HECHLER'S THEOREM FOR TALL ANALYTIC P-IDEALS

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Abstract. We prove the following version of Hechler's classical theorem: For each partially ordered set $(Q, \leq)$ with the property that every countable subset of Q has a strict upper bound in Q, there is a ccc forcing notion such that in the generic extension for each tall analytic P-ideal $\mathcal{I}$ (coded in the ground model) a cofinal subset of $(\mathcal{I}, \subseteq^*)$ is order isomorphic to $(Q, \leq)$.

1. Introduction

A partially ordered set $(Q, \leq)$ is *σ-directed* if each countable subset of Q has a strict upper bound in Q. If $f, g \in \omega^\omega$, then we write $f \leq^* g$ and say g *almost dominates* f if the set $\{n \in \omega : f(n) > g(n)\}$ is finite. Hechler's original theorem is the following statement:

Theorem 1.1. ([5],[2]) *Let $(Q, \leq)$ be a σ-directed partially ordered set. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ a cofinal subset of $(\omega^\omega, \leq^*)$ is order isomorphic to $(Q, \leq)$.*

In [7] L. Soukup asked if Hechler's Theorem hold for classical σ-ideals as partially ordered sets with the inclusion. T. Bartoszyński, M.R. Burke, and M. Kada gave the following positive answers. Denote $\mathcal{N}$ the ideal of measure zero subsets of the reals, and $\mathcal{M}$ the ideal of meager subsets of the reals.

Theorem 1.2. ([3]) *Let $(Q, \leq)$ be a σ-directed partially ordered set. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ a cofinal subset of $(\mathcal{N}, \subseteq)$ is order isomorphic to $(Q, \leq)$.*

Theorem 1.3. ([1]) *Let $(Q, \leq)$ be a σ-directed partially ordered set. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ a cofinal subset of $(\mathcal{M}, \subseteq)$ is order isomorphic to $(Q, \leq)$.*

Using the model of [3] we prove the following theorem in Section 4.

Theorem 1.4. *Let $(Q, \leq)$ be a σ-directed partially ordered set. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ for each tall analytic P-ideal $\mathcal{I}$ coded in V a cofinal subset of $(\mathcal{I}, \subseteq^*)$ is order isomorphic to $(Q, \leq)$.*

Remark 1.5. Tallness is not really necessary in Theorem 1.4. It is enough to assume that $\mathcal{I}$ can be represented by $\mathrm{Exh}(\varphi)$ (see below) such that $\{n \in \omega : \varphi(\{n\}) < \varepsilon\} \notin \mathcal{I}$ for each $\varepsilon > 0$. This property of $\mathrm{Exh}(\varphi)$ is really weaker than tallness.

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We always assume that if $\mathcal{I}$ is an ideal on ω then the ideal is *proper*, i.e. $\omega \notin \mathcal{I}$, and $\mathcal{I}$ contains all finite subsets of ω so in particular $\mathcal{I}$ is *non-principal*.

An ideal $\mathcal{I}$ on ω is *analytic (Borel etc.)* if $\mathcal{I} \subseteq \mathcal{P}(\omega) \simeq 2^\omega$ is an analytic (Borel etc.) set in the usual product topology of the Cantor-set. $\mathcal{I}$ is a *P-ideal* if for each countable $\mathcal{C} \subseteq \mathcal{I}$ there is an $A \in \mathcal{I}$ such that $I \subseteq^* A$ for each $I \in \mathcal{C}$, where $A \subseteq^* B$ iff $A \backslash B$ is finite. $\mathcal{I}$ is *tall* (or *dense*) if each infinite subset of ω contains an infinite element of $\mathcal{I}$.

The following families are well-known examples of tall analytic P-ideals: the *density zero ideal*: $\mathcal{Z} = \big\{A \subseteq \omega : \lim_{n\to\infty} \frac{|A\cap n|}{n} = 0\big\}$, and the *summable ideal*: $\mathcal{I}_{\frac{1}{n}} = \big\{A \subseteq \omega : \sum_{n\in A} \frac{1}{n+1} < \infty\big\}$.

A function $\varphi : \mathcal{P}(\omega) \to [0,\infty]$ is a *submeasure on* ω iff $\varphi(\emptyset) = 0$, $\varphi(A) \le \varphi(B)$ for $A \subseteq B \subseteq \omega$, $\varphi(A\cup B) \le \varphi(A) + \varphi(B)$ for $A, B \subseteq \omega$, and $\varphi(\{n\}) < \infty$ for $n \in \omega$. A submeasure φ is *lower semicontinuous* (lsc in short) iff $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for each $A \subseteq \omega$. Note that if φ is an lsc submeasure on ω then it is σ-subadditive, i.e. $\varphi(\bigcup_{n\in\omega} A_n) \le \sum_{n\in\omega} \varphi(A_n)$ holds for $A_n \subseteq \omega$. We assign an ideal to an lsc submeasure φ as follows

$$\mathrm{Exh}(\varphi) = \big\{A \subseteq \omega : \lim_{n\to\infty} \varphi(A\backslash n) = 0\big\}.$$

$\mathrm{Exh}(\varphi)$ is an $F_{\sigma\delta}$ P-ideal or equal to $\mathcal{P}(\omega)$. It is straightforward to see that $\mathrm{Exh}(\varphi)$ is tall iff $\lim_{n\to\infty} \varphi(\{n\}) = 0$. Furthermore, we can assume without changing $\mathrm{Exh}(\varphi)$ that $\varphi(\{k\}) > 0$ for each $k \in \omega$ because if $\varphi'(A) = \varphi(A) + \sum_{k\in A} 2^{-k}$, then φ' is also an lsc submeasure on ω, $\varphi'(\{k\}) > 0$ for each $k \in \omega$, and $\mathrm{Exh}(\varphi') = \mathrm{Exh}(\varphi)$.

Theorem 1.6. ([6], Theorem 3.1) *If $\mathcal{I}$ is an analytic P-ideal then $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lsc φ.*

Therefore each analytic P-ideal is $F_{\sigma\delta}$ (i.e. $\mathbf{\Pi}^0_3$) so it is a Borel subset of 2^ω.

In Section 2. we recall the definition of *slaloms* and prove that if a forcing notion $\mathbb{P}$ adds a slalom capturing all ground model real, then for each tall analytic P-ideal $\mathcal{I}$ coded in the ground model, $\mathbb{P}$ adds a new element of $\mathcal{I}$ which almost contains old elements of $\mathcal{I}$.

In Section 3. we recall the model of [3] and its main properties. At last, in Section 4. we prove our main Theorem 1.4.

2. Dominating analytic P-ideals

If φ is an lsc submeasure on ω, then clearly φ is determined by $\varphi \restriction [\omega]^{<\omega}$ so we can talk about the "same" analytic P-ideal in forcing extensions without using analytic absoluteness.

Definition 2.1. Let $\mathcal{I}$ be an analytic ideal on ω. A forcing notion $\mathbb{P}$ is *$\mathcal{I}$-dominating* if $\mathbb{P}$ adds a new element of $\mathcal{I}$ which almost contains all elements of $\mathcal{I} \cap V$, in other words $\mathcal{I} \cap V$ is bounded in $(\mathcal{I} \cap V^{\mathbb{P}}, \subseteq^*)$, i.e.

$$\Vdash_{\mathbb{P}} \exists\, B \in \mathcal{I} \cap V[\dot{G}]\ \forall\, A \in \mathcal{I} \cap V\ \ (A \subseteq^* B).$$

Let $\mathcal{S} = \mathsf{X}_{n\in\omega}[\omega]^{\le n}$ be the set of *slaloms*. If $f \in \omega^\omega$ and $S \in \mathcal{S}$ then we say S *almost captures* f and write $f \sqsubseteq^* S$ iff $\forall^\infty\, n\ f(n) \in S(n)$.

Definition 2.2. A forcing notion $\mathbb{P}$ adds a *slalom over the ground model* if $\mathbb{P}$ adds a new element of $\mathcal{S}$ which almost captures all ground model reals, i.e.

$$\Vdash_{\mathbb{P}} \exists\, S \in \mathcal{S} \cap V[\dot{G}]\ \forall\, f \in \omega^\omega \cap V\ \ (f \sqsubseteq^* S).$$

First of all we mention the following known result on the connection between slaloms and measure zero sets.

Theorem 2.3. ([4], 534I) *A forcing notion $\mathbb{P}$ adds a slalom over V iff in $V^{\mathbb{P}}$ the union of null sets coded in V has measure zero, i.e.* $\Vdash_{\mathbb{P}} \bigcup(\mathcal{N} \cap V) \in \mathcal{N}$.

Let $\mathcal{I} = \mathrm{Exh}(\varphi)$ be an analytic P-ideal, and in the rest of the paper fix a bijection $e : \omega \to [\omega]^{<\omega}$. If S is a slalom, then let

$$(*) \qquad I(S) = \bigcup_{n \in \omega} \bigcup \big\{e(k) : k \in S(n) \wedge \varphi(e(k)) < 2^{-n}\big\}.$$

The following Proposition is the core of our main Theorem 1.4.

Proposition 2.4. *Assume that a forcing notion $\mathbb{P}$ adds a slalom S over V. Then $I(S) \in \mathcal{I} \cap V^{\mathbb{P}}$ and $I(S)$ almost contains all elements of $\mathcal{I} \cap V$ so $\mathbb{P}$ is $\mathcal{I}$-dominating for each analytic P-ideal $\mathcal{I}$.*

Proof. For each n the set $\bigcup\big\{e(k) : k \in S(n) \wedge \varphi(e(k)) < 2^{-n}\big\}$ is finite and has measure less then $\frac{n}{2^n}$ so $I(S) \in \mathcal{I}$.

Assume $A \in \mathcal{I} \cap V$. Then let $d_A(n) = \min\{k \in \omega : \varphi(A \setminus k) < 2^{-n}\}$ and

$$f_A(n) = e^{-1}\big(A \cap [d_A(n), d_A(n+1))\big).$$

Clearly $\varphi\big(e(f_A(n))\big) < 2^{-n}$ and $f_A \in \omega^\omega \cap V$. Because S is a slalom over V, there is an N such that $f_A(n) \in S(n)$ for each $n \geq N$ so $A \setminus d_A(N) \subseteq I(S)$. We are done. □

We recall the definition of the *localization forcing.* Let $\mathcal{T} = \bigcup_{n\in\omega} \mathsf{X}_{k<n} [\omega]^{\leq k}$ be the tree of initial slaloms. $p \in \mathbb{LOC}$ iff $p = (s^p, F^p)$ where

(1) $s^p \in \mathcal{T}$ and $F^p \subseteq \omega^\omega$,
(2) $|F^p| \leq |s^p|$.

$q \leq p$ iff

(a) $s^q \supseteq s^p$ and $F^q \supseteq F^p$,
(b) $\forall\, n \in |s^q| \setminus |s^p|\ \forall\, f \in F^p\ f(n) \in s^q(n)$.

Lemma 2.5. (Folklore) $\mathbb{LOC}$ *is σ-n-linked for each n (so ccc) and adds a slalom over the ground model. More explicitly, if G is $\mathbb{LOC}$-generic over V then $S = \bigcup\{s^p : p \in G\} \in V^{\mathbb{LOC}}$ is a slalom over V.*

We will use a special version of the localization forcing (see [3], Definition 3.1): $p \in \mathbb{LOC}^*$ iff $p = (s^p, w^p, F^p)$ where

(1) $s^p \in \mathcal{T}$, $w^p \in \omega$, $F^p \subseteq \omega^\omega$,
(2) $|F^p| \leq w^p \leq |s^p|$,

$q \leq p$ iff

(a) $s^q \supseteq s^p$, $w^q \geq w^p$, and $F^q \supseteq F^p$,
(b) $\forall\, n \in |s^q| \setminus |s^p|\ \forall\, f \in F^p\ f(n) \in s^q(n)$,
(c) $w^q \leq w^p + |s^q| - |s^p|$,
(d) $\forall\, n \in |s^q| \setminus |s^p|\ |s^q(n)| \leq w^p + n - |s^p|$.

Lemma 2.6. ([3], Lemma 3.2, 3.3, and 3.4) $\mathbb{LOC}^*$ *is σ-linked (so ccc) and adds a slalom over the ground model.*

3. The forcing notion

In this section, we recall the model of [3] and its main properties.

Let $(Q, \leq)$ be a partially ordered set such that each countable subset of Q has a strict upper bound in Q. Let $Q^* = Q \cup \{Q\}$ and extend the partial order to this set with $x < Q$ for each $x \in Q$.

Fix a well-founded cofinal $R \subseteq Q$ and a rank function on $R^* = R \cup \{Q\}$, $\varrho : R^* \to \mathrm{On}$. Extend ϱ to Q^* by letting $\varrho(x) = \min\{\varrho(y) : y \in R^*, x < y\}$ for $x \in Q \setminus R$. For $x, y \in Q^*$ define $x \ll y$ iff $x < y$ and $\varrho(x) < \varrho(y)$. Further notations:

- $Q_x = \{y \in Q : y \ll x\}$ for $x \in Q^*$,
- $D_\xi = \{x \in D : \varrho(x) = \xi\}$ for $D \subseteq Q$ and $\xi \in \mathrm{On}$,
- $D_{<\xi} = \{x \in D : \varrho(x) < \xi\}$ for $D \subseteq Q$ and $\xi \in \mathrm{On}$,
- $D_{\leq x} = \{y \in D : \varrho(y) = \varrho(x), y \leq x\}$ for $D \subseteq Q$ and $x \in Q$.

If $E \subseteq D \subseteq Q$, we say that E *is downward closed in* D, $E \subseteq_{\text{d.c.}} D$ in short, if $y \in E$ whenever $y \in D$ and $y \leq x \in E$ for some x.

Definition 3.1. ([3], Definition 3.1) The forcing notions $\mathbb{N}_a$ for $a \in Q^*$ are defined by recursion on $\varrho(a)$.

$p = \{(s^p_x, w^p_x, F^p_x) : x \in D^p\} \in \mathbb{N}_a$ where $D^p \in [Q_a]^{<\omega}$ if the following hold:

(I) for $x \in D^p$, $s^p_x \in \mathcal{T}$, $w^p_x \in \omega$, and F^p_x is a set of nice $\mathbb{N}_x$-names for elements of ω^ω with $|F^p_x| \leq w^p_x$;

(II) for $x \in D^p$, $\sum\{w^p_z : z \in D^p_{\leq x}\} \leq |s^p_x|$;

(III) for each $\xi \in \varrho'' D^p$ there is an $\ell^p_\xi \in \omega$ such that $|s^p_x| = \ell^p_\xi$ for each $x \in D^p_\xi$.

If $p \in \mathbb{N}_a$ and $b \in Q_a$, define $p \restriction b \in \mathbb{N}_b$ by letting

$$p \restriction b = \{(s^p_x, w^p_x, F^p_x) : x \in D^p \cap Q_b\}.$$

$p \leq_{\mathbb{N}_a} q$ iff

(A) $D^p \supseteq D^q$;

(B) $\forall\, x \in D^q \ \big(s^p_x \supseteq s^q_x \wedge w^p_x \geq w^q_x \wedge F^p_x \supseteq F^q_x\big)$;

(C) $\forall\, x \in D^q \ \forall\, n \in |s^p_x| \setminus |s^q_x| \ \forall\, \dot{f} \in F^q_x \ \big(p \restriction x \Vdash_{\mathbb{N}_x} \dot{f}(n) \in s^p_x(n)\big)$;

(D) $\forall\, \xi \in \varrho'' D^q \ \forall\, x, y \in D^q_\xi \ \big(x < y \Rightarrow \forall\, n \in \ell^p_\xi \setminus \ell^q_\xi \ s^p_x(n) \subseteq s^p_y(n)\big)$;

(E) $\forall\, \xi \in \varrho'' D^q$

$$\sum\{w^p_x : x \in D^p_\xi\} \leq \sum\{w^q_x : x \in D^q_\xi\} + (\ell^p_\xi - \ell^q_\xi);$$

(F) $\forall\, \xi \in \varrho'' D^q \ \forall\, E \subseteq_{\text{d.c.}} D^q_\xi \ \forall\, n \in \ell^p_\xi \setminus \ell^q_\xi$

$$\big|\bigcup\{s^p_x(n) : x \in E\}\big| \leq \sum\{w^q_x : x \in E\} + (n - \ell^q_\xi).$$

Proposition 3.2. ([3], Proposition 4.3)

(a) *If* $p, q \in \mathbb{N}_a$, $p \leq_{\mathbb{N}_a} q$, *and* $b \in Q_a$, *then* $p \restriction b \leq_{\mathbb{N}_b} q \restriction b$.

(b) $\leq_{\mathbb{N}_a}$ *is a partial order.*

(c) *If* $a, b \in Q^*$ *and* $p, q \in \mathbb{N}_a \cap \mathbb{N}_b$, *then* $p \leq_{\mathbb{N}_a} q \iff p \leq_{\mathbb{N}_b} q$.

From now on we write $\leq$ $(= \leq_{\mathbb{N}_Q})$ instead of $\leq_{\mathbb{N}_a}$.

Definition 3.3. ([3], Definition 4.4) For an $A \subseteq_{\text{d.c.}} Q$, let $\mathbb{N}_A = \{p \in \mathbb{N}_Q : D^p \subseteq A\}$, and for $p \in \mathbb{N}_Q$, let $p \restriction A = \{(s^p_x, w^p_x, F^p_x) : x \in D^p \cap A\} \in \mathbb{N}_A$. Furthermore, if $\xi \in \mathrm{On}$ then let $\mathbb{N}_\xi = \mathbb{N}_{Q_{<\xi}}$, $p \restriction \xi = p \restriction Q_{<\xi} \in \mathbb{N}_\xi$, and $p \restriction [\xi, \infty) = \{(s^p_x, w^p_x, F^p_x) : x \in D^p \setminus Q_{<\xi}\}$.

So we have $\mathbb{N}_a = \mathbb{N}_{Q_a}$ for each $a \in Q^*$, and $\mathbb{N}_Q$ has the same meaning if we consider Q either as an element of Q^* or as a subset of Q.

Before the following lemma we recall the definition of *complete subforcing*: Assume $\mathbb{P} = (P, \leq_{\mathbb{P}})$ is a subforcing of $\mathbb{Q} = (Q, \leq_{\mathbb{Q}})$, i.e. $P \subseteq Q$ and $\leq_{\mathbb{P}} = \leq_{\mathbb{Q}} \restriction P$. Then we say that $\mathbb{P}$ is a complete subforcing of $\mathbb{Q}$ and write $\mathbb{P} \leq_c \mathbb{Q}$ if maximal antichains of $\mathbb{P}$ are maximal antichains in $\mathbb{Q}$ as well.

Lemma 3.4. ([3], Lemma 4.6) *If $A, B \subseteq_{d.c.} Q$ and $A \subseteq B$, then $\mathbb{N}_A \leq_c \mathbb{N}_B$.*

Remark 3.5. In [3] Lemma 4.6, exactly the following stronger result was proved: If $p \in \mathbb{N}_B$, $r \in \mathbb{N}_A$, and $r \leq p \restriction A$ then there is a $q \in \mathbb{N}_B$ satisfying $q \leq p, r$.

Lemma 3.6. ([3], Lemma 4.10) *$\mathbb{N}_Q$ has ccc.*

We will use the following density arguments.

Lemma 3.7. ([3], Lemma 5.1, 5.2, 5.3, and 5.4) *If $a \in A \subseteq_{d.c.} Q$, $\xi \in \mathrm{On}$, $N \in \omega$, and $\dot{f}$ is a nice $\mathbb{N}_a$-name for an element of ω^ω, then the following sets are dense in $\mathbb{N}_A$:*

(i) $\{p \in \mathbb{N}_A : a \in D^p\}$;
(ii) $\{p \in \mathbb{N}_A : \xi \in \varrho'' D^p \wedge \ell^p_\xi \geq N\}$;
(iii) $\{p \in \mathbb{N}_A : a \in D^p \wedge w^p_a \geq |F^p_a| + 1\}$;
(iv) $\{p \in \mathbb{N}_A : a \in D^p \wedge \dot{f} \in F^p_a\}$.

For an $a \in Q$, let $\dot{S}_a$ be an $\mathbb{N}_Q$-name such that

$$\Vdash_{\mathbb{N}_Q} \dot{S}_a = \bigcup \{s^p_a : p \in \dot{G}\}.$$

Using (i) and (ii) from Lemma 3.7, $\Vdash_{\mathbb{N}_Q} \dot{S}_a \in \mathcal{S}$ for each $a \in Q$. Furthermore using (iv) and the definition of $\mathbb{N}_Q$ we know that $\dot{S}_a$ is a slalom over $V[\dot{G} \cap \mathbb{N}_a]$, i.e.

$$(\sharp_1) \qquad \Vdash_{\mathbb{N}_Q} \forall\, f \in \omega^\omega \cap V[\dot{G} \cap \mathbb{N}_a]\; f \sqsubseteq^* \dot{S}_a.$$

At last, using the definition of $\mathbb{N}_Q$ it is clear that if $\varrho(a) = \varrho(b)$ and $a < b$ then

$$(\dagger) \qquad \Vdash_{\mathbb{N}_Q} \forall^\infty\, n\; \dot{S}_a(n) \subseteq \dot{S}_b(n).$$

4. Proof of the main Theorem 1.4

Let $\mathcal{I} = \mathrm{Exh}(\varphi)$ be a tall analytic P-ideal. We will use $(*)$ and Proposition 2.4: for a slalom $S \in \mathcal{S}$, let

$$I(S) = \bigcup_{n \in \omega} \bigcup \{e(k) : k \in S(n) \wedge \varphi(e(k)) < 2^{-n}\} \in \mathcal{I}.$$

We prove that in $V^{\mathbb{N}_Q}$ the set $\{I(\dot{S}_a) : a \in Q\} \subseteq \mathcal{I}$ is

(i) cofinal, i.e. $\forall\, I \in \mathcal{I} \cap V^{\mathbb{N}_Q}\; \exists\, a \in Q\; I \subseteq^* I(\dot{S}_a)$;
(ii) order isomorphic to $(Q, \leq)$, i.e. $I(\dot{S}_a) \subseteq^* I(\dot{S}_b)$ iff $a \leq b$.

The only difficult step is to show that $a \not\leq b$ implies $I(\dot{S}_a) \not\subseteq^* I(\dot{S}_b)$.

It is clear from $(\sharp_1)$ and Proposition 2.4 that for each $a \in Q$

$$(\sharp_2) \qquad \Vdash_{\mathbb{N}_Q} \forall\, I \in \mathcal{I} \cap V[\dot{G} \cap \mathbb{N}_a]\; I \subseteq^* I(\dot{S}_a).$$

Lemma 4.1. $\Vdash_{\mathbb{N}_Q}$ *"$\{I(\dot{S}_a) : a \in Q\}$ is cofinal in $(\mathcal{I}, \subseteq^*)$".*

Proof. Let $\dot{I}$ be a nice $\mathbb{N}_Q$-name for an element of $\mathcal{I}$. Using that $\mathbb{N}_Q$ is ccc and that each countable subset of Q is (strictly) bounded in Q, there is an $a \in Q$ such that $\dot{I}$ is an $\mathbb{N}_a$-name. Then $\Vdash_{\mathbb{N}_Q} I \subseteq^* I(\dot{S}_a)$ by ($\sharp_2$). $\square$

Lemma 4.2. *Assume $a, b \in Q$ and $a \leq b$. Then $\Vdash_{\mathbb{N}_Q} I(\dot{S}_a) \subseteq^* I(\dot{S}_b)$.*

Proof. If $a \ll b$ then $\Vdash_{\mathbb{N}_Q} I(\dot{S}_a) \in \mathcal{I} \cap V[\dot{G} \cap \mathbb{N}_b]$ so we are done by ($\sharp_2$). If $\varrho(a) = \varrho(b)$ then we are done by (†). $\square$

We will need the following version of Lemma 3.7 (ii) which says that we can extend conditions in a natural way.

Lemma 4.3. *Assume $p \in \mathbb{N}_Q$, $\xi \in \varrho'' D^p$, and $m \geq \ell^p_\xi$. Then there is a $q \leq p$ such that $D^q_\xi = D^p_\xi$ and $q \restriction \xi$ forces that $\forall\ b \in D^q_\xi\ \forall\ n \in [\ell^p_\xi, m]$*

$$s^q_b(n) = \big\{\dot{f}(n) : \dot{f} \in \bigcup\{F^p_{b'} : b' \in D^p_{\leq b}\}\big\}.$$

Proof. First we choose an $r \in \mathbb{N}_\xi$, $r \leq p \restriction \xi$ which decides $\dot{f} \restriction [\ell^p_\xi, m]$ for each $\dot{f} \in \bigcup\{F^p_{b'} : b' \in D^p_{\leq b}\}$: $r \Vdash_{\mathbb{N}_\xi} \dot{f} \restriction [\ell^p_\xi, m] = g_{\dot{f}}$ for some $g_{\dot{f}} \in \omega^{[\ell^p_\xi, m]}$.

Now let q be the following condition:

(i) $q \restriction \xi = r$, $q \restriction [\xi + 1, \infty) = p \restriction [\xi + 1, \infty)$, and $D^q_\xi = D^p_\xi$;
(ii) if $b \in D^q_\xi$ then let $|s^q_b| = m + 1$, $s^q_b \restriction \ell^p_\xi = s^p_b$, $w^q_b = w^p_b$, and $F^q_b = F^p_b$;
(iii) if $b \in D^q_\xi$ and $n \in [\ell^p_\xi, m]$ then let

$$s^q_b(n) = \big\{g_{\dot{f}}(n) : \dot{f} \in \bigcup\{F^p_{b'} : b' \in D^p_{\leq b}\}\big\}.$$

Clearly $q \in \mathbb{N}_Q$. We have to show that $q \leq p$. (A), (B), (C), (D), and (E) hold trivially.

To see (F) assume $E \subseteq_{\text{d.c.}} D^p_\xi$ and $n \in [\ell^p_\xi, m]$ $(m + 1 = \ell^q_\xi)$. Then

$$\big|\bigcup\{s^q_x(n) : x \in E\}\big| = \big|\big\{g_{\dot{f}}(n) : \dot{f} \in \bigcup\{F^p_x : x \in E\}\big\}\big| \leq \sum\{|F^p_x| : x \in E\} \leq$$

$$\sum\{w^p_x : x \in E\} \leq \sum\{w^p_x : x \in E\} + (n - \ell^p_\xi).$$

$\square$

In Lemma 4.4 we will use the following notation: if $s \in \mathcal{T}$ is an initial slalom then let

$$I(s) = \bigcup_{n<|s|} \bigcup\{e(k) : k \in s(n) \wedge \varphi(e(k)) < 2^{-n}\} \in [\omega]^{<\omega}.$$

Clearly, if $p \in \mathbb{N}_Q$ and $a \in D^p$, then $p \Vdash_{\mathbb{N}_Q} I(s^p_a) \subseteq I(\dot{S}_a)$.

Lemma 4.4. *Assume $a, b \in Q$ and $a \not\leq b$. Then $\Vdash_{\mathbb{N}_Q} I(\dot{S}_a) \not\subseteq^* I(\dot{S}_b)$.*

Proof. Let $p \in \mathbb{N}_Q$ and $N \in \omega$. We have to find a $q \leq p$ such that $q \Vdash_{\mathbb{N}_Q} I(\dot{S}_a) \setminus N \not\subseteq I(\dot{S}_b)$. Using Lemma 3.7 (i) and (iii) we can assume that $a, b \in D^p$ and $|w^p_a| \geq |F^p_a| + 1$.

Let $M = \max\{|s^p_a|, |s^p_b|\}$. Using Lemma 3.7 we can assume that M is large enough such that $\varphi(\{k\}) \geq 2^{-M}$ for each $k < N$. For each $m \in \omega$ let

$$X_m = \{k \in \omega : 2^{-m-1} \leq \varphi(\{k\}) < 2^{-m}\}.$$

Let $\xi = \varrho(b)$. Using that $\mathbb{N}_{b'} \leq_c \mathbb{N}_b$ if $b' \in D^p_{\leq b}$ by Lemma 3.4, we can define a descending sequence in $\mathbb{N}_b$: $p \restriction b \geq r_M \geq r_{M+1} \geq \dots$ such that r_m decides $\dot{f} \restriction [\ell^p_\xi, m]$ for each $\dot{f} \in \bigcup\{F^p_{b'} : b' \in D^p_{\leq b}\}$. Let $I_m : [\ell^p_\xi, m] \to [\omega]^{<\omega}$ be defined by

$$r_m \Vdash_{\mathbb{N}_b} I_m(n) = \bigcup \{e(\dot{f}(n)) : \dot{f} \in \bigcup\{F^p_{b'} : b' \in D^p_{\leq b}\} \wedge \varphi\big(e(\dot{f}(n))\big) < 2^{-n}\}.$$

Claim. *There is an $m \geq M$ such that $X_m \nsubseteq I(s^p_b) \cup \bigcup\{I_m(n) : n \in [\ell^p_\xi, m]\}$.*

Proof of the Claim. Assume on the contrary that there is no such an m. Then

$$X_m \subseteq I(s^p_b) \cup \bigcup\{I_m(n) : n \in [\ell^p_\xi, m]\}$$

for each $m \geq M$. Clearly $\omega \subseteq^* \bigcup_{m \geq M} X_m$ by tallness[1], the sets $I(s^p_b)$ and $I_m(n)$ are finite, and if $n \leq m_1 \leq m_2$ then $I_{m_1}(n) = I_{m_2}(n)$ so we have

$$\omega \subseteq^* I(s^p_b) \cup \bigcup_{m \geq M} \bigcup_{n=\ell^p_\xi}^{m} I_m(n) \subseteq^* \bigcup_{m \geq M} I_m(m).$$

Using that $\varphi(I_m(n)) \leq |D^p_{\leq b}| \frac{n}{2^n}$ we obtain that $\omega \in \mathcal{I}$, a contradiction. □

Assume m is suitable in the Claim and let $r = r_m$. Fix a $k \in X_m \setminus \big(I(s^p_b) \cup \bigcup\{I_m(n) : n \in [\ell^p_\xi, m]\}\big)$. Then there is a $\tilde{k}$ such that $e(\tilde{k}) = \{k\}$. Let $\dot{g}$ be the canonical $\mathbb{N}_a$-name for the constant function with value $\tilde{k}$. Denote $p' \in \mathbb{N}_Q$ the condition which extends p by putting $\dot{g}$ into F^p_a (this is really a condition extending p because of our assumption $|w^p_a| \geq |F^p_a| + 1$). We know that $p \restriction b = p' \restriction b$ because $a \notin Q_b$ so $r \leq p' \restriction b$.

Using Remark 3.5 for $Q_b \subseteq Q_{<\xi}$, $r \in \mathbb{N}_b$, and $p' \restriction \xi \in \mathbb{N}_\xi$ we can find a $q' \in \mathbb{N}_\xi$ with $q' \leq r, p' \restriction \xi$. Let $p'' = q' \cup p' \restriction [\xi, \infty) \leq p$.

At last using Lemma 4.3 we can extend p'' to a q such that $D^q_\xi = D^{p''}_\xi (= D^p_\xi)$ and $q \restriction \xi \Vdash_{\mathbb{N}_\xi} \forall\, n \in [\ell^p_\xi, m]\ s^q_b(n) = \{\dot{f}(n) : \dot{f} \in \bigcup\{F^p_{b'} : b' \in D^p_{\leq b}\}\}$.

Because $q \restriction b \leq r$ we obtain that

$$q \Vdash_{\mathbb{N}_Q} I(s^q_b) \subseteq I(s^p_b) \cup \bigcup\{I_m(n) : n \in [\ell^p_\xi, m]\}.$$

By the choice of k and p' it is clear that $\tilde{k} \in s^q_a(m)$ and $\varphi(e(\tilde{k})) = \varphi(\{k\}) < 2^{-m}$ so $k \in I(s^q_a)$ which implies that $q \Vdash_{\mathbb{N}_Q} k \in I(\dot{S}_a) \setminus N$.

To show that $q \Vdash_{\mathbb{N}_Q} k \notin I(\dot{S}_b)$ we know that $k \notin I(s^q_b)$ and if there would be a $\bar{q} \leq q$ such that $k \in I(s^{\bar{q}}_b)$, then there would be an $n > m$ and a $k' \in s^{\bar{q}}_b(n)$ such that $k \in e(k') \subseteq I(s^{\bar{q}}_b)$ but then $2^{-n} > \varphi(e(k')) \geq \varphi(\{k\}) \geq 2^{-m-1}$ would give a contradiction because $n \geq m + 1$. The proof of Lemma 4.4 is done. □

Now we have finished the proof of our main Theorem 1.4.

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[1]This is the only point in the proof where we used tallness of the ideal. As we mentioned in Remark 1.5, it would be enough to assume that $\bigcup_{m \geq M} X_m = \{k \in \omega : \varphi(\{k\}) < 2^{-M}\} \notin \mathcal{I}$.

References

[1] Tomek Bartoszynski and Masaru Kada: *Hechler's theorem for the meager ideal*, Topology Appl. **146-147** (2005), pages 429-435.

[2] Maxim R. Burke: *A proof of Hechler's theorem on embedding $\aleph_1$-directed sets cofinally into* $(\omega^\omega, <^*)$, Arch. Math. Logic **36** (1997), pages 399-403.

[3] Maxim R. Burke and Masaru Kada: *Hechler's theorem for the null ideal*, Arch. Math. Logic **43** (2004), pages 703-722.

[4] David H. Fremlin: *Measure Theory. Set-theoretic Measure Theory.* Torres Fremlin, Colchester, England, 2004. Available at http://www.essex.ac.uk/maths/staff/fremlin/mt.html

[5] S.H. Hechler: *On the existence of certain cofinal subsets of* ${}^\omega\omega$, Axiomatic set theory, editor: Jech, Thomas, pages 155-173, Amer. Math. Soc., 1974.

[6] Słamowir Solecki: *Analytic P-ideals and their applications*, Ann. Pure Appl. Logic **99** (1999), pages 51-72.

[7] Lajos Soukup: *Pcf theory and cardinal invariants of the reals*, unpublished notes (2001)

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