $Q - K \cdot (P-1) \geq Q - K \cdot P$ elements C_{k_l}'s. □

The following lemma is standard, cf. Baumgartner [3, Lemmas 1.5–1.8].

Lemma 2.6. *Let $s \in \mathbb{S}^\kappa$, $N \in \omega$ and $F_N \in [\mathrm{dom}(s)]^{<\omega}$.*

a) $\left\{s|\sigma\colon\ \sigma \in l(F_N, N, s)\right\}$ is an antichain in $\mathbb{S}^\kappa$ and $s = \bigcup_{\sigma \in l(F_N,N,s)} s|\sigma$.
b) If $\sigma \in l(F_N, N, s)$ and $p \leq s|\sigma$, then there exists $q \leq_{F,N} s$ such that $q|\sigma = p$.
c) If $D \subseteq \mathbb{S}^\kappa$ is open dense below s, then there exists $q \leq_{F_N,N} s$ such that $q|\sigma \in D$ for every $\sigma \in l(F_N, N, s)$.

□

Lemma 2.7. *Let $A_0, \ldots, A_k$, M, $\langle \dot\mu_n\colon\ n \in \omega\rangle$, $\dot t$ and s be as in the assumptions of Lemma 2.4. Let $N \in \omega$ and $F_N \in [\mathrm{dom}(s)]^{<\omega}$. Put $K = |l(F_N, N, s)|$ and enumerate $l(F_N, N, s) = \langle \sigma_i\colon\ 1 \leq i \leq K\rangle$.*

Then, there exist a condition $s^* \leq_{F_N,N} s$, a sequence $B_1, \ldots, B_K$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\bigvee_{j=0}^k A_j$ and a sequence $n_K > \ldots > n_1 > M$ such that for every $1 \leq i \leq K$ the condition $s^*|\sigma_i$ forces that:

- $\left|\dot\mu_{n_i}\left(\check B_i\right)\right| > \sum_{j=0}^k \left|\dot\mu_{n_i}\left(\check A_j\right)\right| + \sum_{j=1}^{i-1} \left|\dot\mu_{n_i}\left(\check B_j\right)\right| + \check N + 2$,
- $\left|\dot\mu_{n_i}\right|\left(\bigvee_{j=i+1}^K \check B_j\right) < 1$,
- $\dot t \notin \bigcup_{i=1}^K \left[\check B_i\right]$.

Proof. The proof basically goes by induction in K steps — each step for one σ_i ($1 \leq i \leq K$). We start simply as follows — by Lemmas 2.4 and 2.6.b) there exist a condition $s_1 \leq_{F_N,N} s$, a family $\mathscr{B}_1^1 = \left\{B_1^1, \ldots, B_K^1\right\}$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\bigvee_{j=0}^k A_j$, a sequence $n_K^1 > \ldots > n_1^1 > M$ of natural numbers and a natural number $P_1 > 0$ such that for every $1 \leq j \leq K$ we have:

$$s_1|\sigma_1 \Vdash \left|\dot\mu_{n_j^1}\left(\check B_j^1\right)\right| > \sum_{l=0}^k \left|\dot\mu_{n_j^1}\left(\check A_l\right)\right| + \check N + 2,$$

$$s_1|\sigma_1 \Vdash \check K \cdot \left\|\dot\mu_{n_j^1}\right\| < \check P_1, \text{ and}$$

$$s_1|\sigma_1 \Vdash \dot t \notin \bigcup_{B \in \check{\mathscr{B}}_1^1} [B].$$

Assume now that for some $1 \leq L < K$ we have found:

- a sequence of conditions $s_L \leq_{F_N,N} \ldots \leq_{F_N,N} s_1 \leq_{F_N,N} s$,
- for every $1 \leq i \leq L$ a sequence of families $\mathscr{B}_L^i \subseteq \ldots \subseteq \mathscr{B}_i^i \subseteq \mathscr{B}^i \subseteq \mathcal{A}$ of pairwise disjoint non-zero elements of $\mathcal{A}$ with $\mathscr{B}_L^i \neq \emptyset$ and $\mathscr{B}^i = \left\{B_1^i, \ldots, B_K^i\right\}$,

- a sequence of natural numbers $n^L_K > \ldots > n^L_1 > n^{L-1}_K > \ldots > n^{L-1}_1 > \ldots > n^1_K > \ldots > n^1_1 > M$, and
- a sequence of natural numbers $P_L > \ldots > P_1 > 0$,

such that:

(i) for every $1 \le i \le L$ and $1 \le j \le K$ we have:

$$s_i|\sigma_i \Vdash \left|\dot\mu_{n^i_j}\left(\check B^i_j\right)\right| > \sum_{l=0}^{k}\left|\dot\mu_{n^i_j}\left(\check A_l\right)\right| + \sum_{l=1}^{i-1}\sum_{B\in\check{\mathscr{B}}^l_i}\left|\dot\mu_{n^i_j}(B)\right| + \check N + 2, \text{ and} \tag{1}$$

$$s_i|\sigma_i \Vdash \check K\cdot\left\|\dot\mu_{n^i_j}\right\| < \check P_i; \tag{2}$$

(ii) for every $1 \le j \le i \le L$ we have:

$$s_i|\sigma_j \Vdash \dot t \notin \bigcup_{l=1}^{i}\bigcup_{B\in\check{\mathscr{B}}^l_i}[B]\,; \tag{3}$$

(iii) for every $1 \le l < i \le L$, $1 \le j \le K$ and $B \in \mathscr{B}^i$ we have:

$$s_i|\sigma_l \Vdash \left|\dot\mu_{n^l_j}\right|(\check B) < 1/\check K. \tag{4}$$

Let us now construct $s_{L+1} \le_{F_N,N} s_L$, $\mathscr{B}^1_{L+1} \subseteq \mathscr{B}^1_L, \ldots, \mathscr{B}^L_{L+1} \subseteq \mathscr{B}^L_L, \mathscr{B}^{L+1}_{L+1} \subseteq \mathscr{B}^{L+1} \subseteq \mathcal{A}$, $n^{L+1}_K > \ldots > n^{L+1}_1 > n^L_K$ and $P_{L+1} > P_L$ satisfying also the properties (i)–(iii).

First, we modify a bit the condition s_L. By density, there exists $p \le s_L|\sigma_{L+1}$ such that for every $1 \le i \le L$ either there exists unique $1 \le j_i \le K$ such that $p \Vdash \dot t \in \left[\check B^i_{j_i}\right]$, or for every $B \in \mathscr{B}^i_L$ we have $p \Vdash \dot t \notin \left[\check B\right]$. In the former case put $\mathscr{B}^i_{L+1} = \mathscr{B}^i_L \setminus \left\{B^i_{j_i}\right\}$, in the latter — $\mathscr{B}^i_{L+1} = \mathscr{B}^i_L$. By Lemma 2.6.b), there exists $q \le_{F_N,N} s_L$ such that $q|\sigma_{L+1} = p$. Note that

$$q|\sigma_{L+1} \Vdash \dot t \notin \bigcup_{j=0}^{k}\left[\check A_j\right] \cup \bigcup_{l=1}^{L}\bigcup_{B\in\check{\mathscr{B}}^l_{L+1}}[B]\,. \tag{5}$$

By Lemmas 2.4 and 2.6.b), there exist a condition $r \le_{F_N,N} q$, a family $\mathscr{C} = \left\{C_1, \ldots, C_Q\right\}$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\left(\bigvee_{j=1}^k A_j \vee \bigvee_{l=1}^L \bigvee \mathscr{B}^l_{L+1}\right)$, where $Q = K\cdot L\cdot P_L + K$, a sequence $m_Q > \ldots > m_1 > n^L_K$ of natural numbers and a natural number $P_{L+1} > P_L$ such that for every $1 \le j \le Q$ we have:

$$r|\sigma_{L+1} \Vdash \left|\dot\mu_{m_j}\left(\check C_j\right)\right| > \sum_{l=0}^{k}\left|\dot\mu_{m_j}\left(\check A_l\right)\right| + \sum_{l=1}^{L}\sum_{B\in\check{\mathscr{B}}^l_{L+1}}\left|\dot\mu_{m_j}(B)\right| + \check N + 2, \tag{6}$$

$$r|\sigma_{L+1} \Vdash \check K\cdot\left\|\dot\mu_{m_j}\right\| < \check P_{L+1}, \text{ and}$$

$$r|\sigma_{L+1} \Vdash \dot t \notin \bigcup_{j=1}^{Q}\left[\check C_j\right]. \tag{7}$$

We now define s_{L+1} out of r in two steps. In the first step, by induction, the inequality (2) and Lemmas 2.5 and 2.6.b), we get a sequence $\mathscr{C}_L \subseteq \ldots \subseteq \mathscr{C}_1 \subseteq \mathscr{C}$ with $\left|\mathscr{C}_L\right| = K$, a sequence $k_K > \ldots > k_1$ of natural numbers and a sequence of conditions $p_L \le_{F_N,N} \ldots \le_{F_N,N} p_1 \le_{F_N,N} r$ such that $\mathscr{C}_L = \left\{C_{k_1}, \ldots, C_{k_K}\right\}$ and for every $1 \le i \le L$, $1 \le j \le K$ and $C \in \mathscr{C}_i$ we have:

$$p_i|\sigma_i \Vdash \left|\dot\mu_{n^i_j}\right|(\check C) < 1/\check K. \tag{8}$$

For every $1 \le j \le K$ write $B^{L+1}_j = C_{k_j}$ and $n^{L+1}_j = m_{k_j}$, and put $\mathscr{B}^{L+1} = \left\{B^{L+1}_1, \ldots, B^{L+1}_K\right\}$.

In the second step, by induction and again Lemma 2.6.b), we get a sequence $t_L \le_{F_N,N} \ldots \le_{F_N,N} t_1 \le_{F_N,N} p_L$ such that for every $1 \le i \le L$ either there exists $1 \le j_i \le K$ such that $t_i|\sigma_i \Vdash \dot{t} \in \big[\check{B}^{L+1}_{j_i}\big]$, or for every $1 \le j \le K$ we have $t_i|\sigma_i \Vdash \dot{t} \notin \big[\check{B}^{L+1}_j\big]$. Put:

$$\mathscr{B}^{L+1}_{L+1} = \mathscr{B} \setminus \big\{B^{L+1}_{j_i}\colon\ t_i|\sigma_i \Vdash \dot{t} \in \big[\check{B}^{L+1}_{j_i}\big], 1 \le i \le L\big\} \tag{9}$$

and

$$s_{L+1} = t_L.$$

Note that by (7) and (9), for every $1 \le i \le L+1$ we have:

$$s_{L+1}|\sigma_i \Vdash \dot{t} \notin \bigcup_{B \in \check{\mathscr{B}}^{L+1}_{L+1}} [B]. \tag{10}$$

After the K-th step of the induction has been finished, we are left with the non-empty collections $\mathscr{B}^1_K, \ldots, \mathscr{B}^K_K$ (some of them may be singletons), the sequence $n^K_K > n^K_{K-1} > \ldots > n^1_2 > n^1_1 > M$ and the conditions $s_K \le_{F_N,N} \ldots \le_{F_N,N} s_1 \le_{F_N,N} s$. From each $\mathscr{B}^i_K$ pick one element $B^i_{l_i}$. Then, for every $1 \le i \le K$ by (1) and (6) we have:

$$s_K|\sigma_i \Vdash \big|\dot{\mu}_{n^i_{l_i}}\big(\check{B}^i_{l_i}\big)\big| > \sum_{j=0}^{k} \big|\dot{\mu}_{n^i_{l_i}}\big(\check{A}_j\big)\big| + \sum_{j=1}^{i-1} \big|\dot{\mu}_{n^i_{l_i}}\big(\check{B}^j_{l_j}\big)\big| + \check{N} + 2,$$

and by (4) and (8):

$$s_K|\sigma_i \Vdash \big|\dot{\mu}_{n^i_{l_i}}\big|\Big(\bigvee_{j=i+1}^{K} \check{B}^j_{l_j}\Big) = \sum_{j=i+1}^{K} \big|\dot{\mu}_{n^i_{l_i}}\big|\big(\check{B}^i_{l_i}\big) < \check{K} \cdot 1/\check{K} = 1,$$

and finally by (3), (5) and (10):

$$s_K|\sigma_i \Vdash \dot{t} \notin \bigcup_{j=1}^{K} \Big[\check{B}^j_{l_j}\Big].$$

Put:

$$s^* = s_K$$

and for every $1 \le i \le K$:

$$B_i = B^i_{l_i} \quad \text{and} \quad n_i = n^i_{l_i}.$$

□

By Lemma 2.6.a) we immediately obtain the following corollary.

Corollary 2.8. *Let $A_0, \ldots, A_k$, K, M, N, $\langle \dot{\mu}_n\colon\ n \in \omega\rangle$, $\dot{t}$, s and F_N be as in the assumptions of Lemma 2.7.*

Then, there exist a condition $s^* \le_{F_N,N} s$, a sequence $B_1, \ldots, B_K$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $\bigvee_{j=0}^k A_j$ and a sequence $n_K > \ldots > n_1 > M$ such that s^* forces that $\dot{t} \notin \bigcup_{i=1}^K \big[\check{B}_i\big]$ and that there exists $1 \le i \le \check{K}$ for which it holds:

$$\big|\dot{\mu}_{n_i}\big(\check{B}_i\big)\big| > \sum_{j=0}^{k} \big|\dot{\mu}_{n_i}\big(\check{A}_j\big)\big| + \sum_{j=1}^{i-1} \big|\dot{\mu}_{n_i}\big(\check{B}_j\big)\big| + \check{N} + 2$$

and

$$\big|\dot{\mu}_{n_i}\big|\Big(\bigvee_{j=i+1}^{K} \check{B}_j\Big) < 1.$$

□

Proposition 2.9. *Let $\langle \dot\mu_n\colon\ n\in\omega\rangle$ be a sequence of names for measures on $\mathcal{A}$. Let $s\in\mathbb{S}^\kappa$ force that $\langle \dot\mu_n\colon\ n\in\omega\rangle$ is anti-Nikodym.*

Then, there exists:

- *an increasing sequence $\langle K_N\colon\ N\in\omega\rangle$ of natural numbers,*
- *a sequence $\langle B_i^N\colon\ 1\le i\le K_N,\ N\in\omega\rangle$ of pairwise disjoint elements of $\mathcal{A}$,*
- *a sequence $\langle n_i^N\colon\ 1\le 1\le K_N,\ N\in\omega\rangle$ in ω such that $n_1^N > n_{K_M}^M > \ldots > n_1^M$ for every $N>M$, and*
- *a condition $s^*\le s$ forcing for every $N\in\omega$ that there exists $1\le i\le \check K_N$ such that:*

$$\left|\dot\mu_{n_i^N}(\check B_i^N)\right| > \sum_{M=0}^{N-1}\sum_{j=1}^{K_M}\left|\dot\mu_{n_i^N}(\check B_j^M)\right| + \sum_{j=1}^{i-1}\left|\dot\mu_{n_i^N}(\check B_j^N)\right| + \check N + 2$$

and

$$\left|\dot\mu_{n_i^N}\right|\Big(\bigvee_{j=i+1}^{K_N}\check B_j^N\Big) < 1.$$

Proof. The conclusion follows by the inductive use of Corollary 2.8 (to obtain an appropriate fusion sequence $\langle s_N\colon\ N\in\omega\rangle$ of conditions in $\mathbb{S}^\kappa$) and the ultimate use of the fusion lemma (to obtain a fusion condition $s^*\in\mathbb{S}^\kappa$ such that $s^*\le_{F_N,N} s_N$ for every $N\in\omega$; see Baumgartner [3, Lemma 1.8]). □

3. Main result

Throughout this section $\mathcal{A}$ is a ground model σ-complete Boolean algebra, i.e. $\mathcal{A}\in V$ and $\mathcal{A}$ is σ-complete in V.

Lemma 3.1. *Let $X\in[\omega]^\omega$ and $X=\bigcup_{k\in\omega}X_k$ be an infinite partition of X into infinite subsets. For every measure μ on $\mathcal{A}$ and an antichain $\langle B_N\colon\ N\in\omega\rangle$ in $\mathcal{A}$ there exists $L\in\omega$ such that*

$$|\mu|\Big(\bigvee_{N\in X_k}B_N\Big)<1$$

for every $k>L$.

Proof. Since μ is finitely additive and bounded, we have:

$$\sum_{k\in\omega}|\mu|\Big(\bigvee_{N\in X_k}B_N\Big)\le|\mu|\Big(\bigvee_{N\in\omega}B_N\Big)\le|\mu|(\mathbf{1}_{\mathcal{A}})<\infty.$$

□

Lemma 3.2. *Let $\langle B_N\colon\ N\in\omega\rangle\in V$ be an antichain in $\mathcal{A}$ and $X\in[\omega]^\omega\cap V$. Let $s\in\mathbb{S}^\kappa$ be a condition, $N\in\omega$, $F_N\subseteq[\mathrm{dom}(s)]^{<\omega}$ and $\dot\mu_1,\ldots,\dot\mu_K$ names for measures on $\mathcal{A}$. Then, there exist a condition $s^*\le_{F_N,N}s$ and a set $X'\in[X]^\omega\cap V$ such that for every $1\le i\le K$ we have:*

$$s^*\Vdash|\dot\mu_i|\Big(\bigvee_{M\in\check X'}\check B_M\Big)<1.$$

Proof. Let $X=\bigcup_{k\in\omega}X_k$ be an infinite partition of X into infinite sets. By Lemma 3.1 the following set is open dense below s:

$$D=\Big\{p\le s\colon\ \forall\ 1\le i\le K\ \exists\ L\in\omega\ \forall\ k>L\colon\ p\Vdash|\dot\mu_i|\Big(\bigvee_{M\in\check X_k}\check B_M\Big)<1\Big\}.$$

By Lemma 2.6.c) there exists $s^* \leq_{F_N,N} s$ such that $s^*|\sigma \in D$ for every $\sigma \in l(F_N, N, s)$. Hence, for every $\sigma \in l(F_N, N, s)$ there exists $L_\sigma \in \omega$ such that for every $k > L_\sigma$ the condition $s^*|\sigma$ forces that:

$$\left|\dot{\mu}_i\right|\Big(\bigvee_{M\in\check{X}_k} \check{B}_M\Big) < 1.$$

Let $L = \max\big(L_\sigma\colon\ \sigma \in l(F_N, N, s)\big) + 1$. Put $X' = X_L$ and appeal to Lemma 2.6.a). □

We are now in the position to prove the main theorem of this paper.

Theorem 3.3. *Let G be an $\mathbb{S}^\kappa$-generic filter over V. Then, in $V[G]$ the Boolean algebra $\mathcal{A}$ has the Nikodym property.*

Proof. Working in $V[G]$ assume that $\mathcal{A}$ does not have the Nikodym property. Then, there exists an anti-Nikodym sequence $\langle\mu_n\colon\ n \in \omega\rangle$ of measures on $\mathcal{A}$. Let $t \in K_\mathcal{A}$ be its Nikodym concentration point.

Now and to the end of the proof, let us work in the ground model V. Let $\langle\dot{\mu}_n\colon\ n \in \omega\rangle$ be a sequence of names for measures in the sequence $\langle\mu_n\colon\ n \in \omega\rangle$ and $\dot{t}$ a name for t. There exists a condition $s \in G$ forcing that $\langle\dot{\mu}_n\colon\ n \in \omega\rangle$ is anti-Nikodym on $\check{\mathcal{A}}$ and $\dot{t}$ is its Nikodym concentration point.

Let $\langle K_N\colon\ N \in \omega\rangle$, $\langle B_i^N\colon\ 1 \leq i \leq K_N,\ N \in \omega\rangle$, $\langle n_i^N\colon\ 1 \leq i \leq K_N,\ N \in \omega\rangle$ and $s^* \leq s$ be given by Proposition 2.9. We will find a condition $s^{**} \leq s^*$ and a set $Y \in [\omega]^\omega \cap V$ such that s^{**} forces that

$$\dot{B} = \bigvee_{N\in Y} \bigvee_{i=1}^{K_N} \check{B}_i^N \in \check{\mathcal{A}}$$

and

$$\sup_{n\in\omega} \left|\dot{\mu}_n(\dot{B})\right| = \infty,$$

which will contradict the fact that s forces that $\langle\dot{\mu}_n\colon\ n \in \omega\rangle$ is pointwise bounded.

To obtain s^{**} and Y we follow by induction and use Lemma 3.2 to construct a fusion sequence $\langle s_N\colon\ N \in \omega\rangle$ of conditions such that $s_0 = s^*$ and for every $N \in \omega$ we have $s_{N+1} \leq_{F_N,N} s_N$, where $F_N = \{\alpha_i^k\colon\ i, k < N\}$ and $\operatorname{dom}(s_N) = \{\alpha_k^N\colon\ k \in \omega\}$, and a decreasing sequence $\langle X_N\colon\ N \in \omega\rangle$ of infinite subsets of ω such that:

- $X_0 = \omega$ and for every $N \in \omega$ we have $\min X_N < \min X_{N+1}$, and
- for every $N \in \omega$ and $L = \min X_N$ the condition s_N forces that:

  $$\left|\dot{\mu}_{n_i^L}\right|\Big(\bigvee_{M\in\check{X}_{N+1}} \bigvee_{j=1}^{K_M} \check{B}_j^M\Big) < 1$$

  for every $1 \leq i \leq K_L$.

Let $s^{**} \in \mathbb{S}^\kappa$ be such a condition that $s^{**} \leq_{F_N,N} s_N$ for every $N \in \omega$ (see Baumgartner [3, Lemma 1.8]). Put:

$$Y = \big\{\min X_N\colon\ N \in \omega\big\}$$

and

$$B = \bigvee_{N\in Y} \bigvee_{i=1}^{K_N} B_i^N.$$

Then, $B \in \mathcal{A}$ and, since $\langle X_N \colon N \in \omega \rangle$ is decreasing, s^{**} forces that for every $N \in Y$ and $1 \le i \le K_N$ the following inequality holds:

$$\left|\dot{\mu}_{n_i^N}\right|\Big(\bigvee_{\substack{M \in Y\\ M > N}} \bigvee_{j=1}^{K_M} \check{B}_j^M\Big) < 1.$$

Finally, since $s^{**} \le s^*$, s^{**} forces for every $N \in Y$ that there exists $1 \le i \le K_N$ such that

$$\left|\dot{\mu}_{n_i^N}(\check{B}_i^N)\right| > \sum_{\substack{M \in Y\\ M < N}} \sum_{j=1}^{K_M} \left|\dot{\mu}_{n_i^N}(\check{B}_j^M)\right| + \sum_{j=1}^{i-1} \left|\dot{\mu}_{n_i^N}(\check{B}_j^N)\right| + \check{N} + 2$$

and

$$\left|\dot{\mu}_{n_i^N}\right|\Big(\bigvee_{j=i+1}^{K_N} \check{B}_j^N\Big) < 1,$$

and hence:

$$\begin{aligned}\left|\dot{\mu}_{n_i^N}(\check{B})\right| &= \Big|\dot{\mu}_{n_i^N}\Big(\bigvee_{\substack{M \in Y\\ M < N}} \bigvee_{j=1}^{K_M} \check{B}_j^M\Big) + \dot{\mu}_{n_i^N}\Big(\bigvee_{j=1}^{i-1} \check{B}_j^N\Big) + \dot{\mu}_{n_i^N}(\check{B}_i^N) + \\ &\quad + \dot{\mu}_{n_i^N}\Big(\bigvee_{j=i+1}^{K_N} \check{B}_j^N\Big) + \dot{\mu}_{n_i^N}\Big(\bigvee_{\substack{M \in Y\\ M > N}} \bigvee_{j=1}^{K_M} \check{B}_j^M\Big)\Big| \ge \\ &\ge \left|\dot{\mu}_{n_i^N}(\check{B}_i^N)\right| - \sum_{\substack{M \in Y\\ M < N}} \sum_{j=1}^{K_M} \left|\dot{\mu}_{n_i^N}(\check{B}_j^M)\right| - \sum_{j=1}^{i-1} \left|\dot{\mu}_{n_i^N}(\check{B}_j^N)\right| - \\ &\quad - \left|\dot{\mu}_{n_i^N}\right|\Big(\bigvee_{j=i+1}^{K_N} \check{B}_j^N\Big) - \left|\dot{\mu}_{n_i^N}\right|\Big(\bigvee_{\substack{M \in Y\\ M > N}} \bigvee_{j=1}^{K_M} \check{B}_j^M\Big) \ge \\ &\ge \check{N} + 2 - 1 - 1 = \check{N}.\end{aligned}$$

Thus, s^{**} forces that for every $N \in \omega$ there exists n such that $\left|\dot{\mu}_n(\check{B})\right| > N$ and hence s^{**} forces that $\sup_{n \in \omega} \left|\dot{\mu}_n(\check{B})\right| = \infty$. □

Since the forcing $\mathbb{S}^\kappa$ preserves ω_1 and $\kappa = \mathfrak{c}$ in any $\mathbb{S}^\kappa$-generic extension (see Baumgartner [3, Theorems 1.11 and 1.14]), we immediately obtain the following corollary.

Corollary 3.4. *Assume that V is a model of ZFC+CH. If G is an $\mathbb{S}^\kappa$-generic filter, then in $V[G]$ the relations $\omega_1 < \kappa = \mathfrak{c}$ hold and $\mathcal{A}$ is an example of a Boolean algebra with the Nikodym property and of cardinality ω_1.*

4. Concluding remarks

4.1. **The Vitali–Hahn–Sacks property.** Schachermayer [12, Theorem 2.5] proved that a Boolean algebra $\mathcal{A}$ has simultaneously the Nikodym property and the Grothendieck property if and only if $\mathcal{A}$ has the *Vitali–Hahn-Saks property*, i.e. every pointwise convergent sequence of measures on $\mathcal{A}$ is uniformly exhaustive. Thus, Theorem 3.3 and Brech's result [4, Theorem 3.1] imply together that if $\mathcal{A}$ is a σ-complete Boolean algebra in the ground model V, then it has the Vitali–Hahn–Saks property in the $\mathbb{S}^\kappa$-generic extension $V[G]$. In particular, as in Corollary 3.4, this yields a simple consistent example of a Boolean algebra with the Vitali–Hahn–Saks property and of cardinality strictly less than $\mathfrak{c}$.

4.2. **Cardinal characteristics of the continuum.** In Sobota [14], the first author studied relations between the Nikodym property and cardinal characteristics of the continuum. In particular, the Nikodym number $\mathfrak{n}$ denoting the smallest size of an infinite Boolean algebra with the Nikodym property was introduced and the inequality $\mathfrak{n} \geq \max\big(\mathfrak{b}, \mathfrak{s}, \operatorname{cov}(\mathcal{M})\big)$ was established in ZFC, where $\mathfrak{b}$ denotes the bounding number, $\mathfrak{s}$ — the splitting number and $\operatorname{cov}(\mathcal{M})$ — the covering of category. It was also proved in ZFC, however in a quite complicated manner, that $\mathfrak{n} \leq \kappa$ for all cardinal numbers κ such that $\operatorname{cof}(\mathcal{N}) \leq \kappa = \operatorname{cof}\big([\kappa]^\omega\big)$, where $\operatorname{cof}(\mathcal{N})$ denotes the cofinality of measure. Since $\operatorname{cof}(\mathcal{N}) = \omega_1$ in the side-by-side Sacks forcing extensions and $\operatorname{cof}([\omega_n]^\omega) = \omega_n$ in ZFC for all $n \in \omega$, it follows that $\mathfrak{n} = \omega_1$ in the side-by-side Sacks model and the algebra constructed in Sobota [14] witnesses this fact. However, Theorem 3.3 provides much simpler examples, namely all infinite ground model σ-complete Boolean algebras.

Since $\mathfrak{n} \geq \max\big(\mathfrak{b}, \mathfrak{s}, \operatorname{cov}(\mathcal{M})\big)$, the natural question about the relation of the dominating number $\mathfrak{d}$ and the Nikodym number $\mathfrak{n}$ arises. Obviously, under Martin's axiom it holds that $\omega_1 < \mathfrak{n} = \mathfrak{d} = \mathfrak{c}$ and Theorem 3.3 yields consistently that $\mathfrak{n} = \mathfrak{d} = \omega_1 < \mathfrak{c}$. However, we know neither whether any of the two relations $\mathfrak{d} < \mathfrak{n}$ and $\mathfrak{d} > \mathfrak{n}$ may be consistently true, nor whether any of the relations $\mathfrak{d} \leq \mathfrak{n}$ and $\mathfrak{d} \geq \mathfrak{n}$ holds in ZFC.

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