FORCING INDESTRUCTIBLE EXTENSIONS OF ALMOST DISJOINT FAMILIES

BARNABÁS FARKAS

ABSTRACT. We prove the following theorem answering a question of L. Soukup: If $\mathbb{F}$ is a forcing notion and $\mathcal{A}$ is an infinite AD family on ω, then $\mathcal{A}$ can be extended to an $\mathbb{F}$-indestructible MAD family in a ccc forcing extension.

A family $\mathcal{A} \subseteq [\omega]^\omega = \{A \subseteq \omega : |A| = \omega\}$ is *almost disjoint* (AD) if $|A \cap B| < \omega$ for each distinct $A, B \in \mathcal{A}$. An infinite AD family $\mathcal{A}$ is *maximal* (MAD) if

$$\forall\, X \in [\omega]^\omega \ \exists\, A \in \mathcal{A}\ |X \cap A| = \omega,$$

i.e. $\mathcal{A}$ is $\subseteq$-maximal among AD families. Using Zorn's Lemma each infinite AD family can be extended to a MAD family. It is easy to see that there are AD families with cardinality $\mathfrak{c} = 2^\omega$ and that each MAD family is uncountable.

Assume $\mathcal{A}$ is a MAD family and $\mathbb{F}$ is a forcing notion. We say that $\mathcal{A}$ is $\mathbb{F}$-*indestructible* if $\Vdash_{\mathbb{F}}$"$\mathcal{A}$ is a MAD family".

Kunen [1, Ch.VIII, Theorem 2.3] constructed a Cohen-indestructible MAD family assuming CH. His method was later extended to other forcing notions and there were proved a lot of similar indestructibility results assuming typically that some cardinal invariant of the continuum is equal to $\mathfrak{c}$.

In general, if $\mathbb{F}$ is a classical forcing notion (such as the Cohen, the random, the Sacks, the Laver, or the Miller forcing), then the existence of an $\mathbb{F}$-indestructible MAD family (in ZFC) is still an open problem.

Let $\mathbb{C}_\kappa$ denote the Cohen forcing which adds κ many Cohen reals. The motivation of this paper is based on the following theorem and the related question after that.

Theorem 1. [2, Theorem 11] *In $V^{\mathbb{C}_{\omega_1}}$ there are AD families $\mathcal{A}$ and $\mathcal{B}$ such that, in any generic extension of $V^{\mathbb{C}_{\omega_1}}$ by a ccc forcing notion $\mathbb{P}$ such that $\mathbb{P} \in V$, $\mathcal{A}$ cannot be extended to a Cohen-indestructible MAD family and $\mathcal{B}$ cannot be extended to a random-indestructible MAD family.*

L. Soukup asked if any AD family could be extended to a Cohen-indestructible MAD family in a ccc forcing extension. Using Kunen's idea we show that the answer is yes not only for the Cohen forcing but for any fixed forcing notion from the ground model.

We will use the following trivial observation:

Observation 2. *Let $V \subseteq W$ be transitive models of (a large enough finite fragment of) ZFC and $\mathbb{F} \in V$ be a forcing notion. Assume furthermore that $p \in \mathbb{F}$, $S \in \mathcal{P}(\omega) \cap V$, and $\dot{X} \in V$ is a nice $\mathbb{F}$-name for a subset of ω. Then*

2000 *Mathematics Subject Classification.* 03E40.

Key words and phrases. Maximal almost disjoint families; forcing indestructibility.

The preparation of this paper was supported by Hungarian National Foundation for Scientific Research grant no 68262.

(i) $V \models p \Vdash_{\mathbb{F}} |\dot{X}| = \omega$ *if and only if* $W \models p \Vdash_{\mathbb{F}} |\dot{X}| = \omega$,
(ii) $V \models p \Vdash_{\mathbb{F}} S \cap \dot{X} \neq \emptyset$ *if and only if* $W \models p \Vdash_{\mathbb{F}} S \cap \dot{X} \neq \emptyset$.

Theorem 3. *Assume $\mathbb{F}$ is a forcing notion and $\mathcal{A}$ is an infinite AD family. Then there is a ccc forcing extension in which $\mathcal{A}$ can be extended to an $\mathbb{F}$-indestructible MAD family. Moreover, if either $|\mathbb{F}| < \mathfrak{c}$ or $|\mathbb{F}| = \mathfrak{c}$ and $\mathbb{F}$ is ccc, then there is a σ-centered forcing extension in which $\mathcal{A}$ can be extended to an $\mathbb{F}$-indestructible MAD family.*

Proof. Let $\kappa = |\mathbb{F}|$. By recursion on κ^+ we will define a finite support iteration of ccc forcing notions $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa^+, \beta < \kappa^+ \rangle$ and a sequence $\langle \dot{A}_\alpha : \alpha < \kappa^+ \rangle$ such that $\dot{A}_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name, $\Vdash_\alpha$"$\mathcal{A} \cup \{\dot{A}_\beta : \beta < \alpha\}$ is an AD family", and $|\mathbb{P}_{\kappa^+}| \leq 2^\kappa$. In $V^{\mathbb{P}_{\kappa^+}}$ the family $\mathcal{A} \cup \{\dot{A}_\beta : \beta < \kappa^+\}$ will be an $\mathbb{F}$-indestructible MAD family.

At stage α we will work with a condition p from $\mathbb{F}$ such that each $p \in \mathbb{F}$ will be worked at cofinal many stages in κ^+.

Assume $\mathbb{P}_\alpha$ and $\{\dot{A}_\beta : \beta < \alpha\}$ are done and we have a $p \in \mathbb{F}$. From now on we are working in $V^{\mathbb{P}_\alpha}$. Let $\mathcal{A}_\alpha = \mathcal{A} \cup \{\dot{A}_\beta : \beta < \alpha\}$ and $\mathcal{X}_\alpha$ be the set of all $\dot{X}$ nice $\mathbb{F}$-names for an infinite subsets of ω such that $p \Vdash_{\mathbb{F}}$"$\mathcal{A}_\alpha \cup \{\dot{X}\}$ is an AD family". Let $\mathbb{Q}_\alpha$ be the following forcing notion:

$(n, s, F, \mathcal{B}, \mathcal{Y}) \in \mathbb{Q}_\alpha$ iff

(1) $n \in \omega$ and $s \subseteq n$;
(2) F is a finite subset of $\{q \in \mathbb{F} : q \leq_{\mathbb{F}} p\} \times \omega$;
(3) $\mathcal{B}$ is a finite subset of $\mathcal{A}_\alpha$;
(4) $\mathcal{Y}$ is a finite subset of $\mathcal{X}_\alpha$.

$(n_1, s_1, F_1, \mathcal{B}_1, \mathcal{Y}_1) < (n_0, s_0, F_0, \mathcal{B}_0, \mathcal{Y}_0)$ iff

(a) $n_1 \geq n_0$ and $s_1 \cap n_0 = s_0$,
(b) $F_1 \supseteq F_0$, $\mathcal{B}_1 \supseteq \mathcal{B}_0$, and $\mathcal{Y}_1 \supseteq \mathcal{Y}_0$;
(c) $(s_1 \setminus s_0) \cap \bigcup \mathcal{B}_0 = \emptyset$;
(d) $\forall\ (q, k) \in F_0\ \forall\ \dot{X} \in \mathcal{Y}_0\ \exists\ r \leq_{\mathbb{F}} q\ r \Vdash_{\mathbb{F}} (s_1 \setminus k) \cap \dot{X} \neq \emptyset$.

Notation: $c = (n^c, s^c, F^c, \mathcal{B}^c, \mathcal{Y}^c) \in \mathbb{Q}_\alpha$.

Claim. *$\mathbb{Q}_\alpha$ is σ-centered, $|\mathbb{Q}_\alpha| \leq 2^\kappa$, and the following sets are dense in $\mathbb{Q}_\alpha$:*

(i) $\{c \in \mathbb{Q}_\alpha : s^c \setminus k \neq \emptyset\}$ *for each* $k \in \omega$;
(ii) $\{c \in \mathbb{Q}_\alpha : (q, k) \in F^c\}$ *for each* $(q, k) \in \{q \in \mathbb{P} : q \leq p\} \times \omega$;
(iii) $\{c \in \mathbb{Q}_\alpha : B \in \mathcal{B}^c\}$ *for each* $B \in \dot{\mathcal{A}}_\alpha$;
(iv) $\{c \in \mathbb{Q}_\alpha : \dot{X} \in \mathcal{Y}^c\}$ *for each* $\dot{X} \in \mathcal{X}_\alpha$.

Proof. σ-centeredness: We show that conditions with the same first and second coordinates are compatible. Let $(n, s, F_0, \mathcal{B}_0, \mathcal{Y}_0), (n, s, F_1, \mathcal{B}_1, \mathcal{Y}_1) \in \mathbb{Q}_\alpha$ and let

$$(F_0 \times \mathcal{Y}_0) \cup (F_1 \times \mathcal{Y}_1) = \{\langle (q_\ell, k_\ell), \dot{X}_\ell \rangle : \ell < L\}$$

be an enumeration. For each $\ell < L$ we know that $q_\ell \Vdash_{\mathbb{F}}$"$\dot{X}_\ell \setminus \bigcup (\mathcal{B}_0 \cup \mathcal{B}_1)$ is infinite" so we can choose an $r_\ell \leq_{\mathbb{F}} q_\ell$ and a $k'_\ell > \max\{n, k_\ell\}$ such that $r_\ell \Vdash_{\mathbb{F}} k'_\ell \in \dot{X}_\ell \setminus \bigcup (\mathcal{B}_0 \cup \mathcal{B}_1)$. Let $s' = s \cup \{k'_\ell : \ell < L\}$, $n' = \max(s') + 1$, $F' = F_0 \cup F_1$, $\mathcal{B}' = \mathcal{B}_0 \cup \mathcal{B}_1$, and $\mathcal{Y}' = \mathcal{Y}_0 \cup \mathcal{Y}_1$. Then $(s', n', F', \mathcal{B}', \mathcal{Y}')$ is a common extension of our two conditions.

$|\mathbb{Q}_\alpha| \leq 2^\kappa$ is trivial. (i) can be proved as σ-centeredness. (ii), (iii), and (iv) are trivial. □

Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathbb{Q}_\alpha$, and $\dot{A}_\alpha$ be a $\mathbb{P}_{\alpha+1}$-name for the set $\bigcup\{s^c : c \in \dot{H}_\alpha\}$ where $\dot{H}_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name for the $\dot{\mathbb{Q}}_\alpha$-generic filter. Then

- $V^{\mathbb{P}_{\alpha+1}} \models$"$\mathcal{A} \cup \{\dot{A}_\beta : \beta \leq \alpha\}$ is an AD family" because of (i), (iii), and (c);
- $V^{\mathbb{P}_{\alpha+1}} \models p \Vdash_{\mathbb{F}} |\dot{A}_\alpha \cap \dot{X}| = \omega$ holds for each $\dot{X} \in \mathcal{X}_\alpha$ by (ii), (iv), and (d).

At last we prove that $\dot{\mathcal{A}}_{\kappa^+} = \mathcal{A} \cup \{\dot{A}_\alpha : \alpha < \kappa^+\}$ is an $\mathbb{F}$-indestructible MAD family in $V^{\mathbb{P}_{\kappa^+}}$.

Assume on the contrary that there is a $\mathbb{P}_{\kappa^+}$-generic filter G_{κ^+}, a $p \in \mathbb{F}$, and a nice $\mathbb{F}$-name $\dot{X} \in V[G_{\kappa^+}]$ for an infinite subset of ω such that

$$V[G_{\kappa^+}] \models p \Vdash_{\mathbb{F}} \text{"}\dot{\mathcal{A}}_{\kappa^+} \cup \{\dot{X}\} \text{ is an AD family"}.$$

The there is an $\alpha < \kappa^+$ such that at the stage α we worked with p and $\dot{X} \in \mathcal{X}_\alpha$ (because $|\dot{X}| \leq \kappa$). Then in particular $V[G_{\kappa^+} \cap \mathbb{P}_{\alpha+1}] \models p \Vdash_{\mathbb{F}} |\dot{A}_\alpha \cap \dot{X}| = \omega$ so this holds in $V[G_{\kappa^+}]$ as well, a contradiction.

If $|\mathbb{F}| < \mathfrak{c}$, then $\kappa^+ \leq \mathfrak{c}$ so $\mathbb{P}_{\kappa^+}$ is σ-centered because of the well-known fact that the limit of a $\mathfrak{c}$ stage finite support iteration of σ-centered forcing notions is σ-centered.

If $|\mathbb{F}| = \mathfrak{c}$ and $\mathbb{F}$ is ccc, then it is enough to work with a $\mathfrak{c}$ stage finite support iteration because each nice $\mathbb{F}$-name for an infinite subset of ω is a countable object so it will appear at a stage less than $\mathfrak{c}$. □

Corollary 4. *Assume $\mathcal{A}$ is an infinite AD family. Then there is a σ-centered forcing extension in which $\mathcal{A}$ can be extended to a Cohen-indestructible MAD family.*

Unfortunately our theorem does not say anything about definable forcing notions so the following question is still open.

Question 5. Assume $\mathcal{A}$ is a infinite AD family. Does there exist a ccc (or nice enough) forcing extension in which $\mathcal{A}$ can be extended to a random-indestructible MAD family? What can we say about other classical real forcings such as the Sacks, Laver, or the Miller forcing?

References

[1] K. Kunen: *Set Theory, An Introduction to Independence Proofs*, North Holland, Amsterdam, New York, Oxford 1980.

[2] S. Fuchino, S. Geschke, L. Soukup: *How to drive our families mad*, submitted to Arch. Math. Logic.

Budapest University of Technology and Economics (BME)
E-mail address: barnabasfarkas@gmail.com