# SPECIAL ARONSZAJN TREES AND KUREPA TREES

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ABSTRACT. Assuming $\omega$ many supercompact cardinals, we show that it is consistent that for all $0 < n < \omega$, all $\aleph_n$-Aronszajn trees are special and there exist such, and there are no $\aleph_n$-Kurepa trees. We also extend this result to a global version.

## 1. INTRODUCTION

In this paper, we construct models in which all Aronszajn trees of some heights are special, there are such, and there are no Kurepa trees of certain heights.

Laver and Shelah showed in their paper [LS81] that it is consistent with CH that all $\aleph_2$-Aronszajn trees are special and there exists one. Golshani and Hayut extended this result in [GH20] with a similar but more involved technique to show that it is consistent that for all successors of regular cardinals all Aronszajn trees are special and there exists one. So the questions about Suslin trees and Aronszajn trees are settled in this model. What about Kurepa trees?

In [Bau84] Baumgartner showed that PFA implies that there are no $\aleph_2$-Aronszajn trees and no $\aleph_1$-Kurepa trees. Cummings proved in [Cum18], assuming a weakly compact cardinal, that there is a generic extension in which there are no $\aleph_2$-Aronszajn trees and there is an $\aleph_1$-Kurepa tree.

Motivated by [GH20] and [Cum18], we have worked on models in which all Aronszajn trees on some cardinals are special and they exist, and there are no Kurepa trees of some heights. It turned out that in the forcing extension as in [GH20] $\aleph_n$-Kurepa trees actually exist, so we had to change the forcing iteration: Instead of using a product of Lévy collapses we use an iteration of these collapses in the beginning of the iteration. In fact, we have constructed a model in which for every $0 < n < \omega$, all $\aleph_n$-Aronszajn trees are special, there are such, and there exists no $\aleph_n$-Kurepa tree. To get this, we start with $\omega$ many supercompact cardinals which we collapse with an iteration of Lévy collapses to the $\aleph_n$'s. These collapses are followed by

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specializing forcings for the Aronszajn trees. We use supercompact embeddings to show that the specializing forcings have a suitable chain condition. Then we argue that there is no Kurepa tree in the final model. If there were such a tree, a small regular subforcing would capture it, which can be seen using supercompact embeddings. In the extension by the small subforcing, the tree cannot have many branches, and an analysis of the quotient shows that no branches are added and therefore the tree is not a Kurepa tree in the final model.

## 2. Preliminaries

**Definition 2.1.** A *tree* is a set of nodes $T$, together with an order $<$, with the following properties:

(1) There exists a *root*, i.e., an $r \in T$ with $r = s$ or $r < s$ for all $s \in T$.
(2) $(\{t \in T \mid t < s\}, <)$ is a well-order for every $s \in T$.

For a tree $T$ with the order $<$ we use the following notation:

- For $s \in T$ the *length of* $s$ is the order type of $(\{t \in T \mid t < s\}, <)$; we denote the length of $s$ by $|s|$.
- The *$\xi$th level of* $T$, denoted $T_\xi$, is the set of nodes in $T$ which have length $\xi$.
- The *height of* $T$ is the smallest ordinal $\xi$ such that $T_\xi = \emptyset$.
- A *cofinal branch* of $T$ is a chain in $(T, <)$ whose order type is equal to the height of the tree.

Now we can define Kurepa trees and Aronszajn trees:

**Definition 2.2.**

(1) A $\kappa$-Aronszajn tree is a tree of height $\kappa$ all whose levels are smaller than $\kappa$ which has no cofinal branch.
(2) A $\kappa^+$-Aronszajn tree $T$ is special if there exists a function $f: T \to \kappa$ such that if $x < y$ then $f(x) \neq f(y)$.
(3) A $\kappa$-Kurepa tree is a tree of height $\kappa$ all whose levels are smaller than $\kappa$ which has more than $\kappa$ many cofinal branches.

The following is easy to see:

**Remark 2.3.** If there exists a $\kappa^+$-(non-special)-Aronszajn tree or a $\kappa^+$-Kurepa tree, then there exists one with $T_\xi \subseteq \{\xi\} \times \kappa$ for each $\xi < \kappa^+$.

Therefore we will always assume that our trees of successor cardinal height satisfy $T_\xi \subseteq \{\xi\} \times \kappa$.

Kurepa trees and Aronszajn trees have been studied a lot. In the following we will present some of the classical results.

The existence of an $\aleph_1$-Kurepa tree is independent from ZFC. On the one hand, if $V = L$, then there exists an $\aleph_1$-Kurepa tree, on the other hand, it is consistent with ZFC that there exists no $\aleph_1$-Kurepa tree. Since the proof of the second uses ingredients which we will use later, we will give the proof of it here.

First we define the Lévy collapse and prove its basic properties.

**Definition 2.4.** Let $\lambda$ be an inaccessible cardinal and $\kappa$ a regular cardinal with $\kappa < \lambda$. The *Lévy collapse* of $\lambda$ to $\kappa^+$, written as $\mathrm{Col}(\kappa, {<}\lambda)$ is defined as follows: For each cardinal $\kappa < \alpha < \lambda$ let $\mathrm{Col}(\kappa, \alpha)$ be the set of partial functions of size $< \kappa$ from $\kappa$ to $\alpha$, ordered by $q \leq p$ if $q \supseteq p$. Now let $\mathrm{Col}(\kappa, {<}\lambda) := \prod_{\kappa<\alpha<\lambda} \mathrm{Col}(\kappa, \alpha)$ with ${<}\kappa$-support.

**Lemma 2.5.** *Let $\lambda$ be an inaccessible cardinal and $\kappa$ a regular cardinal with $\kappa < \lambda$. The Lévy collapse* $\mathrm{Col}(\kappa, {<}\lambda)$ *is ${<}\kappa$-closed.*

*Proof.* Let $\mu < \kappa$ and let $\langle p_i \mid i < \mu\rangle$ be a decreasing sequence in $\mathrm{Col}(\kappa, {<}\lambda)$. Define a condition $p$ by letting $p(\alpha) := \bigcup_{i<\mu} p_i(\alpha)$ for every cardinal $\kappa < \alpha < \lambda$; it is easy to see that $p \in \mathrm{Col}(\kappa, {<}\lambda)$ by the following argument. Since $\mu < \kappa$ and for each $i < \mu$ and each $\kappa < \alpha < \lambda$ each $p_i(\alpha)$ is a partial function from $\kappa$ to $\alpha$ with $|\mathrm{dom}(p_i(\alpha))| < \kappa$ and $p_j(\alpha) \subseteq p_i(\alpha)$ for $j < i$, also $p(\alpha)$ is such a function. Moreover $\mathrm{supp}(p) = \bigcup_{i<\mu} \mathrm{supp}(p_i)$, so it is a union of less than $\kappa$ many sets of size $< \kappa$. Since $\kappa$ is regular, this is of size $< \kappa$. So $p \in \mathrm{Col}(\kappa, {<}\lambda)$ and clearly $p \leq p_i$ for each $i < \mu$. □

**Lemma 2.6.** *Let $\lambda$ be an inaccessible cardinal and $\kappa$ a regular cardinal with $\kappa < \lambda$. Then the Lévy collapse* $\mathrm{Col}(\kappa, {<}\lambda)$ *has the $\lambda$-c.c..*

*Proof.* Let $A \subseteq \mathrm{Col}(\kappa, {<}\lambda)$ be an antichain. By induction on $\alpha < \kappa$ we define $A_\alpha \subseteq A$ and $S_\alpha \subseteq \lambda$ such that $\langle A_\alpha \mid \alpha < \kappa\rangle$ and $\langle S_\alpha \mid \alpha < \kappa\rangle$ are $\subseteq$-increasing, as follows. Pick one element $p_0 \in A$. Let $A_0 := \{p_0\}$ and $S_0 := \mathrm{supp}(p_0)$. Assume we have defined $A_\alpha$ and $S_\alpha$, now define $A_{\alpha+1}$ and $S_{\alpha+1}$. For each $p \in \mathrm{Col}(\kappa, {<}\lambda)$ with $\mathrm{supp}(p) \subseteq S_\alpha$ pick $q \in A$ such that $q \restriction S_\alpha = p$, if such a $q$ exists. Let $A_{\alpha+1}$ be $A_\alpha$ together with these conditions $q$. Let $S_{\alpha+1} := \bigcup\{\mathrm{supp}(p) \mid p \in A_{\alpha+1}\}$. For limit ordinals $\alpha$, let $A_\alpha := \bigcup_{\beta<\alpha} A_\beta$ and $S_\alpha := \bigcup_{\beta<\alpha} S_\beta$ . Let $S := \bigcup_{\alpha<\kappa} S_\alpha$

Now we show that $A = \bigcup_{\alpha<\kappa} A_\alpha$. Let $p \in A$. Since $\mathrm{supp}(p) < \kappa$ it follows that there exists $\alpha < \kappa$ such that $\mathrm{supp}(p \restriction S) \subseteq S_\alpha$. Now one $q \in A$ with $q \restriction S_\alpha = p \restriction S_\alpha$ was picked to be in $A_{\alpha+1}$ (we know that one such $q$ exists, because $p$ is a witness for it). Since outside of $S_\alpha$ the support of $q$ and the support of $p$ are disjoint, $p$ and $q$ are compatible. Since they are both in $A$, they are the same, so $p \in A_{\alpha+1}$.

Now we show by induction on $\alpha$ that $|A_\alpha|, |S_\alpha| < \lambda$ for every $\alpha < \kappa$. First note that $|A_0| = 1$ and $|S_0| = |\mathrm{supp}(p_0)| < \kappa < \lambda$. Now we assume $|A_\alpha|$ and $|S_\alpha| < \lambda$ we show that the same holds true for $\alpha + 1$. Conditions $p$ with $\mathrm{supp}(p) \subseteq S_\alpha$ are partial functions from $S_\alpha$ to $\bigcup_{\beta\in S_\alpha} \mathrm{Col}(\kappa,\beta)$ with $p(\beta) \in \mathrm{Col}(\kappa,\beta)$. Since each $\mathrm{Col}(\kappa,\beta)$ has size $< \lambda$ and $|S_\alpha| < \lambda$ and $\lambda$ is inaccessible there exist $< \lambda$ many such functions, hence $|A_{\alpha+1}| < \lambda$. Since each condition has a support of size $< \kappa$, also $|S_{\alpha+1}| < \lambda$. For limits $\alpha$ it follows that $|A_\alpha|$ and $|S_\alpha| < \lambda$ because the union of less than $\lambda$ many sets of size smaller than $\lambda$ is still smaller than $\lambda$.

Hence $A$ is the union of $\kappa$ many sets of size smaller than $\lambda$, so $|A|$ is smaller than $\lambda$. □

**Lemma 2.7.** *Let $G$ be a generic filter over $V$ for* $\mathrm{Col}(\kappa, {<}\lambda)$. *Then all cardinals $\alpha$ from $V$ with $\alpha \leq \kappa$ or $\alpha \geq \lambda$ are cardinals in $V[G]$ and $V[G] \models \kappa = |\beta|$ for all cardinals $\kappa < \beta < \lambda$ and $V[G] \models \kappa^+ = \lambda$.*

*Proof.* The Lévy collapse $\mathrm{Col}(\kappa, {<}\lambda)$ is ${<}\kappa$-closed by Lemma 2.5, hence it does not change cardinals $\leq \kappa$. By Lemma 2.6 $\mathrm{Col}(\kappa, {<}\lambda)$ has the $\lambda$-c.c., hence it does not change cardinals above $\lambda$. For each $\kappa < \beta < \lambda$ the forcing $\mathrm{Col}(\kappa, \beta)$ adds a surjection from $\kappa$ to $\beta$, so it forces that $\kappa = |\beta|$, hence this holds in $V[G]$. Since $\lambda$ is a cardinal in $V[G]$, it follows that $\kappa^+ = \lambda$. □

The following lemma is essentially due to Silver [Sil71] and generalizes his lemma:

**Lemma 2.8.** *Assume there exists some $\mu < \mathrm{cf}(\lambda)$ such that $2^\mu \geq \lambda$. Let $\mathbb{P}$ be a ${<}\lambda$-closed forcing and $T$ a $\lambda$-tree. Then forcing with $\mathbb{P}$ does not add a new cofinal branch to $T$.*

*Proof.* Assume $\dot{b}$ is a name for a new cofinal branch. Let $p_{\langle\rangle} \in \mathbb{P}$, $x_{\langle\rangle} \in T$ and $\alpha_0$ be such that $p_{\langle\rangle} \Vdash \dot{b}(\alpha_0) = x_{\langle\rangle}$.

Now continue inductively: Assume $p_w$, $x_w$ and $\alpha_{|w|}$ have been defined such that $p_w \Vdash \dot{b}(\alpha_{|w|}) = x_w$. Since $\dot{b}$ is a new branch, there exists $\alpha_{|w|+1} > \alpha_{|w|}$ such that for every $v \in 2^{|w|}$ there exist two conditions $p_{v^\frown 0}, p_{v^\frown 1} \leq p_v$ and $x_{v^\frown 0} \neq x_{v^\frown 1}$ such that $p_{v^\frown 0} \Vdash \dot{b}(\alpha_{|w|+1}) = x_{v^\frown 0}$ and $p_{v^\frown 1} \Vdash \dot{b}(\alpha_{|w|+1}) = x_{v^\frown 1}$. For $w \in 2^{\leq\mu}$ of limit length, let $\alpha_{|w|} > \alpha_\delta$ for all $\delta < |w|$ and let $x_w \in T$ and $p_w$ be such that $p_w$ is a lower bound of $\langle p_{w\restriction\delta} \mid \delta < |w|\rangle$ such that $p_w \Vdash \dot{b}(\alpha_w) = x_w$; such a condition $p_w$ exists because $\mathbb{P}$ is ${<}\lambda$-closed. It follows easily that $x_w \neq x_v$ for $w \neq v$ of the same length. Therefore $|\{x_w \mid w \in 2^\mu\}| = 2^\mu \geq \lambda$. But $\{x_w \mid w \in 2^\mu\} \subseteq T_{\alpha_\mu}$, which contradicts the fact that all levels of $T$ are of size $< \lambda$. □

Let us fix the following notation: If $\mathbb{P}$ is a forcing, we denote with $G(\mathbb{P})$ a generic filter for $\mathbb{P}$, and $V[\mathbb{P}]$ is a shorthand for $V[G(\mathbb{P})]$.

**Theorem 2.9** (Silver)**.** *Let $k \geq 1$ and $\lambda$ be an inaccessible cardinal and $\mathbb{L}_\lambda = \mathrm{Col}(\aleph_k, {<}\lambda)$. Then there is no $\aleph_k$-Kurepa tree in $V[\mathbb{L}_\lambda]$.*

The following proposition is a generalization of Silver's theorem:

**Proposition 2.10.** *Let $k \geq 1$ and $\lambda$ be an inaccessible cardinal and $\mathbb{L}_\lambda = \mathrm{Col}(\aleph_k, {<}\lambda)$ . In $V[\mathbb{L}_\lambda]$ let $\mathbb{Q}$ be a forcing of size $\leq \aleph_k$ such that either $\mathbb{Q}$ is*

*$<\aleph_k$-distributive or $\mathbb{Q}$ has the $\aleph_k$-c.c.. Then there is no $\aleph_k$-Kurepa tree in $V[\mathbb{L}_\lambda * \mathbb{Q}]$.*

*Proof.* Let $\dot{\mathbb{Q}}$ be an $\mathbb{L}_\lambda$-name for $\mathbb{Q}$. Since $\mathbb{Q}$ is a set of size $\leq \aleph_k$ and $\mathbb{L}_\lambda$ has the $\lambda$-c.c., we can assume that $\dot{\mathbb{Q}}$ has size less than $\lambda$. Conditions in $\mathbb{L}_\lambda$ have a support of size $< \aleph_k$, therefore there exists $\mu_{\mathbb{Q}} < \lambda$ such that the $<\lambda$ many conditions in $\dot{\mathbb{Q}}$ belong to $\prod_{\alpha<\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha)$. Hence $\mathbb{Q} \in V[G \cap \prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha)]$. Therefore $\mathbb{L}_\lambda * \mathbb{Q}$ is equivalent to $\prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha<\lambda} \mathrm{Col}(\aleph_k, \alpha))$.

A similar argument works for an $\aleph_k$-tree: In $V[\mathbb{L}_\lambda * \mathbb{Q}]$ let $T$ be an $\aleph_k$-tree. Since $T$ with its order is an object of size $\aleph_k$ and $\mathbb{L}_\lambda * \mathbb{Q}$ has the $\lambda$-c.c., there exists a name $\dot{T}$ for it, of size less than $\lambda$. Conditions in $\mathbb{L}_\lambda$ have a support of size $< \aleph_k$, therefore there exists $\mu_{\mathbb{Q}} < \mu_T < \lambda$ such that the $<\lambda$ many conditions in $\dot{T}$ belong to $\prod_{\alpha<\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha\leq\mu_T} \mathrm{Col}(\aleph_k, \alpha))$. Hence $T \in V[G \cap \prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha\leq\mu_T} \mathrm{Col}(\aleph_k, \alpha))]$. In $V[G \cap \prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha\leq\mu_T} \mathrm{Col}(\aleph_k, \alpha))]$ still $2^{\aleph_k} < \lambda$, hence $T$ has less than $\lambda$ many branches here.

If $\mathbb{Q}$ is $<\aleph_k$-distributive, then $\prod_{\alpha>\mu_T} \mathrm{Col}(\aleph_k, \alpha)$ is still $<\aleph_k$-closed in $V[G \cap \prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \mu_{\mathbb{Q}}) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha\leq\mu_T} \mathrm{Col}(\aleph_k, \alpha))]$, hence by Lemma 2.8 it does not add branches to $T$. So $T$ is not a Kurepa tree in $V[\mathbb{L}_\lambda * \mathbb{Q}]$.

If $\mathbb{Q}$ has the $\aleph_k$-c.c., it follows by Lemma 3.8 below that $\prod_{\alpha>\mu_T} \mathrm{Col}(\aleph_k, \alpha)$ does not add branches to $T$. So $T$ is not a Kurepa tree in $V[\mathbb{L}_\lambda * \mathbb{Q}]$. □

**Proposition 2.11.** *If $2^{\aleph_n} = \aleph_{n+1}$, then there exists a special $\aleph_{n+2}$-Aronszajn tree.*

*Proof.* Let $Q$ be the set of those $x \in \aleph_{n+1}^{\aleph_n}$ with $|\{\alpha \in \aleph_n \mid x(\alpha) \neq 0\}| < \aleph_n$ with the lexicographical ordering. For every $x < y \in Q$ and every $\gamma < \aleph_{n+1}$ there exists an increasing sequence $\langle z_i \mid i < \gamma \rangle$ with $x < z_i < y$ for every $i < \alpha$: Let $\alpha$ be large enough such that for all $\beta \geq \alpha$ $x(\beta) = y(\beta) = 0$. Let $z_i(\alpha) = i$ and $z_i(\beta) = x(\beta)$ for $\beta \neq \alpha$. It is easy to see that the $z_i$ are as desired. It follows that $|Q| = \aleph_{n+1}$ using that $2^{\aleph_n} = \aleph_{n+1}$.

Now we construct $T$ by induction on the levels, such that for each $\alpha < \aleph_{n+2}$ the following holds:

(1) For each $\beta < \alpha$, for each $s \in T_\beta$ and each $x \in Q$ with $x > \sup(s)$ there exists $t \in T_\alpha$ such that $s < t$ and $x \geq \sup(t)$.

$T$ will consist of increasing sequences in $Q$.

Let $T_0 = \{\langle\rangle\}$. Assume now that $T_\alpha$ has been constructed. Let $T_{\alpha+1} := \{s^\frown x \mid s \in T_\alpha \wedge x > \sup(s)\}$. To see that (1) holds for $\alpha+1$, first let $s \in T_\alpha$ and $x \in Q$ with $x > \sup(s)$. So $s^\frown x \in T_{\alpha+1}$ is a witness for (1). Now let $\beta < \alpha$, $s \in T_\beta$ and $x \in Q$ with $x > \sup(s)$. Let $x' \in Q$ be such that $\sup(s) < x' < x$. By (1), there exists $t \in T_\alpha$ with $s < t$ and $x' \geq \sup(t)$. So $t^\frown x \in T_{\alpha+1}$ is a witness for (1). Now let $\alpha$ be a limit and assume that $T_\beta$ has been defined for every $\beta < \alpha$. Using (1) and the fact that $\alpha$ can be embedded into every interval, we get that for each $s \in \bigcup_{\beta<\alpha} T_\beta$ and each $x > \sup(s)$ there is an increasing sequence $t$ of length $\alpha$ such that $s < t$ and $x \geq \sup(t)$ and $t \restriction \beta \in T_\beta$ for every $\beta < \alpha$. For each $s \in \bigcup_{\beta<\alpha} T_\beta$ and each $x > \sup(s)$, pick one such $t$, let this set be $\tilde{T}_\alpha$. Then let $T_\alpha := \{s^\frown x \mid s \in \tilde{T}_\alpha \wedge x > \sup(s)\}$. To see that (1) holds for $\alpha$, let $\beta < \alpha$ and $s \in T_\beta$ and $x \in Q$ with $x > \sup(s)$. Let $x > x' > \sup(s)$. By construction there is $t \in \tilde{T}_\alpha$ with $x' \geq \sup(t)$ and $t^\frown x \in T_\alpha$ is a witness for (1).

Let $T := \bigcup_{\alpha<\aleph_{n+2}} T_\alpha$. Note that every $s \in T$ is an increasing sequence in $Q$ of successor length.

Now we show that $|T_\alpha| < \aleph_{n+2}$ for every $\alpha < \aleph_{n+2}$ by induction: This is obvious for $\alpha = 0$. The successor step $\alpha+1$ follows by $|T_{\alpha+1}| = |T_\alpha| \cdot |\aleph_{n+1}^{<\aleph_n}| = |T_\alpha| \cdot \aleph_{n+1} = \aleph_{n+1}$ since $|T_\alpha| < \aleph_{n+2}$. Now let $\alpha$ be a limit: $|\tilde{T}_\alpha| \leq |\{(s, x) \mid s \in \bigcup_{\beta<\alpha} T_\beta \wedge x \in Q\}| = \alpha \cdot \aleph_{n+1} \cdot \aleph_{n+1} = \aleph_{n+1}$. As in the successor step, it follows that $|T_\alpha| = \aleph_{n+1}$.

To see that $T$ has no branches we can argue as follows. There are no increasing sequences of length $\aleph_{n+2}$ in $Q$ with the lexicographical ordering: Assume $\langle x_\alpha \in Q \mid \alpha < \aleph_{n+2}\rangle$ is increasing. Since $|\{\alpha \in \aleph_n \mid x(\alpha) \neq 0\}| < \aleph_n$ for every $\alpha < \aleph_{n+2}$, for cofinally many of them the bound is the same, hence we can assume that they all have the same bound $\lambda < \aleph_n$. For each $\alpha < \aleph_{n+2}$, there exists $\xi_\alpha < \lambda$ such that $x_\alpha(\xi_\alpha) < x_{\alpha+1}(\xi_\alpha)$. By the pigeonhole principle, there exists one $\xi < \lambda$ such that for $\xi = \xi_\alpha$ for unboundedly many $\alpha < \aleph_{n+2}$, again we assume it holds for all $\alpha$. This means $\langle x_\alpha(\xi) \mid \alpha < \aleph_{n+2}\rangle$ is an increasing sequence of length $\aleph_{n+2}$ in $\aleph_{n+1}$, a contradiction.

To see that $T$ is special, let $\phi\colon Q \to \omega_{n+1}$ be a bijection. For $s \in T$ with $x$ the last element of $s$, let $f(s) = \phi(x)$. If $s < t$ then $t$ extends $s$, since $t$ is increasing it follows that $f(s) \neq f(t)$. □

## 3. General lemmata

In this section, we give some general lemmata, which will be needed in the later sections.

**Definition 3.1.** $\mathbb{P}$ is a regular subforcing of $\mathbb{Q}$ if $\mathbb{P} \subseteq \mathbb{Q}$ and every maximal antichain of $\mathbb{P}$ is a maximal antichain in $\mathbb{Q}$.

**Lemma 3.2.** *If $\mathbb{P}$ is a forcing and $\lambda$ a cardinal in $V$, then $(2^\lambda)^{V[\mathbb{P}]} \leq (2^{|\mathbb{P}|\cdot\lambda})^V$.*

*Proof.* Every subset of $\lambda$ in $V[\mathbb{P}]$ has a name $\dot{a}$ of the form $\{(\check{\alpha}, p) \mid \alpha \in \lambda \wedge p \in \mathbb{P}\}$. There are $(2^{|\mathbb{P}|\cdot\lambda})^V$ many such names. Hence $(2^\lambda)^{V[\mathbb{P}]} \leq (2^{|\mathbb{P}|\cdot\lambda})^V$. □

**Lemma 3.3.** *If $\mathbb{P}$ is a forcing and $\kappa$ and $\lambda$ are cardinals with $|\mathbb{P}| = \kappa$, and $\lambda \geq \kappa$, then $(2^\lambda)^V = (2^\lambda)^{V[\mathbb{P}]}$.*

*Proof.* This follows directly from Lemma 3.2. □

**Lemma 3.4.** *Let $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$ and $\bar{\mathbb{P}}$ a regular subforcing of $\mathbb{P}$ and $\mathbb{P} \Vdash$ "$\dot{\bar{\mathbb{Q}}}$ is a regular subforcing of $\dot{\mathbb{Q}}$" and $\bar{\mathbb{R}} = \bar{\mathbb{P}} * \dot{\bar{\mathbb{Q}}}$. Let $G$ be an $\bar{\mathbb{R}}$-generic filter and $H$ and $K$ the corresponding $\bar{\mathbb{P}}$- and $\dot{\bar{\mathbb{Q}}}$-generic filters. Then the following holds true: $\mathbb{R}/G = (\mathbb{P}/H) * (\dot{\mathbb{Q}}/K)$.*

*Proof.* First we show that $\mathbb{R}/G \subseteq (\mathbb{P}/H) * (\dot{\mathbb{Q}}/K)$. Let $(p, \dot{q}) \in \mathbb{R}/G$. That means for every $p' \in H$ for every $\dot{q}' \in K$ there exists an $r \in \mathbb{P}$, stronger than $p$ and $p'$ such that $r \Vdash \dot{q} || \dot{q}'$. Thus no $p' \in H$ can force that $\dot{q}$ and $\dot{q}'$ are incompatible. So by the Truth Lemma, $V[H] \models \dot{q}$ is compatible with $\dot{q}'$ for every $\dot{q}' \in K$ and therefore $V[H] \models \dot{q} \in \dot{\mathbb{Q}}/K$. So $V[G] \models (p, \dot{q}) \in \mathbb{P}/H * \dot{\mathbb{Q}}/K$.

Now we show that $(\mathbb{P}/H) * (\dot{\mathbb{Q}}/K) \subseteq \mathbb{R}/G$. Let $(p, \dot{q}) \in \mathbb{P}/H * \dot{\mathbb{Q}}/K$, so $p \in \mathbb{P}/H$ and $p \Vdash \dot{q} \in \dot{\mathbb{Q}}/K$, so clearly $(p, \dot{q}) \in \mathbb{R}/G$. □

The following lemma of Silver is useful to lift elementary embeddings to forcing extensions. A proof can be found in [Cum10].

**Lemma 3.5** (Lifting Lemma)**.** *Let $j \colon V \to M$ be an elementary embedding and $\mathbb{P} \in V$ a forcing. Let $G(\mathbb{P})$ be generic for $\mathbb{P}$ and let $G(j(\mathbb{P}))$ be generic for $j(\mathbb{P})$. The following are equivalent:*

*(1) $j[G(\mathbb{P})] \subseteq G(j(\mathbb{P}))$.*
*(2) There exists an elementary embedding $j' \colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ such that $j'(G(\mathbb{P})) = G(j(\mathbb{P}))$ and $j' \restriction V = j$.*

**Lemma 3.6.** *If $\mathbb{P} * \dot{\mathbb{Q}}$ is a two step iteration which has the $\kappa$-c.c., then $\mathbb{P}$ has the $\kappa$-c.c., and $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ has the $\kappa$-c.c.".*

*Proof.* Since it is easy to see that $\mathbb{P}$ has the $\kappa$-c.c., we only show that $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ has the $\kappa$-c.c.". Let $p_0 \in \mathbb{P}$ and $\dot{A}$ a $\mathbb{P}$-name such that $p_0 \Vdash$ "$\dot{A} \subseteq \dot{\mathbb{Q}}$ and $\dot{A}$ has size $\kappa$". Furthermore, let $\dot{f}$ be a $\mathbb{P}$-name such that $p_0 \Vdash$ "$\dot{f} \colon \kappa \to \dot{A}$ is a bijection". For every $\alpha \in \kappa$ there exist $p_\alpha \leq p_0$ and $\dot{q}_\alpha$ such that $p_\alpha \Vdash \dot{f}(\alpha) = \dot{q}_\alpha$. Since $\mathbb{P} * \dot{\mathbb{Q}}$ has the $\kappa$-c.c., there exist $\alpha, \beta \in \kappa$ such that $(p_\alpha, \dot{q}_\alpha)$ and $(p_\beta, \dot{q}_\beta)$ are compatible. Let $(p, \dot{q})$ be a witness for the compatibility. It follows that $p \Vdash$ "$\dot{f}(\alpha) = \dot{q}_\alpha \in \dot{A} \wedge \dot{f}(\beta) = \dot{q}_\beta \in \dot{A} \wedge \dot{q} \leq \dot{q}_\alpha, \dot{q}_\beta$". In particular $p \Vdash$ "$\dot{A}$ is not an antichain". □

**Lemma 3.7.** *Let $\mathbb{P}$ be a forcing where $\mathbb{P} \times \mathbb{P}$ has the $\kappa$-c.c., and $\mathbb{P}'$ a regular subforcing of $\mathbb{P}$. Then $(\mathbb{P}/\mathbb{P}') \times (\mathbb{P}/\mathbb{P}')$ has the $\kappa$-c.c..*

*Proof.* First note that $(\mathbb{P}/\mathbb{P}') \times (\mathbb{P}/\mathbb{P}') = (\mathbb{P} \times \mathbb{P})/(\mathbb{P}' \times \mathbb{P}')$ by Lemma 3.4. In other words, $\mathbb{P} \times \mathbb{P}$ is the two step iteration $(\mathbb{P}' \times \mathbb{P}') * ((\mathbb{P}/\mathbb{P}') \times (\mathbb{P}/\mathbb{P}'))$. Hence, by Lemma 3.6, $(\mathbb{P}/\mathbb{P}') \times (\mathbb{P}/\mathbb{P}')$ has the $\kappa$-c.c.. □

The following lemma can be found in [Ung12] for the case where $\mathbb{P}$ has the $\mu^+$-c.c. and $\mathbb{R}$ is $<\mu^+$-closed:

**Lemma 3.8.** *Let $\lambda$ be a regular cardinal and $\mu < \lambda$ with $2^\mu \geq \lambda$. Let $\mathbb{P}$ be a forcing which has the $\lambda$-c.c. and $\mathbb{R}$ a forcing which is $<\lambda$-closed and $\dot{T}$ a $\mathbb{P}$-name for a $\lambda$-tree. Then forcing with $\mathbb{R}$ over $V[\mathbb{P}]$ does not add branches to $T$.*

*Proof.* In $V[\mathbb{P}]$ let $\dot{b}$ be an $\mathbb{R}$-name for a new branch through $T$.

**Claim.** *For all $r_1, r_2 \in \mathbb{R}$ the set $D_{r_1,r_2}$ of conditions $p \in \mathbb{P}$ with the following properties is dense.*

(1) *$p \Vdash$ "there are $r_1' \leq r_1$ and $r_2' \leq r_2$ and $\gamma < \lambda$ such that $r_1'$ and $r_2'$ decide $\dot{b}(\gamma)$ in different ways".*
(2) *$p$ decides $\gamma$, $r_1'$ and $r_2'$.*

*Proof.* Let $p^* \in \mathbb{P}$ and let $G_\mathbb{P}$ be generic for $\mathbb{P}$ with $p^* \in G_\mathbb{P}$. In $V[G_\mathbb{P}]$ the conditions $r_1$ and $r_2$ cannot decide all of $\dot{b}$, because it is a new branch. Hence there exists $\gamma$ such that $r_1$ and $r_2$ do not decide $\dot{b}(\gamma)$. So there exist conditions $r_1' \leq r_1$ and $r_2' \leq r_2$ which decide $\dot{b}(\gamma)$ differently. So there exists a condition $p' \in G_\mathbb{P}$ which forces this and decides $\gamma$, $r_1'$ and $r_2'$. Since both $p^*$ and $p'$ are in $G_\mathbb{P}$, they are compatible. Any witness of the compatibility is in $D_{r_1,r_2}$ and stronger than $p^*$. □

**Claim.** *For every condition $r \in \mathbb{R}$ there exists a maximal antichain $A$ in $\mathbb{P}$ and conditions $r_1, r_2 \leq r$ and $\gamma < \lambda$ such that for all $p \in A$, $p \Vdash$ "$r_1$ and $r_2$ decide $\dot{b}(\gamma)$ differently".*

*Proof.* By induction define increasing antichains $A_\alpha$ and decreasing sequences $r_0^\alpha$ and $r_1^\alpha$ and an increasing sequence $\gamma_\alpha$ such that for each $p \in A_\alpha$, $p \Vdash r_0^\alpha$ and $r_1^\alpha$ decide $\dot{b}(\gamma_\alpha)$ differently.

Let $r \in \mathbb{R}$ and $p_0 \in D_{r,r}$ and $r_1^0$, $r_2^0$ and $\gamma_0 < \lambda$ witnesses for this. Let $A_0 := \{p_0\}$.

For the successor step assume $A_\alpha$, $r_1^\alpha$, $r_2^\alpha$ and $\gamma_\alpha$ have been defined. If $A_\alpha$ is a maximal antichain, we stop the construction here. If $p \in \mathbb{P}$ is incompatible to every condition in $A_\alpha$, let $p' \leq p$ with $p' \in D_{r_1^\alpha, r_2^\alpha}$ and let $r_1^{\alpha+1}$, $r_2^{\alpha+1}$ and $\gamma_{\alpha+1}$ be witnesses for this. Let $A_{\alpha+1} := A_\alpha \cup \{p'\}$.

For the limit step $\alpha$ let $A_\alpha := \bigcup_{\beta<\alpha} A_\beta$. Note that $A_\alpha$ is an antichain of size $\alpha$, since $\mathbb{P}$ has the $\lambda$-c.c., it follows that $\alpha < \lambda$. Let $r_1^\alpha$ and $r_2^\alpha$ be lower bounds of the sequences $\langle r_0^\beta \mid \beta < \alpha \rangle$ and $\langle r_1^\beta \mid \beta < \alpha \rangle$ (such lower bounds exist because $\mathbb{R}$ is $<\lambda$-closed). Let $\gamma_\alpha := \sup\{\gamma_\beta \mid \beta < \alpha\}$, it follows that each $p \in A_\alpha$ forces that $r_0^\alpha$ and $r_1^\alpha$ decide $\dot{b}(\gamma_\alpha)$ differently.

Since $\mathbb{P}$ has the $\lambda$-c.c., for some $\alpha < \lambda$ the antichain $A_\alpha$ will be maximal. Then we stop the induction and define $A := A_\alpha$, $r_1 := r_1^\alpha$, $r_2 := r_2^\alpha$ and $\gamma := \gamma_\alpha$.

To see that the claim is fulfilled, let $p \in A$. Hence $p \in A_{\beta+1}$ for some $\beta + 1 \leq \alpha$, so $p \Vdash r_1^{\beta+1}$ and $r_0^{\beta+1}$ decide $\dot{b}(\gamma_{\beta+1})$ differently. Since $r_1 \leq r_1^{\beta+1}$, $r_1 \leq r_1^{\beta+1}$ and $\gamma > \gamma_{\beta+1}$ it follows that $p \Vdash r_1$ and $r_0$ decide $\dot{b}(\gamma)$ differently. □

Let $\mu < \lambda$ be minimal with $2^\mu \geq \lambda$. For every $w \in 2^{<\mu}$ we construct $r_w$, $x_w$ and $\alpha_i$ for every $i < \mu$ such that $r_w \Vdash \dot{b}(\alpha_{|w|}) = x_w$, for $w, w' \in 2^{<\mu}$ of the same length, $x_w \neq x'_w$ and $\alpha_i > \alpha_j$ for $i > j$: Let $r_{\langle\rangle}$ be a condition in $\mathbb{R}$ such that there exists $\alpha_0$ and $x_{\langle\rangle}$ such that $r_{\langle\rangle} \Vdash \dot{b}(\alpha_0) = x_{\langle\rangle}$.

Now use the above claim inductively to get $r_{w^\frown 0} \in \mathbb{R}$ and $r_{w^\frown 1} \in \mathbb{R}$, together with $x_{w^\frown 0}$, $x_{w^\frown 1}$, $\alpha_{|w|+1} > \alpha_{|w|}$ and a maximal antichain $A_w$ such that each $p \in A_w$ forces that $r_{w^\frown 0}$ and $r_{w^\frown 1}$ decide $\dot{b}(\alpha_{|w|+1})$ differently. For $w \in 2^{\leq\mu}$ of limit length let $\alpha_{|w|} > \alpha_\delta$ for all $\delta < |w|$ and $r_w$, be a lower bounds of $\langle r_{w\restriction\delta} \mid \delta < |w|\rangle$. Such $r_w$ exist using the closure of $\mathbb{R}$.

Let $\alpha := \sup\{\alpha_\delta \mid \delta \leq \mu\}$. Since $\lambda$ is regular and $\mu < \lambda$, it follows that $\alpha < \lambda$.

Let $G$ be generic for $\mathbb{P}$. In $V[G]$ for all $v \neq w \in 2^\mu$, $r_v$ and $r_w$ decide $\dot{b}(\alpha)$ differently: Let $\delta < \mu$ be minimal such that $v(\delta) \neq w(\delta)$. Hence, every $p \in A_{w\restriction\delta}$ forces that $r_{v\restriction\delta+1}$ and $r_{w\restriction\delta+1}$ decide $\dot{b}(\alpha_{\delta+1})$ differently. $A_{w\restriction\delta}$ is a maximal antichain, so there exists $p \in G \cap A_{w\restriction\delta}$. So $V[G] \models r_{v\restriction\delta+1}$ and $r_{w\restriction\delta+1}$ decide $\dot{b}(\alpha_{\delta+1})$ differently. Since $r_v \leq r_{v\restriction\delta+1}$ and $r_w \leq r_{w\restriction\delta+1}$ and $\alpha > \alpha_\delta$ and $\dot{T}$ a tree, it follows that $V[G] \models r_v$ and $r_w$ decide $\dot{b}(\alpha)$ differently.

Hence $V[G] \models |\dot{T}_\alpha| \geq 2^\mu \geq \lambda$, contradicting that $\dot{T}$ is a name for a $\lambda$-tree. □

The following lemma is essentially due to Mitchell [Mit73]:

**Lemma 3.9.** *Let $\lambda$ be a regular cardinal and $\mathbb{P}$ be a forcing where $\mathbb{P}\times\mathbb{P}$ has the $\lambda$-c.c. and $T$ a tree of height $\lambda$. Then forcing with $\mathbb{P}$ does not add a new cofinal branch (of length $\lambda$) to $T$.*

*Proof.* Assume there is a new branch. Let $\dot{b}$ be a name for it. We inductively build a sequence of conditions $\{(p_i, q_i) \in \mathbb{P} \times \mathbb{P} \mid i < \lambda\}$ and a strictly increasing sequence $\{\gamma_i \mid i < \lambda\}$ of ordinals, such that:

(1) There exists $x_i$ such that $p_i, q_i \Vdash \dot{b}(\gamma_i) = x_i$.
(2) There exist $y_i \neq z_i$ such that $p_i \Vdash \dot{b}(\gamma_{i+1}) = y_i$ and $q_i \Vdash \dot{b}(\gamma_{i+1}) = z_i$.

Let $\gamma_0 = 0$ and $p \in \mathbb{P}$ and $x_0$ such that $p \Vdash \dot{b}(0) = x_0$. Since $\dot{b}$ is a name for a new branch, there exists $\alpha < \lambda$ such that $p$ does not decide $\dot{b}(\alpha)$. Find $p_0$ and $q_0$ and $y_0 \neq z_0$ with $p_0, q_0 \leq p$ and $p_0 \Vdash \dot{b}(\alpha) = y_0$ and $q_0 \Vdash \dot{b}(\alpha) = z_0$ and set $\gamma_1 := \alpha$.

Assume $p_i$, $q_i$, $\gamma_{i+1}$, $x_i$, $y_i$ and $z_i$ have been defined. Let $x_{i+1} = y_i$ and $\gamma_{i+2} > \gamma_{i+1}$ such that $p_i$ does not decide $\dot{b}(\gamma_{i+2})$. Find $p_{i+1}, q_{i+1} \leq p_i$ and $y_{i+2} \neq z_{i+2}$ such that $p_{i+1} \Vdash \dot{b}(\gamma_{i+2}) = y_{i+1}$ and $q_{i+1} \Vdash \dot{b}(\gamma_{i+2}) = z_{i+1}$.

Assume $j$ is a limit and $p_i$, $q_i$, $\gamma_i$, $x_i$, $y_i$ and $z_i$ have been defined for every $i < j$. Let $\gamma_j = \sup \gamma_i + 1$ and $\gamma_{j+1}$ such that no $p_i$ decides $\dot{b}(\gamma_{j+1})$. That is possible because $j < \lambda$ and the height of $T$ is $\lambda$, so $\gamma_i < \lambda$. Now let $p_j, q_j$ and $x_j$, $y_j$ and $z_j$ be such that $p_j, q_j \Vdash \dot{b}(\gamma_j) = x_j$ and $p_j \Vdash \dot{b}(\gamma_{j+1}) = y_j$ and $q_j \Vdash \dot{b}(\gamma_{j+1}) = z_j$. This finishes the construction.

Since $\mathbb{P} \times \mathbb{P}$ has the $\lambda$-c.c., there exist $i < j < \lambda$ such that $(p_i, q_i)$ and $(p_j, q_j)$ are compatible. Let $(p, q) \leq (p_i, q_i), (p_j, q_j)$. $p \leq p_j$ and $q \leq q_j$ so $p, q \Vdash \dot{b}(\gamma_j) = x_j$, $p \leq p_i$ and $q \leq q_i$, so $p \Vdash \dot{b}(\gamma_{i+1}) = y_i$ and $q \Vdash \dot{b}(\gamma_{i+1}) = z_i$. So in $T$ there are two nodes $y_i$ and $z_i$ on level $\gamma_{i+1} \leq \gamma_j$ with $y_i, z_i \leq x_j$, this contradicts the fact that $T$ is a tree. We conclude that there is no new branch in $V[\mathbb{P}]$. □

## 4. $\aleph_2$-TREES

From the existence of a supercompact cardinal and an inaccessible above, we prove that it is consistent that all $\aleph_2$-Aronszajn trees are special, there are such, and there are no $\aleph_1$- or $\aleph_2$-Kurepa trees.

4.1. **Definition of the forcing.** Let $\kappa_2 < \kappa_3$, $\kappa_2$ supercompact and $\kappa_3$ inaccessible.

**Definition 4.1.** For a $\kappa_2$-Aronszajn tree $T$ let $\mathbb{S}(T)$ be the forcing to specialize $T$, defined as follows: $\mathbb{S}(T)$ consists of partial multi-valued functions $f$ from $T$ into $\omega_1$ such that $|\mathrm{dom}(f)| \leq \omega$, $|f(\sigma)| \leq \omega$ for each $\sigma \in \mathrm{dom}(f)$, and $f(\sigma) \cap f(\tau) = \emptyset$ whenever $\sigma, \tau \in \mathrm{dom}(f)$ are comparable in $T$. The order is given by $g \leq f$ if $g \supseteq f$.

We use multi-valued functions instead of single-valued functions, to make some of the proofs easier.

Let $\mathbb{L}_2 = \mathrm{Col}(\omega_1, {<}\kappa_2)$ and $\mathbb{L}_3 = \mathrm{Col}(\kappa_2, {<}\kappa_3)^{V[\mathbb{L}_2]}$.

Let $\dot{T}_0$ be an $\mathbb{L}_2 * \mathbb{L}_3$-name for a $\kappa_2$-Aronszajn tree and $\mathbb{S}(\dot{T}_0)$ an $\mathbb{L}_2 * \mathbb{L}_3$-name for the forcing to specialize $\dot{T}_0$ as in Definition 4.1.

Let $\mathbb{P}_1 := \mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}(\dot{T}_0)$ and $\mathbb{S}_1 := \mathbb{S}(\dot{T}_0)$.

Assume $\mathbb{P}_i$ has been defined. Continue the iteration in the same way, i.e., let $\dot{T}_i$ be a $\mathbb{P}_i$-name for a $\kappa_2$-Aronszajn tree and $\mathbb{S}(\dot{T}_i)$ the forcing to specialize $\dot{T}_i$ as in the case of $\dot{T}_0$. Let $\mathbb{P}_{i+1} := \mathbb{P}_i * \mathbb{S}(\dot{T}_i)$ and $\mathbb{S}_{i+1} := \mathbb{S}_i * \mathbb{S}(\dot{T}_i)$. Continue this as a countable support iteration for $\kappa_3$ many steps, using a bookkeeping function for the nice names of $\kappa_2$-Aronszajn trees. Let $\mathbb{P}_{\kappa_3}^{\aleph_2}$ be this forcing iteration. We will show that in $V[\mathbb{L}_2 * \mathbb{L}_3]$ the forcing iteration to specialize $\kappa_2$-Aronszajn trees has the $\kappa_2$-c.c. (see Lemma 4.11). Recall that by Lemma 2.5, $\kappa_2 = \aleph_2$ in $V[\mathbb{L}_2]$. By Lemma 2.6, $\mathbb{L}_3$ is ${<}\kappa_2$-closed, so it does not collapse $\kappa_2$ and, since the forcing iteration to specialize $\kappa_2$-Aronszajn trees has the $\kappa_2$-c.c., also this forcing does not collapse $\kappa_2$. Thus $V[\mathbb{P}_{\kappa_3}^{\aleph_2}] \models \kappa_2 = \aleph_2$.

Since $\mathbb{L}_2 * \mathbb{L}_3$ has the $\kappa_3$-c.c., it follows that $\mathbb{P}^{\aleph_2}_{\kappa_3}$ has the $\kappa_3$-c.c., therefore $\kappa_3$ is preserved and every $\kappa_2$-Aronszajn tree in $V[\mathbb{P}^{\aleph_2}_{\kappa_3}]$ has a nice $\mathbb{P}^{\aleph_2}_{\kappa_3}$-name of size smaller than $\kappa_3$ and so there are only $\kappa_3$ many nice names for $\kappa_2$-Aronszajn trees. Hence the bookkeeping function can make sure that all $\kappa_2$-Aronszajn trees have been specialized in $V[\mathbb{P}^{\aleph_2}_{\kappa_3}]$.

### 4.2. **Chain condition.**

**Definition 4.2.** For two forcing notions $\mathbb{P}$ and $\mathbb{Q}$, a function $\iota\colon \mathbb{P} \to \mathbb{Q}$ is a regular embedding, if the following holds:

(1) If $p \leq_{\mathbb{P}} p'$ then $\iota(p) \leq_{\mathbb{Q}} \iota(p')$.
(2) $p \perp_{\mathbb{P}} p'$ iff $\iota(p) \perp_{\mathbb{Q}} \iota(p')$.
(3) For every $q \in \mathbb{Q}$ there exists a reduction $p \in \mathbb{P}$, i.e., for each $p' \in \mathbb{P}$ if $p' \leq p$ then $\iota(p') \not\perp_{\mathbb{Q}} q$.

**Definition 4.3.** For $\mathbb{P}$ a suborder of $\mathbb{Q}$, we say that $\pi\colon \mathbb{Q} \to \mathbb{P}$ is a reduction map, if whenever $q \in \mathbb{Q}$ and $p \leq_{\mathbb{P}} \pi(q)$ then $p$ and $q$ are compatible in $\mathbb{Q}$.

**Lemma 4.4.** *Let $\mathbb{P} \subseteq \mathbb{Q}$. $\mathbb{P}$ is a regular subforcing of $\mathbb{Q}$ if*

*(1) there exists a reduction map $\pi\colon \mathbb{Q} \to \mathbb{P}$ and*
*(2) if two conditions $p, q \in \mathbb{P}$ are compatible in $\mathbb{Q}$, then they are compatible in $\mathbb{P}$.*

*Proof.* See [Kun11, III.3.72] □

The following theorem is useful to represent some forcings as an easier forcing followed by a quotient which has a good closure. A good source for it is [Cum10].

**Theorem 4.5** (Absorption)**.** *Let $\kappa$ be regular and $\lambda \geq \kappa$. Let $\mathbb{P}$ be separative, $\kappa$-closed and $|\mathbb{P}| < \lambda$. Then there is a regular embedding $\iota\colon \mathbb{P} \to \mathrm{Col}(\kappa, {<}\lambda)$ such that if $G$ is $\mathbb{P}$-generic over $V$, then $\mathrm{Col}(\kappa, {<}\lambda)$ is forcing equivalent to $\mathrm{Col}(\kappa, {<}\lambda)/\iota[G]$.*

From now on let $j\colon V \to M$ be a supercompact embedding for $\kappa_2$ with $j(\kappa_2) > \kappa_3, |\mathbb{L}_3|$ and ${}^{\leq\kappa_3}M \subseteq M$.

**Lemma 4.6.** *There exists an $\mathbb{L}_2$-name $\mathbb{L}_3^*$ such that $\mathbb{L}_3^*$ is a regular subforcing of $j(\mathbb{L}_2)/G(\mathbb{L}_2)$ in $V[\mathbb{L}_2]$, $(j(\mathbb{L}_2)/G(\mathbb{L}_2))/G(\mathbb{L}_3^*)$ is $\sigma$-closed, $\mathbb{L}_3^*$ is forcing equivalent to $\mathbb{L}_3$ and $|\mathbb{L}_3^*| < j(\kappa_2)$.*

*Proof.* Let $G(\mathbb{L}_2)$ be generic for $\mathbb{L}_2$. We apply Theorem 4.5: $V[G(\mathbb{L}_2)] \models \mathbb{L}_3$ is $\sigma$-closed and $|\mathbb{L}_3| < j(\kappa_2)$, thus there exists a regular embedding $\iota\colon \mathbb{L}_3 \to j(\mathbb{L}_2)/G(\mathbb{L}_2)$ such that $j(\mathbb{L}_2)/G(\mathbb{L}_2)$ is equivalent to $(j(\mathbb{L}_2)/G(\mathbb{L}_2))/\iota[G(\mathbb{L}_3)]$. So in particular $(j(\mathbb{L}_2)/G(\mathbb{L}_2))/\iota[G(\mathbb{L}_3)]$ is $\sigma$-closed.

$\mathbb{L}_3^* := \iota[\mathbb{L}_3]$ is the forcing we are looking for. □

**Corollary 4.7.** *There exists a reduction map* $\pi\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to \mathbb{L}_2 * \mathbb{L}_3^*$.

*Proof.* Clearly there exists a reduction map $\pi_1\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to j(\mathbb{L}_2)$. $j(\mathbb{L}_2)$ is forcing equivalent to $\mathbb{L}_2 * (j(\mathbb{L}_2)/\mathbb{L}_2)$. By Lemma 4.6 $j(\mathbb{L}_2)/\mathbb{L}_2$ has $\mathbb{L}_3^*$ as a regular subforcing, hence there exists a reduction map $\pi_2\colon \mathbb{L}_2 * (j(\mathbb{L}_2)/\mathbb{L}_2) \to \mathbb{L}_2 * \mathbb{L}_3^*$. This shows that there exists a reduction map $\pi\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to \mathbb{L}_2 * \mathbb{L}_3^*$. □

To be able to use the supercompact embeddings, we have to lift them to the forcing extensions. To lift a supercompact embedding for $\kappa$ to the extension by a Lévy collapse for some larger cardinal $\kappa'$ we use absorption, i.e., the fact that the Lévy collapse for $\kappa'$ contains the collapse for $\kappa$ as a regular subforcing:

**Lemma 4.8.** *Let* $G(\mathbb{L}_2)$ *be generic for* $\mathbb{L}_2$ *and* $G(\mathbb{L}_3)$ *generic for* $\mathbb{L}_3$ *over* $V[G(\mathbb{L}_2)]$. *The supercompact embedding* $j$ *can be lifted to*

$$j\colon V[G(\mathbb{L}_2 * \mathbb{L}_3)] \to M[G(j(\mathbb{L}_2) * j(\mathbb{L}_3))].$$

*Proof.* Let $\iota\colon \mathbb{L}_3 \to j(\mathbb{L}_2)/G(\mathbb{L}_2)$ be a regular embedding as in Lemma 4.6.

We can choose $G(j(\mathbb{L}_2))$ such that $G(j(\mathbb{L}_2)) \cap \mathrm{range}(\iota) = \iota[G(\mathbb{L}_3)]$, thus $\iota[G(\mathbb{L}_3)] \in V[G(j(\mathbb{L}_2))]$ and $G(\mathbb{L}_2) \subseteq G(j(\mathbb{L}_2))$; that is possible because $\mathbb{L}_2 * \iota[\mathbb{L}_3]$ is a regular subforcing of $j(\mathbb{L}_2)$. Thus it follows that $\iota[G(\mathbb{L}_3)] \in V[G(j(\mathbb{L}_2))]$ and since $\iota, j \restriction \mathbb{L}_3 \in V[G(j(\mathbb{L}_2))]$ it follows that $j[G(\mathbb{L}_3)] \in V[G(j(\mathbb{L}_2))]$. Since $M$ is closed under subsets of size $\leq \kappa_3$ the same holds for $M[G(j(\mathbb{L}_2))]$ and therefore $j[G(\mathbb{L}_3)] \in M[G(j(\mathbb{L}_2))]$.

$j[G(\mathbb{L}_3)] \subseteq j[\mathbb{L}_3] \subseteq j(\mathbb{L}_3)$, $j[G(\mathbb{L}_3)]$ is a directed set of size $< j(\kappa_2)$ and $j(\mathbb{L}_3)$ is $<j(\kappa_2)$-directed closed, therefore there exists a mastercondition $p \in j(\mathbb{L}_3)$ for $j[G(\mathbb{L}_3)]$. Let $G(j(\mathbb{L}_3))$ be generic for $j(\mathbb{L}_3)$ with $p \in G(j(\mathbb{L}_3))$. It follows that $j[G(\mathbb{L}_3)] \subseteq G(j(\mathbb{L}_3))$.

Now we can use the Lifting Lemma (Lemma 3.5) to lift $j$ to an embedding $j\colon V[G(\mathbb{L}_2)][G(\mathbb{L}_3)] \to M[G(j(\mathbb{L}_2))][G(j(\mathbb{L}_3))]$. □

As a summary of the previous lemmata we get the following:

**Corollary 4.9.** *There exists a regular subforcing* $\mathbb{L}^*$ *of* $j(\mathbb{L}_2 * \mathbb{L}_3)$*, with the following properties:*

*(1)* $\mathbb{L}^*$ *is also a regular subforcing of* $j(\mathbb{L}_2)$.
*(2)* $\mathbb{L}^*$ *is forcing equivalent to* $\mathbb{L}_2 * \mathbb{L}_3$.
*(3) There exists a reduction map* $\pi\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to \mathbb{L}^*$.
*(4)* $j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}^*)$ *is* $\sigma$*-closed.*

*Moreover, there exists a lifting of the supercompact embedding for* $\kappa_2$ *to* $j\colon V[\mathbb{L}_2 * \mathbb{L}_3] \to M[j(\mathbb{L}_2 * \mathbb{L}_3)]$.

One of the main technical parts of the proof is to show that the forcing iteration has a good chain condition. The main work lies in the following lemma, which deals with the successor step of the iteration.

Note that $\mathbb{L}_2 * \mathbb{L}_3$ with $\mathbb{L}_2 * \mathbb{L}_3^*$ as a subforcing of $j(\mathbb{L}_2 * \mathbb{L}_3)$ fulfills the requirements of the following lemma.

**Lemma 4.10.** *Assume $\mathbb{P}$ is a forcing with $V[G(\mathbb{P})] \models \kappa_2 = \omega_2$ and $\mathbb{P}^*$ is forcing equivalent to $\mathbb{P}$, and $\mathbb{P}^*$ is a regular subforcing of $j(\mathbb{P})$ with reduction map $\pi$ and $j(\mathbb{P})/G(\mathbb{P}^*)$ is $\sigma$-closed. Let $j\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ be a lifting of the supercompact embedding for $\kappa_2$ and $\mathbb{S} = \mathbb{S}(T)$ a specializing forcing of a $\kappa_2$-Aronszajn tree $T$ in $V[G(\mathbb{P})]$. Then the following hold:*

*(1) There exists a regular subforcing $\mathbb{P}^* * \mathbb{S}^*$ of $j(\mathbb{P}) * j(\mathbb{S})$ with a reduction map $\pi^*\colon j(\mathbb{P}) * j(\mathbb{S}) \to \mathbb{P}^* * \mathbb{S}^*$ such that the first component of $\pi^*(p, s)$ extends $\pi(p)$.*

*(2) $\mathbb{P} \Vdash \mathbb{S}$ has the $\kappa_2$-c.c.*

*(3) $\mathbb{P} * \mathbb{S}$ is forcing equivalent to $\mathbb{P}^* * \mathbb{S}^*$.*

*(4) The supercompact embedding $j$ can be lifted to*

$$j\colon V[G(\mathbb{P} * \mathbb{S})] \to M[G(j(\mathbb{P}) * j(\mathbb{S}))].$$

*(5) $j(\mathbb{P} * \mathbb{S})/G(\mathbb{P}^* * \mathbb{S}^*)$ is $\sigma$-closed.*

*Proof.* The main work is to prove (1).

**Proof of (1)**: Let $(p, s) \in j(\mathbb{P}) * j(\mathbb{S})$. Let $p' \leq p, \pi(p)$ such that $p'$ decides $s$, that means in $V$ there exists a countable partial function $f\colon \omega_1 \times j(\kappa_2) \to [\omega_1]^{\leq\omega}$ such that $p \Vdash s = f$. If $p'' \leq \pi(p')$, then $p''$ is compatible with $p'$ and therefore with $\pi(p)$, thus $\pi(p)$ and $\pi(p')$ are compatible in $j(\mathbb{P})$. Since $\mathbb{P}^*$ is a regular subforcing of $j(\mathbb{P})$, $\pi(p)$ and $\pi(p')$ are compatible in $\mathbb{P}^*$. Let $\hat{p} \in \mathbb{P}^*$ with $\hat{p} \leq \pi(p), \pi(p')$.

Then choose a generic $G(\mathbb{P}^*)$ containing $\hat{p}$ and let $G(\mathbb{P})$ be the corresponding generic for $\mathbb{P}$, i.e., $V[G(\mathbb{P})] = V[G(\mathbb{P}^*)]$; that is possible because $\mathbb{P}$ and $\mathbb{P}^*$ are forcing equivalent.

Since $T \in V[G(\mathbb{P})]$, it follows that $T \in V[G(\mathbb{P}^*)]$. Let $T^*$ be a $\mathbb{P}^*$-name for $T$ and let $\mathbb{S}^* := \mathbb{S}(T^*)$, the specializing forcing of $T^*$ in $V[G(\mathbb{P}^*)]$.

We assume that the nodes on the $\alpha$th level $T_\alpha$ of $T$ are the elements of $\omega_1 \times \{\alpha\}$ and all the levels are of size $<\kappa_2$, therefore $T = j[T] = j(T) \restriction \kappa_2$.

We can assume that for each $\sigma \in \operatorname{dom}(s) \cap j(T)_{>\kappa_2}$ there exists a $\sigma' \in \operatorname{dom}(s)$ on level $\kappa_2$ such that $p' \Vdash \sigma' \leq \sigma$.

Let $\bar{s} := s \restriction T$, $\{\sigma_n \mid n \in \omega\} = \operatorname{dom}(s) \cap T_{\kappa_2}$ and $C_n := \bigcup\{s(\tau) \mid \tau \supseteq \sigma_n, \tau \in \operatorname{dom}(s)\}$ the set of colors which $s$ assigns to nodes which are in $\operatorname{dom}(s)$ and equal to or above $\sigma_n$.

Let $\mathbb{Q} := j(\mathbb{P})/G(\mathbb{P}^*)$. Define a tree $\mathcal{T}$ of height $\omega$ inductively as follows:

- The root of $\mathcal{T}$ is $(p_{\langle\rangle}, \bar{\sigma}_0)$ where $p_{\langle\rangle} \in \mathbb{Q}$ and $p_{\langle\rangle} \Vdash$ "$\bar{\sigma}_0 \leq \sigma_0 \wedge \bar{\sigma}_0 \in T$". So $\bar{\sigma}_0$ is just some node which is forced by $p_{\langle\rangle}$ to be below $\sigma_0$.

- Let $t \in \mathcal{T}$. By construction $t$ is of the form $(p_w, \tau_w^0, \dots, \tau_w^{n-1})$ for some $n \in \omega$, $w \in 2^n$. With $p_w \Vdash$ "$\bar{\sigma}_k <_T \tau_w^k <_T \sigma_k$, $p_w \in \mathbb{Q}$. Again $\bar{\sigma}_n$ is just some node which is forced by $p_w$ to be below $\sigma_n$.

  $T$ is an Aronszajn tree in $V[G(\mathbb{P}^*)]$, hence every branch through $T$ in $V[j(\mathbb{P})/G(\mathbb{P}^*)]$ is new. Therefore there exist two conditions $p_{w^\frown 0}$ and $p_{w^\frown 1}$ which decide for every $k < n$ the nodes between $\tau_w^k$ and $\sigma_k$ differently, (between $\bar{\sigma}_n$ and $\sigma_n$ for $n$). In $\mathcal{T}$, $t$ has exactly two successors:

  $$(p_{w^\frown 0}, \tau^0_{w^\frown 0}, \dots, \tau^{n-1}_{w^\frown 0}, \tau^n_{w^\frown 0}) \text{ and } (p_{w^\frown 1}, \tau^0_{w^\frown 1}, \dots, \tau^{n-1}_{w^\frown 1}, \tau^n_{w^\frown 1})$$

  where $p_{w^\frown i}$ and $\tau^k_{w^\frown i}$ are such that $p_{w^\frown i} \Vdash$ "$\bar{\sigma}_k \leq \tau^k_{w^\frown i} \leq \sigma_k, \tau^k_{w^\frown i} \in T$" and $\tau^k_{w^\frown 0}$ is incomparable with $\tau^k_{w^\frown 1}$ in $T$.

For each branch $b$ through $\mathcal{T}$ let $p_b$ be stronger than all $p_{b\restriction k}$ and $\tau_b^n$ such that $p_b \Vdash \tau^n_{b\restriction k} \leq \tau^n_b$. Note that such $\tau^n_b$ exist in $T$, since the height of $T$ is $\kappa_2$, which is a regular cardinal.

Let $s' := \bar{s} \cup \{(\tau_b^n, C_n) \mid n \in \omega, b \in K\}$, where $K$ is the set of elements in $2^\omega$ which have only boundedly many 1's. This is a condition, because for each $n$ the set $C_n$ contains all the colors which appear at or above $\sigma_n$, so they don't appear at nodes below $\sigma_n$ and therefore not at nodes below $\tau^n_b$.

Let $V[G(\mathbb{P}^*)] \models q \leq s'$. We show that there exists a condition in $G(\mathbb{P}^*)$ which forces that $q \not\perp s$. Let $c \in 2^\omega$ be such that no node in $\mathrm{dom}(q)$ extends a $\tau_c^n$ for any $n$. Such a $c$ exists, since $2^\omega$ is uncountable and $\mathrm{dom}(q)$ is countable.

Now $p_c \Vdash$ "$\tau^n_c \leq \sigma_n$" for all $n$, thus $p_c \Vdash$ "$\tau^n_b \not\leq \sigma_n$" for all $n$ and all $b \in K$. Let $t \in \mathrm{dom}(q)$.

Case 1: There is some $n$ with $t < \tau_c^n$. Since $\tau^n_c$ is the limit of some $\tau^n_s$'s and for every $s$ exists a $b \in K$ which extends $s$, there exists some $b \in K$ with $t \leq \tau^n_b$ . Therefore, since $q$ is a condition and $\tau^n_b$ is in its domain, $q(t) \cap q(\tau^n_b) = \emptyset$ and since $q(\tau^n_b) = C_n \supseteq s(\sigma_n)$, it follows that $s$ is compatible with $(t, q(t))$.

Case 2: There is no $n$ with $t < \tau^n_c$. $\tau^n_c \not\leq t$ by the choice of $c$, thus $\tau^n_c$ is incomparable with $t$. Since $p_c \Vdash \tau^n_c \leq \sigma_n$ it follows that $p_c \Vdash t \not< \sigma_n$ and therefore $p_c \Vdash s$ is compatible with $(t, q(t))$.

Since $p_c$ forces for every $t \in \mathrm{dom}(q)$ that $s$ is compatible with $(t, q(t))$ it follows that $p_c \Vdash q$ is compatible with $s$. Thus it holds in $V[G(\mathbb{P}^*)]$ that for every $q \leq s'$ there exists a $p' \leq p$ such that $p' \Vdash q$ is compatible with $s$. Since $G(\mathbb{P}^*)$ is a filter, we can choose a condition $\bar{p} \in G(\mathbb{P}^*)$ below $\hat{p}$ which forces this.

Define $\pi^*(p, s) := (\bar{p}, s')$.

If $(p^*, s^*) \leq \pi^*(p, s)$ then $p^* \leq \pi(p)$ and therefore $p^*$ is compatible with $p$ and it forces that some extension of $p$ forces $s^*$ to be compatible with $s$. So

$(p^*, s^*)$ is compatible with $(p, s)$ and therefore $\pi^*$ is a reduction map such that the first component of $\pi^*(p, s)$ extends $\pi(p)$.

To see that $\mathbb{P}^* * \mathbb{S}^*$ is a regular subforcing of $j(\mathbb{P}) * j(\mathbb{S})$ we also have to show that if two conditions $(p, s), (p', s') \in \mathbb{P}^* * \mathbb{S}^*$ are compatible in $j(\mathbb{P}) * j(\mathbb{S})$, then they are compatible in $\mathbb{P}^* * \mathbb{S}^*$. To see this, note that the set $D$ of conditions $(p, s)$ with the following property is dense in $j(\mathbb{P}) * j(\mathbb{S})$: There exists $s^*$ such that

(1) $p \Vdash s \leq s^*$
(2) $p \Vdash s^* \in \mathbb{S}^*$
(3) If $p \Vdash s \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash s^* \leq \bar{s}$.

If $p$ decides $s$ then $(p, s)$ fulfills this property: Let $s^*$ be $s$ restricted to the nodes on levels below $\kappa_2$. So $p \Vdash s \leq s^* \wedge s^* \in \mathbb{S}^*$ and if $p \Vdash s \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash s^* \leq \bar{s}$, because in this case $\bar{s} \subseteq s^*$. So the set $D$ is dense.

Suppose now that $(p^*_0, s^*_0)$ and $(p^*_1, s^*_1)$ are in $\mathbb{P}^* * \mathbb{S}^*$ and they are compatible in $j(\mathbb{P}) * j(\mathbb{S})$. Let $(p, s)$ be a witness for the compatibility in the dense set $D$ with witness $s^*$. So $(p, s^*)$ is also below $(p^*_0, s^*_0)$ and $(p^*_1, s^*_1)$. Now $(\pi(p), s^*)$ is in $\mathbb{P}^* * \mathbb{S}^*$ and stronger than $(p^*_0, s^*_0)$ and $(p^*_1, s^*_1)$: Since $p \Vdash s^* \in \mathbb{S}^* \wedge s^* \leq s^*_0, s^*_1$ and that depends only on $\mathbb{P}^*$ the same holds for $\pi(p)$.

**Proof of (2)**: Let $G(\mathbb{P}^*)$ be generic for $\mathbb{P}^*$ and $G(\mathbb{P})$ the corresponding generic for $\mathbb{P}$. Let $j \colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ be a lifting of the supercompact embedding for $\kappa_2$. Let $A^*$ be a maximal antichain in $\mathbb{S}^*$. Since $\mathbb{S}^*$ is the same as $\mathbb{S}$, just with a different name, $A^*$ is also a maximal antichain in $\mathbb{S}$. By elementarity $j(A^*)$ is maximal in $j(\mathbb{S})$. Since $j$ is the identity on $\mathbb{S}$ it follows that $A^* \subseteq j(A^*)$. Let $G(j(\mathbb{P})/G(\mathbb{P}^*))$ be generic for $j(\mathbb{P})/G(\mathbb{P}^*)$ and $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models s \in j(\mathbb{S})$.

**Claim.** $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models$ "$\exists a \in A^*$ *which is compatible with* $s$".

*Proof.* Let $p \in G(j(\mathbb{P})/G(\mathbb{P}^*))$ with $p \Vdash s \in j(\mathbb{S})$. The following set is dense in $j(\mathbb{P})/G(\mathbb{P}^*)$ below $p$:

$$\{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \Vdash \exists a \in A^* \text{ which is compatible with } s\}$$

$p \Vdash$ "$s \in j(\mathbb{S})$ and there exists a reduction $s' \in \mathbb{S}^*$ of $s$". Since $s' \in \mathbb{S}^*$ there exists $a \in A^*$ and $q \leq p$ such that $q \Vdash a$ is compatible with $s'$. Since $s'$ is a reduction of $s$, it follows that $q \Vdash$ "$a$ is compatible with $s'$".

Thus there exists a $q \in G(j(\mathbb{P})/G(\mathbb{P}^*)) \cap (\{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \perp p\} \cup \{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \Vdash \exists a \in A^* \text{ which is compatible with } s\})$. And since $p$ is in the generic filter, there exists $q$ in the generic with $q \Vdash$ "$\exists a \in A^*$ which is compatible with $s$". □

Thus it follows that $A^*$ is a maximal antichain in $j(\mathbb{S})$. Since $j(A^*)$ is an antichain and $A^* \subseteq j(A^*)$ it follows that $A^* = j(A^*)$. Thus $|j(A^*)| < j(\kappa_2)$ and by elementarity $|A^*| < \kappa_2$.

**Proof of (3)**: $\mathbb{P}^*$ is forcing equivalent to $\mathbb{P}$, and $\mathbb{S}^*$ in $V[\mathbb{P}^*]$ is the same forcing as $\mathbb{S}$ in $V[\mathbb{P}]$.

**Proof of (4)**: Let $G(j(\mathbb{S}))$ be generic for $j(\mathbb{S})$ over $M[G(j(\mathbb{P}))]$. Since $\mathbb{S}$ is a regular subforcing of $j(\mathbb{S})$, $G(j(\mathbb{S}))$ contains a generic filter $G(\mathbb{S})$ for $\mathbb{S}$. Thus, by the Lifting Lemma (Lemma 3.5), $j$ can be lifted to an embedding from $V[G(\mathbb{P})][G(\mathbb{S})]$ to $M[G(j(\mathbb{P}))][G(j(\mathbb{S}))]$.

**Proof of (5)**: $j(\mathbb{P} * \mathbb{S})/G(\mathbb{P}^* * \mathbb{S}^*) = j(\mathbb{P})/G(\mathbb{P}^*) * j(\mathbb{S})/G(\mathbb{S}^*)$. So we just have to show that $j(\mathbb{S})/G(\mathbb{S}^*)$ is $\sigma$-closed. Conditions in $j(\mathbb{S})/G(\mathbb{S}^*)$ are conditions in $j(\mathbb{S})$ with domain above $\kappa_2$ which are compatible with $G(\mathbb{S}^*)$. And since the countable union of conditions in $j(\mathbb{S})$ which are compatible with $G(\mathbb{S}^*)$ is also compatible with $G(\mathbb{S}^*)$, this is $\sigma$-closed. □

Now we are ready to prove the $\kappa_2$-c.c. of $\mathbb{S}_{\kappa_3}$.

**Lemma 4.11.** *Let $\pi^*\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to \mathbb{L}_2 * \mathbb{L}_3^*$ be a reduction map and $j\colon V[G(\mathbb{L}_2 * \mathbb{L}_3)] \to M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$ a lifting of the supercompact embedding for $\kappa_2$, and in $V[\mathbb{L}_2 * \mathbb{L}_3]$ let $\mathbb{S}_\alpha$ be an iteration of length $\alpha \leq \kappa_3$ of forcings to specialize $\aleph_2$-Aronszajn trees.*

*Then there exists a regular subforcing $\mathbb{S}^*_\alpha$ of $j(\mathbb{S}_\alpha)$ with the following properties:*

*(1) $|\mathbb{S}^*_\alpha| < j(\kappa_2)$,*
*(2) there exists a reduction map $\pi\colon j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha) \to \mathbb{L}_2 * \mathbb{L}_3^* * \mathbb{S}^*_\alpha$ such that the first two coordinates of the mapping extend $\pi^*$,*
*(3) $j(\mathbb{S}_\alpha)/\mathbb{S}^*_\alpha$ is $\sigma$-closed,*
*(4) $\mathbb{L}_2 * \mathbb{L}_3^* * \mathbb{S}^*_\alpha$ is forcing equivalent to $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha$,*
*(5) $j$ can be lifted to an elementary embedding $j\colon V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha] \to M[j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha)]$.*

*Moreover, in $V[\mathbb{L}_2 * \mathbb{L}_3]$ the forcing $\mathbb{S}_\alpha$ has the $\kappa_2$-c.c..*

*Proof.* The proof is by induction on $\alpha \leq \kappa_3$.

For $\alpha = 0$ there is nothing to show.

$\alpha = \beta + 1$: By induction $\mathbb{S}_\beta$, the first $\beta$ steps of the iteration, fulfills the requirements of Lemma 4.10, so we can apply this lemma to $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\beta * \mathbb{S}(T_\alpha)$ and get what we wanted.

$\alpha$ limit: Let $\mathbb{S}^*_\alpha$ be the iteration $\mathbb{L}_2 * \mathbb{L}_3^* * \mathbb{S}^*_1 * \mathbb{S}^*_2 * \mathbb{S}^*_3 \dots$ with countable support. Let $p \in \mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha$ and let $\{\beta_i \mid i < \omega\}$ be the indices of the support of $p$. Let $\pi_{\beta_i}$ be the reduction map of the iteration of length $\beta_i + 1$. Since these maps cohere, $\pi_{\beta_0}(p \restriction (\beta_0 + 1)) \geq \pi_{\beta_1}(p \restriction (\beta_1 + 1)) \geq \pi_{\beta_2}(p \restriction (\beta_2 + 1)) \dots$ and since $\mathbb{L}_2 * \mathbb{L}_3^* * \mathbb{S}^*_1 * \mathbb{S}^*_2 * \mathbb{S}^*_3 \dots$ is $\sigma$-closed, there exists a lower bound of these reductions, let $\pi_\alpha(p)$ be such a lower bound. So $\pi_\alpha$ is a reduction map which is coherent with the earlier $\pi_\beta$.

Next we show that if two conditions $(p, \vec{s}), (p', \vec{s'}) \in (\mathbb{L}_2 * \mathbb{L}_3^*) * \mathbb{S}_\alpha^*$ are compatible in $j(\mathbb{L}_2 * \mathbb{L}_3) * j(\mathbb{S}_\alpha)$, then they are compatible in $(\mathbb{L}_2 * \mathbb{L}_3^*) * \mathbb{S}_\alpha^*$: To see this, note that the set $D$ of conditions $(p, \vec{s})$ with the following property is dense in $j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha)$: There exists $\vec{s^*}$ such that

(1) $p \Vdash \vec{s} \leq \vec{s^*}$
(2) $p \Vdash \vec{s^*} \in \mathbb{S}_\alpha^*$
(3) If $p \Vdash \vec{s} \leq \bar{s} \wedge \bar{s} \in \mathbb{S}_\alpha^*$ then $p \Vdash \vec{s^*} \leq \bar{s}$.

$(p, \vec{s})$ fulfills this property, if $p$ decides $\vec{s}$: Let $\vec{s^*}$ be the tupel of coordinates of $\vec{s}$ restricted to the nodes on levels below $\kappa_2$. So $p \Vdash \vec{s} \leq \vec{s^*} \wedge \vec{s^*} \in \mathbb{S}_\alpha^*$ and if $p \Vdash \vec{s} \leq \bar{s} \wedge \bar{s} \in \mathbb{S}_\alpha^*$ then $p \Vdash \vec{s^*} \leq \bar{s}$, because in this case every coordinate of $\bar{s}$ is forced to be a subset of the corresponding coordinate if $\vec{s^*}$. So $D$ is dense.

Suppose now that $(p_0^*, \vec{s_0^*})$ and $(p_1^*, \vec{s_1^*})$ are in $(\mathbb{L}_2 * \mathbb{L}_3^*) * \mathbb{S}_\alpha^*$ and they are compatible in $j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha)$. Let $(p, \vec{s})$ be a witness for the compatibility in the dense set. So $(p, \vec{s^*})$ is also below $(p_0^*, \vec{s_0^*})$ and $(p_1^*, \vec{s_1^*})$. Now $(\pi(p), \vec{s^*})$ is in $(\mathbb{L}_2 * \mathbb{L}_3^*) * \mathbb{S}_\alpha^*$ and stronger than $(p_0^*, \vec{s_0^*})$ and $(p_1^*, \vec{s_1^*})$: Since $p \Vdash \vec{s^*} \in \mathbb{S}_\alpha^* \wedge \vec{s^*} \leq \vec{s_0^*}, \vec{s_1^*}$ and that depends only on $\mathbb{L}_2 * \mathbb{L}_3^*$ the same holds for $\pi(p)$.

It remains to show that $\mathbb{L}_2 * \mathbb{L}_3 \Vdash \mathbb{S}_\alpha$ has the $\kappa_2$-c.c.. This follows by the same argument, as (2) of Lemma 4.10:

Let $G(\mathbb{L}_2 * \mathbb{L}_3^*)$ be generic for $\mathbb{L}_2 * \mathbb{L}_3^*$ and $G(\mathbb{L}_2 * \mathbb{L}_3)$ the corresponding generic for $\mathbb{L}_2 * \mathbb{L}_3$. Let $j\colon V[G(\mathbb{L}_2 * \mathbb{L}_3)] \to M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$ be a lifting of the supercompact embedding.

Let $A^*$ be a maximal antichain in $\mathbb{S}_\alpha^*$. Since $\mathbb{S}_\alpha^*$ is the same as $\mathbb{S}_\alpha$, just with a different name, $A^*$ is also a maximal antichain in $\mathbb{S}_\alpha$. By elementarity $j(A^*)$ is maximal in $j(\mathbb{S}_\alpha)$. Since $j$ is the identity on $\mathbb{S}_\alpha$ it follows that $A^* \subseteq j(A^*)$.

Let $G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))$ be generic for $j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*)$. Assume $M[G(\mathbb{L}_2 * \mathbb{L}_3^*)][G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))] \models \vec{s} \in j(\mathbb{S}_\alpha)$.

**Claim.** *$M[G(\mathbb{L}_2 * \mathbb{L}_3^*)][G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))] \models$ "$\exists \vec{a} \in A^*$ which is compatible with $\vec{s}$".*

*Proof.* Let $p \in G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))$ with $p \Vdash \vec{s} \in j(\mathbb{S})$. The following set is dense in $j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*)$ below $p$: $\{q \in j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*) \mid q \Vdash \exists \vec{a} \in A^*$ which is compatible with $\vec{s}\}$: $p \Vdash$ "$\vec{s} \in j(\mathbb{S}_\alpha)$ and there exists a reduction $\vec{s'}$ of $\vec{s}$". Since $\vec{s'} \in \mathbb{S}_\alpha$ there exists $\vec{a} \in A^*$ and $q \leq p$ such that $q \Vdash$ "$\vec{a}$ is compatible with $\vec{s'}$". Since $\vec{s'}$ is a reduction of $\vec{s}$, it follows that $q \Vdash \vec{a}$ is compatible with $\vec{s'}$.

Thus since $p \in G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*)$ there exists a $q \in G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*)) \cap \{q \in j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*) \mid q \Vdash$ "$\exists \vec{a} \in A^*$ which is compatible with $\vec{s}$"$\}$. □

Thus it follows that $A^*$ is a maximal antichain in $j(\mathbb{S}_\alpha)$.

Since $j(A^*)$ is an antichain and $A^* \subseteq j(A^*)$ it follows that $A^* = j(A^*)$. Thus $|j(A^*)| < j(\kappa_2)$ and by elementarity $|A^*| < \kappa_2$.

That $j$ can be lifted, follows by the same proof as (4) of Lemma 4.10: Let $G(j(\mathbb{S}_\alpha))$ be generic for $j(\mathbb{S}_\alpha)$ over $M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$. Since $\mathbb{S}_\alpha$ is a regular subforcing of $j(\mathbb{S}_\alpha)$ and $j[\mathbb{S}_\alpha] = \mathbb{S}_\alpha$, $G(j(\mathbb{S}_\alpha))$ contains a generic filter $G(\mathbb{S}_\alpha)$ for $\mathbb{S}_\alpha$. Thus, by the Lifting Lemma (Lemma 3.5), $j$ can be lifted to $V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{S}_\alpha)] \to M[G(j(\mathbb{L}_2 * \mathbb{L}_3))][G(j(\mathbb{S}_\alpha))]$.

$j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha)/G(\mathbb{L}_2 * \mathbb{L}_3^* * \mathbb{S}_\alpha^*) = j(\mathbb{L}_2)/G(\mathbb{L}_2) * j(\mathbb{L}_3)/G(\mathbb{L}_3^*) * j(\mathbb{S}_0)/G(\mathbb{S}_0^*) * j(\mathbb{S}_1)/G(\mathbb{S}_1^*) \dots$, so by induction this is a countable support iteration of forcings which are $\sigma$-closed, so it is $\sigma$-closed itself. □

The next lemma has been proved in [GH20, Lemma 2.5].

**Lemma 4.12.** *In $V[\mathbb{L}_2 * \mathbb{L}_3]$ the forcing $\mathbb{S}_{\aleph_3} \times \mathbb{S}_{\aleph_3}$ has the $\kappa_2$-c.c..*

*Proof.* Let $\varphi$ be a bookkeeping function such that $\varphi(\mathbb{P}_\alpha)$ is a $\mathbb{P}_\alpha$-name for an $\aleph_2$-Aronszajn tree (if there exists one) for every forcing $\mathbb{P}_\alpha$ and $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{S}}(\varphi(\mathbb{P}_\alpha))$ is a forcing to specialize this tree, i.e., $\varphi$ is a bookkeeping function which gives $\mathbb{S}_{\aleph_3}$ as an iteration. Now define $\tilde{\varphi}$ and let $\tilde{\mathbb{S}}_{\aleph_3}$ be the forcing iteration, given by the bookkeeping $\tilde{\varphi}$. For $\alpha < \aleph_3$ let $\tilde{\varphi}(\tilde{\mathbb{P}}_\alpha) = \varphi(\mathbb{P}_\alpha)$. For $\alpha = \aleph_3 + \beta$ for some $\beta < \aleph_3$, let $\tilde{\varphi}(\tilde{\mathbb{P}}_\alpha) = \varphi(\mathbb{P}_\beta)$, i.e., we repeat the same iteration which was done up to $\aleph_3$ again between $\aleph_3$ and $\aleph_3 + \aleph_3$.

$\tilde{\mathbb{S}}_{\aleph_3+\aleph_3}$ has the $\aleph_2$-c.c. by Lemma 4.11, and since no new countable sets are added by $\mathbb{S}_{\aleph_3}$ it holds true that $\mathbb{S}_{\aleph_3} \times \mathbb{S}_{\aleph_3} = \mathbb{S}_{\aleph_3} * \mathbb{S}_{\aleph_3} = \tilde{\mathbb{S}}_{\aleph_3+\aleph_3}$. □

**Lemma 4.13.** *Let $\alpha < \aleph_3$. In $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha]$, the forcing $(\mathbb{S}_{\aleph_3}/G(\mathbb{S}_\alpha)) \times (\mathbb{S}_{\aleph_3}/G(\mathbb{S}_\alpha))$ has the $\kappa_2$-c.c. (for any $G(\mathbb{S}_\alpha)$ being generic for $\mathbb{S}_\alpha$).*

*Proof.* By Lemma 4.12 $\mathbb{S}_{\aleph_3} \times \mathbb{S}_{\aleph_3}$ has the $\kappa_2$-c.c. and so Lemma 3.7 implies that $(\mathbb{S}_{\aleph_3}/G(\mathbb{S}_\alpha)) \times (\mathbb{S}_{\aleph_3}/G(\mathbb{S}_\alpha))$ has the $\kappa_2$-c.c.. □

**Lemma 4.14.** *For every $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$-name $\dot{T}$ for an $\aleph_1$-tree with level $\alpha$ being $\{\alpha\} \times \omega$ for every $\alpha < \aleph_1$ there exists a regular subforcing $\bar{\mathbb{L}} * \bar{\mathbb{S}}$ of $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$ with the following properties:*

*(1) $\bar{\mathbb{L}} \Vdash |\bar{\mathbb{S}}| < \kappa_2$,*
*(2) $\bar{\mathbb{L}} \Vdash \bar{\mathbb{S}}$ is $\omega$-distributive,*
*(3) $\bar{\mathbb{L}}$ is a regular subforcing of $\mathbb{L}_2$,*
*(4) $\mathbb{L}_2 * \mathbb{L}_3 \Vdash$ "$\bar{\mathbb{S}}$ is a regular subforcing of $\mathbb{S}_{\aleph_3}$ and $\mathbb{S}_{\aleph_3}/\bar{\mathbb{S}}$ is $\sigma$-closed",*
*(5) there exists an $\bar{\mathbb{L}} * \bar{\mathbb{S}}$-name $\dot{T}'$ such that $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3} \Vdash \dot{T} = \dot{T}'$.*

*Proof.* By Corollary 4.9 and Lemma 4.11, there exists a lifting of $j$ to $j \colon V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}] \to M[j(\mathbb{L}_2) * j(\mathbb{L}_3) * j(\mathbb{S}_{\aleph_3})]$ and a regular subforcing $\mathbb{L}^* * \mathbb{S}^*$ of $j(\mathbb{L}_2) * j(\mathbb{L}_3) * j(\mathbb{S}_{\aleph_3})$ of size $< j(\kappa_2)$ such that $j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3})/G(\mathbb{L}^* * \mathbb{S}^*)$ is $\sigma$-closed, $\mathbb{L}^*$ is a regular subforcing of $j(\mathbb{L}_2)$, $\mathbb{S}^*$ is a regular subforcing of $j(\mathbb{S}_{\aleph_3})$ and $\mathbb{L}^* * \mathbb{S}^*$ is equivalent to $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$. In particular $\mathbb{S}^*$ is $\omega$-distributive.

Let $\dot{T}$ be an $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$-name for an $\aleph_1$-tree with level $\alpha$ being $\{\alpha\} \times \omega$. Since the critical point of $j$ is $\kappa_2$, $j(\dot{T})$ is also a $j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3})$-name for an $\aleph_1$-tree. So there exists an $\mathbb{L}^* * \mathbb{S}^*$-name $\dot{T}^*$ such that $j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}) \Vdash j(\dot{T}) = \dot{T}^*$.

Thus we have that there exist regular subforcings $\mathbb{L}^*, \mathbb{S}^*$ of $j(\mathbb{L}_2), j(\mathbb{S}_{\aleph_3})$ such that $\mathbb{S}^*$ is $\omega$-distributive, $|\mathbb{S}^*| < j(\kappa_2)$ and an $\mathbb{L}^* * \mathbb{S}^*$-name $\dot{T}^*$ for $j(\dot{T})$. And $j(\mathbb{L}_2 * \mathbb{L}_3) \Vdash$ "$j(\mathbb{S}_{\aleph_3})/\mathbb{S}^*$ is $\sigma$-closed".

By elementarity of $j$ the same holds for $\mathbb{L}_2$, $\mathbb{L}_3$ and $\mathbb{S}_{\aleph_3}$:

There exist regular subforcings $\bar{\mathbb{L}}, \bar{\mathbb{S}}$ of $\mathbb{L}_2, \mathbb{S}_{\aleph_3}$ such that $\bar{\mathbb{S}}$ is $\omega$-distributive and $|\bar{\mathbb{S}}| < \kappa_2$, and $\mathbb{L}_2 * \mathbb{L}_3 \Vdash$ "$\mathbb{S}_{\aleph_3}/\bar{\mathbb{S}}$ is $\sigma$-closed" and an $\bar{\mathbb{L}} * \bar{\mathbb{S}}$-name $\dot{T}'$ such that $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3} \Vdash \dot{T} = \dot{T}'$. □

**Theorem 4.15.** *It follows from the consistency of a supercompact cardinal and an inaccessible cardinal above that it is consistent that all $\aleph_2$-Aronszajn trees are special, there are such, and there is no $\aleph_1$-Kurepa tree and no $\aleph_2$-Kurepa tree.*

*Proof.* First we show that there are no $\aleph_1$-Kurepa trees: Assume $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$ forces "There exists an $\aleph_1$-Kurepa tree". Let $\dot{T}$ be a name for an $\aleph_1$-tree with level $\alpha$ equal to $\{\alpha\} \times \omega$.

By Lemma 4.14 there exists a regular subforcing $\bar{\mathbb{L}} * \bar{\mathbb{S}}$ of $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$, and an $\bar{\mathbb{L}} * \bar{\mathbb{S}}$-name $\bar{T}$ such that $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3} \Vdash \dot{T} = \bar{T}$. $\bar{\mathbb{S}}$ is $\sigma$-closed, $|\bar{\mathbb{S}}| < \kappa_2$ and $\bar{\mathbb{L}}$ is a regular subforcing of $\mathbb{L}_2$, $\bar{\mathbb{S}}$ is a regular subforcing of $\mathbb{S}_{\aleph_3}$ and $\mathbb{L}_2 * \mathbb{L}_3 \Vdash \mathbb{S}_{\aleph_3}/\bar{\mathbb{S}}$ is $\sigma$-closed. Note that as $\bar{\mathbb{L}}$ is a regular subforcing of $\mathbb{L}_2$, $\bar{T}$ can also be regarded as a $\mathbb{L}_2 * \bar{\mathbb{S}}$-name and $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}$ is equivalent to $\mathbb{L}_2 * \bar{\mathbb{S}} * \mathbb{L}_3 * \mathbb{S}/\bar{\mathbb{S}}$.

By Proposition 2.10 there exists no $\aleph_1$-Kurepa tree in $V[\mathbb{L}_2 * \bar{\mathbb{S}}]$. So $V[\mathbb{L}_2 * \bar{\mathbb{S}}] \models |[\dot{T}]| < \aleph_2$. $\mathbb{L}_3 * \mathbb{S}_{\aleph_3}/\bar{\mathbb{S}}$ is $\sigma$-closed, so it does not add branches to $\dot{T}$, so $V[\mathbb{L}_2 * \bar{\mathbb{S}} * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}/\bar{\mathbb{S}}] \models$ "$\dot{T}$ is not an $\aleph_1$-Kurepa tree".

No $\aleph_2$-Kurepa trees: Work in $V[\mathbb{L}_2 * \mathbb{L}_3]$. Let $\dot{T}$ be an $\mathbb{S}_{\aleph_3}$-name for an $\aleph_2$-tree. Since $\mathbb{S}_{\aleph_3}$ has the $\aleph_3$-c.c. (indeed the $\aleph_2$-c.c.), we can assume that $|\dot{T}| = \aleph_2$. Hence there exists $\alpha < \aleph_3$ such that $\dot{T}$ is an $\mathbb{S}_\alpha$-name. So $T \in V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha]$. Note that $|\mathbb{S}_\alpha| = \aleph_2$ and $\mathbb{S}_\alpha$ has the $\aleph_2$-c.c., hence by Proposition 2.10 there exists no $\aleph_2$-Kurepa tree in $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha]$. Therefore $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha] \models |[T]| < \aleph_3$. By Lemma 4.13 $(\mathbb{S}_{\aleph_3}/\mathbb{S}_\alpha)^2$ has the $\kappa_2$-c.c., so $\mathbb{S}_{\aleph_3}/\mathbb{S}_\alpha$ does not add branches to $\dot{T}$, so $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha * \mathbb{S}_{\aleph_3}/\mathbb{S}_\alpha] \models \dot{T}$ is not an $\aleph_2$-Kurepa tree.

After forcing with $\mathbb{L}_2 * \mathbb{L}_3$, CH holds, and since $\mathbb{S}_{\aleph_3}$ is $\sigma$-closed, CH holds in the final model, which implies the existence of an $\aleph_2$-Aronszajn tree (see Proposition 2.11). □

## 5. Trees for all $\aleph_n$

Now let us continue with the proof of the main result. Some of the proofs in this section are generalizations of proofs from the previous section.

From the existence of $\omega$ many supercompact cardinal, we prove that it is consistent that for all $0 < n < \omega$ all $\aleph_n$-Aronszajn trees are special, there are such, and there are no $\aleph_n$-Kurepa trees.

5.1. **Definition of the forcing.** Let $\langle \kappa_n \mid 1 < n < \omega \rangle$ be an increasing sequence of Laver indestructible supercompact cardinals; for simplicity of notation let $\kappa_1 = \aleph_1$. Let $\delta = (\sup_{0<n<\omega} \kappa_n)^{++}$.

For every $1 < n < \omega$ let $j_n \colon V \to M$ be a supercompactness embedding for $\kappa_n$ with $j_n(\kappa_n) > \delta$. We will often write $j$ instead of $j_n$ if it is clear from context which $n$ is meant.

Define the forcing iteration as follows. We start with an iteration of Lévy collapses of all the supercompact cardinals. For every $n \geq 2$ let

$$\mathbb{L}_n := \mathrm{Col}(\kappa_{n-1}, {<}\kappa_n)^{V[\mathbb{L}_2 * \cdots * \mathbb{L}_{n-1}]}.$$

We will use the following notation. Let $\mathbb{L}_\omega := \mathbb{L}_2 * \mathbb{L}_3 * \mathbb{L}_4 * \ldots$ with countable support, let $\mathbb{L}_{>n} := \mathbb{L}_{n+1} * \mathbb{L}_{n+2} * \mathbb{L}_{n+3} * \ldots$ with countable support, and let $\mathbb{L}_{\leq n} := \mathbb{L}_2 * \mathbb{L}_3 * \cdots * \mathbb{L}_n$ and $\mathbb{L}_{<n} := \mathbb{L}_2 * \mathbb{L}_3 * \cdots * \mathbb{L}_{n-1}$.

To specialize the $\aleph_1$-Aronszajn trees, we use the classical forcing from [BMR70]:

**Definition 5.1.** Let $T$ be an $\aleph_1$-Aronszajn tree. Let $\mathbb{S}(T)$ be the forcing consisting of the following conditions:

(1) $\mathrm{dom}(p) \subseteq T$ is finite
(2) $\mathrm{range}(p) \subseteq \omega$
(3) if $x, y \in \mathrm{dom}(p)$ and $x <_T y$, then $p(x) \neq p(y)$.

The order is given by $q \leq p$ if $q \supseteq p$.

**Lemma 5.2.** *If $T$ is an $\aleph_1$-Aronszajn tree, then $\mathbb{S}(T)$ has the c.c.c..*

*Proof.* See [BMR70] (or [Jec03, Lemma 16.18]). □

Following [GH20] we combine the specializing forcings for all the $\aleph_n$ as follows:

**Definition 5.3** (Specializing names)**.** Assume that $\mathbb{P}$ is a forcing with $1_{\mathbb{P}} \Vdash$ "$\dot{T}$ is an $\aleph_n$-Aronszajn tree with $\dot{T}_\xi = \{\xi\} \times \aleph_{n-1}$". Let $\mathbb{S}_{\mathbb{P}}(\dot{T})$ be the following forcing: Conditions are partial multi-valued functions $f$ from $\dot{T}$ into $\aleph_{n-1}$ of size $< \aleph_{n-1}$ such that $|f(s)| < \aleph_{n-1}$ and if $f(s) \cap f(t) \neq \emptyset$, then $1_{\mathbb{P}} \Vdash$ "$s$ is incomparable to $t$ in $\dot{T}$". The order is given by $g \leq f$ if $g \supseteq f$.

Let us now define the iteration. Let $\{A_n \mid n \in \omega\}$ be a partition of $\delta$ such that every $A_n$ is cofinal in $\delta$. Assume $\mathbb{Q}_\beta$ has been defined for all $\beta < \alpha$.

Let $\mathbb{S}_\alpha^{>n}$ be the iteration of all the $\dot{\mathbb{Q}}_\beta$, with $\beta < \alpha$, where $\dot{\mathbb{Q}}_\beta$ is a forcing to specialize a name for an $\aleph_k$-tree for some $k > n$, with a mixed support such that $X$ is a possible support if $|A_k \cap X| < \kappa_{k-1}$ for each $n < k < \omega$. Analogously define $\mathbb{S}_\alpha^{\geq n}$.

Let $\dot{\mathbb{S}}_\alpha^n$ be an $\mathbb{L}_\omega * \mathbb{S}_\alpha^{>n}$-name for the iteration of all the $\dot{\mathbb{Q}}_\beta$, $\beta < \alpha$, where $\dot{\mathbb{Q}}_\beta$ is a forcing to specialize a name for an $\aleph_n$-tree with $<\kappa_{n-1}$-support.

Let $\dot{\mathbb{S}}_\alpha^{<n}$ be an $\mathbb{L}_\omega * \mathbb{S}_\alpha^{\geq n}$-name for the iteration of all the $\dot{\mathbb{Q}}_\beta$, where $\dot{\mathbb{Q}}_\beta$ is a forcing to specialize a name for an $\aleph_k$-tree for some $k < n$, with a mixed support such that $X$ is a possible support if $|A_k \cap X| < \kappa_{k-1}$ for each $k < n$.

Finally let $\dot{\mathbb{S}}_\alpha := \dot{\mathbb{S}}_\alpha^{>n} * \dot{\mathbb{S}}_\alpha^n * \dot{\mathbb{S}}_\alpha^{<n}$ and let $\mathbb{P}_\alpha := \mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha$. So $\mathbb{P}_\alpha$ has a mixed support: for every $n$, the forcings which specialize names for $\aleph_n$-trees, as a subiteration, have $<\kappa_{n-1}$-support.

If $\alpha \in A_n$, let $\dot{T}_\alpha^n$ be an $\mathbb{L}_\omega * \mathbb{S}_\alpha^{>n} * \mathbb{S}_\alpha^n$-name for an $\aleph_n$-Aronszajn tree, if there exists one, and let $\dot{\mathbb{Q}}_\alpha := \mathbb{S}_{\mathbb{L}_\omega * \mathbb{S}_\alpha^{>n} * \mathbb{S}_\alpha^n}(\dot{T}_\alpha^n)$ be the forcing to specialize the name $\dot{T}_\alpha^n$. Otherwise let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing.

Continue this iteration for length $\delta$, using a bookkeeping function such that every $\aleph_n$-Aronszajn tree will get specialized in one step of the iteration. To see that it is possible to specialize all $\aleph_n$-Aronszajn trees for all $0 < n < \omega$, note that for every $\alpha < \delta$ the size of $\mathbb{S}_\alpha$ is less than $\delta$. Therefore $2^{\aleph_n} < \delta$ in $V[\mathbb{L}_\omega * \mathbb{S}_\alpha]$. So there exist less than $\delta$ many $\aleph_n$-Aronszajn trees with level $\xi$ equal to $\{\xi\} \times \aleph_{n-1}$ in this model. This shows that it is possible to specialize all $\aleph_n$-Aronszajn trees in an iteration of length $\delta$.

**Lemma 5.4.** *Let $\dot{T}$ be a $\mathbb{P}$-name for an $\aleph_n$-Aronszajn tree. For every $(\xi, \beta) \in \aleph_n \times \aleph_{n-1}$ the set $\{g \in \mathbb{S}_{\mathbb{P}}(\dot{T}) \mid (\xi, \beta) \in dom(g)\}$ is dense in $\mathbb{S}_{\mathbb{P}}(\dot{T})$.*

*Proof.* Let $f \in \mathbb{S}_{\mathbb{P}}(\dot{T})$, $\xi \in \aleph_n$ and $\beta \in \aleph_{n-1}$. Since $|\mathrm{dom}(f)| < \aleph_{n-1}$, and $|f(s)| < \aleph_{n-1}$ for every $s \in \mathrm{dom}(f)$, there exists $i \in \aleph_{n-1} \setminus \mathrm{rng}(f)$. Let $g := f \cup ((\xi, \beta), i)$. So $g \leq f$ and $(\xi, \beta) \in \mathrm{dom}(g)$. □

**Lemma 5.5.** *Let $\mathbb{P}$ be a forcing with $1_{\mathbb{P}} \Vdash$ "$\dot{T}$ is an $\aleph_n$-Aronszajn tree". Then $\mathbb{P} * \mathbb{S}_{\mathbb{P}}(\dot{T}) \Vdash$ "$\dot{T}$ is special".*

*Proof.* In $V[\mathbb{P}]$ let $G$ be a generic filter for $\mathbb{S}_{\mathbb{P}}(\dot{T})$. Let $F := \bigcup\{f \in \mathbb{S}_{\mathbb{P}}(\dot{T}) \mid f \in G\}$. It follows from the above lemma that $\mathrm{dom}(F) = \aleph_n \times \aleph_{n-1}$. For $s, t \in \aleph_n \times \aleph_{n-1}$ with $F(s) = F(t)$ we have that $1_{\mathbb{P}} \Vdash$ "$s$ and $t$ are incomparable in $\dot{T}$", hence $F(s) \neq F(t)$ if $s < t$. So $F$ is a specializing function of $\dot{T}$. □

We will show that all $\aleph_n$ are preserved by the forcing iteration after $\mathbb{L}_\omega$ and can thus conclude that in the extension by $\mathbb{P}_\delta$, all $\aleph_n$-Aronszajn trees will be special for all $n \in \omega$.

### 5.2. **Chain condition and closure.**

**Lemma 5.6.** *Let $\alpha \leq \delta$. In $V[\mathbb{L}_{\leq k}]$ let $j_{k+1}\colon V \to M$ be a supercompact embedding for $\kappa_{k+1}$ such that $j_{k+1}(\kappa_{k+1}) > |\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1}|$.*

*There exists a regular subforcing $\mathbb{P}^*$ of $j_{k+1}(\mathbb{L}_{k+1})$ which is forcing equivalent to $\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1}$ such that $j_{k+1}(\mathbb{L}_{k+1})/\mathbb{P}^*$ is equivalent to $j_{k+1}(\mathbb{L}_{k+1})$ and $j_{k+1}(\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1})/\mathbb{P}^*$ is $<j_{k+1}(\kappa_k)$-closed.*

*Proof.* Let $G(\mathbb{L}_{\leq k})$ be generic for $\mathbb{L}_{\leq k}$ and let $j_{k+1}\colon V \to M$ be a supercompact embedding for $\kappa_{k+1}$ such that $j_{k+1}(\kappa_{k+1}) > |\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1}|$. So $V[G(\mathbb{L}_{\leq k})] \models \mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1}$ is $<\kappa_k$-closed and $j_{k+1}(\kappa_{k+1}) > |\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1}|$ and we can apply Theorem 4.5. This gives $\iota\colon \mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1} \to \mathrm{Col}(\kappa_k, <j_{k+1}(\kappa_{k+1}))$ such that if $G(\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1})$ is a generic filter for $\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1}$ over $V[G(\mathbb{L}_{\leq k})]$, then the collapse $\mathrm{Col}(\kappa_k, <j_{k+1}(\kappa_{k+1}))$ is equivalent to the quotient $\mathrm{Col}(\kappa_k, <j_{k+1}(\kappa_{k+1}))/\iota[G(\mathrm{Col}(\kappa_k, <j_{k+1}(\kappa_{k+1})))]$, which is equal to $j_{k+1}(\mathbb{L}_{k+1})/\iota[G(\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1})]$.

Moreover $j_{k+1}(\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1})$ is $<j_{k+1}(\kappa_k)$-closed and therefore also the quotient $j_{k+1}(\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1})/\iota[G(\mathrm{Col}(\kappa_k, <j_{k+1}(\kappa_{k+1})))]$ is $<j_{k+1}(\kappa_k)$-closed, so $\mathbb{P}^* := \iota[\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1}]$ is the forcing we are looking for. □

**Corollary 5.7.** *In $V[\mathbb{L}_{\leq k}]$ the forcing $\mathbb{L}_{k+1}$ has a regular subforcing $\mathbb{P}^{**}$ of size $< \kappa_{k+1}$ such that $\mathbb{L}_{k+1}/\mathbb{P}^{**}$ is equivalent to $\mathbb{L}_{k+1}$ and $(\mathbb{L}_{\geq k+1} * \mathbb{S}_\alpha^{\geq k+1})/\mathbb{P}^{**}$ is $<\kappa_k$-closed.*

*Proof.* This follows directly by elementarity from Lemma 5.6. □

**Corollary 5.8.** *There exists a reduction map $\pi\colon j_{k+1}(\mathbb{L}_{\leq k} * \mathbb{L}_{>k} * \mathbb{S}_\delta^{\geq k+1}) \to \mathbb{L}_{\leq k} * \mathbb{P}^{**}$.*

*Proof.* This follows directly from Lemma 5.6. □

**Lemma 5.9.** $\mathbb{L}_\omega * \mathbb{S}_\alpha^{\geq k+1} \Vdash$ "*There exists no $\aleph_k$-Kurepa tree" for every $\alpha \leq \delta$.*

*Proof.* First note that $\mathbb{L}_\omega * \mathbb{S}_\alpha^{\geq k+1} \Vdash \aleph_k = \kappa_k$.

Assume $\mathbb{L}_\omega * \mathbb{S}_\alpha^{\geq k+1} \Vdash$ "There exists a $\kappa_k$-Kurepa tree", let $\dot{T}$ be a name for this $\kappa_k$-tree. $\mathbb{L}_\omega * \mathbb{S}_\alpha^{\geq k+1}$ is equivalent to $\mathbb{L}_{\leq k} * \mathbb{P}^* * \mathbb{L}_{>k}/\mathbb{P}^* * \mathbb{S}_\alpha^{\geq k+1}/\mathbb{P}^*$.

Since $\kappa_{k+1}$ is Laver indestructible and $|\mathbb{L}_{\leq k} * \mathbb{P}^*| < \kappa_{k+1}$, in the extension $V[\mathbb{L}_{\leq k} * \mathbb{P}^*]$ by this forcing, $\kappa_{k+1}$ is still supercompact, in particular inaccessible, so $V[\mathbb{L}_{\leq k} * \mathbb{P}^*] \models \|[\dot{T}]\| < \kappa_{k+1}$ and $\mathbb{L}_{>k}/\mathbb{P}^* * \mathbb{S}_\alpha^{\geq k+1}/\mathbb{P}^*$ is $<\kappa_k$-closed, so it does not add branches to $\dot{T}$, so $V[\mathbb{L}_{\leq k} * \mathbb{P}^* * \mathbb{L}_{>k}/\mathbb{P}^* * \mathbb{S}_\alpha^{\geq k+1}/\mathbb{P}^*] \models$ "$\dot{T}$ is not a $\kappa_k$-Kurepa tree" □

**Lemma 5.10.** *If $\dot{T}$ is an $\mathbb{L}_\omega * \mathbb{S}_\delta^{>n} * \mathbb{P}^{**}$-name for a $\kappa_k$-tree, then $V[\mathbb{L}_\omega * \mathbb{S}_\delta^{>n} * \mathbb{P}^{**}] \models \|[\dot{T}]\| < \kappa_{k+1}$.*

*Proof.* By Lemma 5.9, $V[\mathbb{L}_\omega * \mathbb{S}_\delta^{>n}] \models \|[\dot{T}]\| < \kappa_{k+1}$ and since $|\mathbb{P}^{**}| < \kappa_k$, in particular $(\mathbb{P}^{**})^2$ has the $\kappa_k$-c.c., so by Lemma 3.9 it does not add branches to $\dot{T}$. □

**Lemma 5.11.** *Let $j\colon V \to M$ be a supercompact embedding for $\kappa_k$ with $j(\kappa_k) > \kappa_{k+1}$ and ${}^{\le\kappa_{k+1}}M \subseteq M$. Let $G(\mathbb{L}_{\le k})$ be generic for $\mathbb{L}_{\le k}$ and $G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})$ be generic for $\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}$. Then there exist generic filters $G(j(\mathbb{L}_{\le k}))$ and $G(j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}))$ for $j(\mathbb{L}_{\le k})$ and $j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})$ such that the supercompact embedding $j$ can be lifted to $j\colon V[G(\mathbb{L}_{\le k})][G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})] \to M[G(j(\mathbb{L}_{\le k}))][G(j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}))]$.*

*Proof.* Let $j\colon V \to M$ be a supercompact embedding with $j(\kappa_k) > \kappa_{k+1}$ and ${}^{\le\kappa_{k+1}}M \subseteq M$. Let $G(\mathbb{L}_k)$ be generic for $\mathbb{L}_k$ and $G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})$ generic for $\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}$ over $V[G(\mathbb{L}_k)]$.

Let $\iota\colon \mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1} \to j(\mathbb{L}_k)/G(\mathbb{L}_k)$ be a regular embedding as in Lemma 5.6.

We can choose $G(j(\mathbb{L}_k))$ such that $G(j(\mathbb{L}_k)) \cap \operatorname{range}(\iota) = \iota[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})]$, thus $\iota[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})] \in V[G(j(\mathbb{L}_k))]$ and $G(\mathbb{L}_k) \subseteq G(j(\mathbb{L}_k))$; that is possible because $\mathbb{L}_k * \iota[\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}]$ is a regular subforcing of $j(\mathbb{L}_k)$.

Thus it follows that $\iota[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})] \subseteq V[G(j(\mathbb{L}_k))]$ and since $\iota, j \restriction \mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1} \in V[G(j(\mathbb{L}_k))]$ it follows that $j[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})] \in V[G(j(\mathbb{L}_k))]$. Since $M$ is closed under subsets of size $\le \kappa_{k+1}$ the same holds for $M[G(j(\mathbb{L}_k))]$ and therefore $j[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})] \in M[G(j(\mathbb{L}_k))]$.

$j[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})] \subseteq j[\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}] \subseteq j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})$, $j[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})]$ is a directed set with $|j[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})]| < j(\kappa_k)$ and $j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})$ is $<j(\kappa_k)$-directed closed, therefore there exists a mastercondition $p \in j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})$ for $j[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})]$. Let $G(j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}))$ be generic for $j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})$ with $p \in G(j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}))$. It follows that $j[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})] \subseteq G(j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}))$.

Now we can use the Lifting Lemma (Lemma 3.5) to lift $j$ to an embedding $j\colon V[G(\mathbb{L}_k)][G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})] \to M[G(j(\mathbb{L}_k))][G(j(\mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}))]$. □

One of the main technical parts of the proof is to show that the forcing iteration has a good chain condition. The main work lies in the following lemma, which deals with the successor step of the iteration. Note that $\mathbb{L}_k * \mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1}$ with $\mathbb{L}_k * \mathbb{P}^*$ as a subforcing of $j(\mathbb{L}_k * \mathbb{L}_{>k} * \mathbb{S}_\delta^{\ge k+1})$ fulfills the requirements of the lemma.

**Lemma 5.12.** *Assume $\mathbb{P}$ is a forcing with $V[G(\mathbb{P})] \models \kappa_k = \aleph_k$ and $\mathbb{P}^*$ is forcing equivalent to $\mathbb{P}$ and $\mathbb{P}^*$ is a regular subforcing of $j_k(\mathbb{P})$ with reduction map $\pi$ and $j_k(\mathbb{P})/G(\mathbb{P}^*)$ is $<\kappa_{k-1}$-closed. Let $j_k\colon V[G(\mathbb{P})] \to M[G(j_k(\mathbb{P}))]$ be a lifting of the supercompact embedding of $\kappa_k$ and $\mathbb{S} = \mathbb{S}_{\mathbb{P}}(T)$ a specializing forcing of a $\kappa_k$-Aronszajn tree in $V[G(\mathbb{P})]$.*

1. *There exists a regular subforcing $\mathbb{P}^* * \mathbb{S}^*$ of $j(\mathbb{P}) * j(\mathbb{S})$ with a reduction map $\pi^*\colon j(\mathbb{P}) * j(\mathbb{S}) \to \mathbb{P}^* * \mathbb{S}^*$ such that the first component of $\pi^*(p, s)$ extends $\pi(p)$.*
2. *$\mathbb{P} \Vdash \mathbb{S}$ has the $\kappa_k$-c.c.*
3. *$\mathbb{P} * \mathbb{S}$ is forcing equivalent to $\mathbb{P}^* * \mathbb{S}^*$.*

(4) *The supercompact embedding $j$ can be lifted to $j\colon V[G(\mathbb{P} * \mathbb{S})] \to M[G(j(\mathbb{P}) * j(\mathbb{S}))]$.*
(5) $j(\mathbb{P} * \mathbb{S})/G(\mathbb{P}^* * \mathbb{S}^*)$ *is* $<\kappa_{k-1}$*-closed.*

*Proof.* The proof is a generalization of the corresponding proof for $\kappa_2$.

**Proof of (1)**: Let $(p, s) \in j(\mathbb{P}) * j(\mathbb{S})$. Let $p' \leq p, \pi(p)$ such that $p'$ decides $s$, that means that in $V$ exists a partial function $f\colon \omega_{k-1} \times j(\kappa_k) \to \omega_{k-1}^{\leq \kappa_{k-2}}$ of size $<\kappa_{k-1}$ such that $p \Vdash s = f$. If $p'' \leq \pi(p')$, $p''$ is compatible with $p'$ and therefore with $\pi(p)$, thus $\pi(p)$ and $\pi(p')$ are compatible in $j(\mathbb{P})$. Since $\mathbb{P}^*$ is a regular subforcing of $j(\mathbb{P})$, $\pi(p)$ and $\pi(p')$ are compatible in $\mathbb{P}^*$. Let $\hat{p} \in \mathbb{P}^*$ with $\hat{p} \leq \pi(p), \pi(p')$.

Then choose a generic $G(\mathbb{P}^*)$ containing $\hat{p}$ and let $G(\mathbb{P})$ be the corresponding generic for $\mathbb{P}$, i.e., $V[G(\mathbb{P})] = V[G(\mathbb{P}^*)]$; that is possible because $\mathbb{P}$ and $\mathbb{P}^*$ are forcing equivalent.

Since $T \in V[G(\mathbb{P})]$, it follows that $T \in V[G(\mathbb{P}^*)]$. Let $T^*$ be a $\mathbb{P}^*$-name for $T$ and let $\mathbb{S}^* = \mathbb{S}(T^*)$, the specializing forcing of $T^*$ in $V[G(\mathbb{P}^*)]$.

We assume that the nodes on the $\alpha$th level $T_\alpha$ of $T$ are the elements of $\omega_{k-1} \times \{\alpha\}$ and all the levels are of size $<\kappa_k$, therefore $T = j[T] = j(T) \restriction \kappa_k$.

We can assume that for each $\sigma \in \operatorname{dom}(s)$ there exists a $\sigma' \in \operatorname{dom}(s)$ on level $\kappa_k$ such that $p' \Vdash \sigma' \leq \sigma$.

Let $\bar{s} = s \restriction T$, $\{\sigma_\alpha \mid \alpha \in \omega_{k-2}\} = \operatorname{dom}(s) \cap T_{\kappa_k}$ and $C_\alpha := \bigcup\{s(\tau) \mid \tau \supseteq \sigma_\alpha, \tau \in \operatorname{dom}(s)\}$ the set of colors which $s$ assings to nodes which are in $\operatorname{dom}(s)$ and equal to or above $\sigma_\alpha$.

Let $\mathbb{Q} = j(\mathbb{P})/G(\mathbb{P}^*)$.

Define a tree $\mathcal{T}$ of height $\omega_{k-2}$ as follows:

- The root of $\mathcal{T}$ is $(p_{\langle\rangle}, (\bar{\sigma}_0))$ where $p_{\langle\rangle} \Vdash$ "$\bar{\sigma}_0 \leq \sigma_0, \bar{\sigma}_0 \in T$", $p_{\langle\rangle} \in \mathbb{Q}$. So $\bar{\sigma}_0$ is just some node which is forced by $p_{\langle\rangle}$ to be below $\sigma_0$.
- Let $t \in \mathcal{T}$. By construction $t$ is of the form $(p_w, (\tau_w^\beta, \mid \beta < \alpha))$ for some $\alpha \in \omega_{k-2}$, $w \in 2^\alpha$.

  With $p_w \Vdash$ "$\bar{\sigma}_k <_T \tau_w^k <_T \sigma_k$, $p_w \in \mathbb{Q}$. Again $\bar{\sigma}_\alpha$ is just some node which is forced by $p_w$ to be below $\sigma_\alpha$.

  As every branch through $T$ in $V[j(\mathbb{P})/G(\mathbb{P}^*)]$ is new, there exist two conditions $p_{w^\frown 0}$ and $p_{w^\frown 1}$ which decide for every $k < \alpha$ the nodes between $\tau_w^k$ and $\sigma_k$ differently, (between $\bar{\sigma_\alpha}$ and $\sigma_\alpha$ for $\alpha$). The node $t$ has exactly two successors: $(p_{w^\frown 0}, (\tau^0_{w^\frown 0}, \ldots, \tau^\alpha_{w^\frown 0}))$ and $(p_{w^\frown 1}, (\tau^0_{w^\frown 1}, \ldots, \tau^\alpha_{w^\frown 1}))$ where $p_{w^\frown i}$ and $\tau^k_{w^\frown i}$ are such that $p_{w^\frown i} \Vdash$ "$\bar{\sigma}_k \leq \tau^k_{w^\frown i} \leq \sigma_k, \tau^k_{w^\frown i} \in T$" and $\tau^k_{w^\frown 0}$ is incompatible to $\tau^k_{w^\frown 1}$ in $T$.

For each branch $b$ through $\mathcal{T}$ let $p_b$ be stronger than all $p_{b \restriction k}$ and $\tau_b^\alpha$ such that $p_b \Vdash \tau^\alpha_{b \restriction k} \leq \tau^\alpha_b$. Note that such $\tau^\alpha_b$ exist in $T$, since the height of $T$ is $\kappa_k$, which is a regular cardinal.

Let $s' = \bar{s} \cup \{(\tau^\alpha_b, C_\alpha) \mid \alpha \in \omega_{k-2}, b \in K\}$, where $K$ is the set of elements in $2^{\omega_{k-2}}$ which have only boundedly many 1's. This is a condition, because for each $\alpha$ the set $C_\alpha$ contains all the colors which appear at or above $\sigma_\alpha$, so they don't appear at nodes below $\sigma_\alpha$ and therefore not at nodes below $\tau^\alpha_b$.

Let $V[G(\mathbb{P}^*)] \models q \leq s'$.

Let $c \in 2^{\omega_{k-2}}$ such that no node in dom$(q)$ extends a $\tau^\alpha_c$ for any $\alpha$. Such a $c$ exists, since $2^{\omega_{k-2}}$ is larger than dom$(q)$.

Now $p_c \Vdash$ "$\tau^\alpha_c \leq \sigma_\alpha$" for all $\alpha$, thus $p_c \Vdash$ "$\tau^\alpha_b \nleq \sigma_\alpha$" for all $\alpha$ and all $b \in K$.

Let $t \in$ dom$(q)$.

Case 1: There is some $\alpha$ with $t < \tau^\alpha_c$. Since $\tau^\alpha_c$ is the limit of some $\tau^\alpha_s$'s and for every $s$ exists a $b \in K$ which extends $s$, there exists some $b \in K$ with $t \leq \tau^\alpha_b$ . Therefore, since $q$ is a condition and $\tau^\alpha_b$ is in its domain, $q(t) \cap q(\tau^\alpha_b) = \emptyset$ and since $q(\tau^\alpha_b) = C_\alpha \supseteq s(\sigma_\alpha)$, it follows that $s$ is compatible with $(t, q(t))$.

Case 2: There is no $\alpha$ with $t < \tau^\alpha_c$. Since $p_c \Vdash \tau^\alpha_c \leq \sigma_\alpha$ it follows that $p_c \Vdash t \nless \sigma_\alpha$ and therefore $p_c \Vdash s$ is compatible with $(t, q(t))$.

Since $p_c$ forces for every $t \in$ dom$(q)$ that $s$ is compatible with $(t, q(t))$ it follows that $p_c \Vdash q$ is compatible with $s$.

Thus in $V[G(\mathbb{P}^*)]$ holds that for every $q \leq s'$ exists a $p' \leq p$ such that $p' \Vdash q$ is compatible with $s$. Now choose a condition $\bar{p} \in G(\mathbb{P}^*)$ below $\hat{p}$ which forces this.

Define $\pi^*(p, s) := (\bar{p}, s')$.

If $(p^*, s^*) \leq \pi^*(p, s)$ then $p^* \leq \pi(p)$ and therefore $p^*$ is compatible with $p$ and it forces that some extension of $p$ forces $s^*$ to be compatible with $s$. So $(p^*, s^*)$ is compatible with $(p, s)$ and therefore $\pi^*$ is a reduction map such that the first component of $\pi^*(p, s) \leq \pi(p)$.

To see that $\mathbb{P}^* * \mathbb{S}^*$ is a regular subforcing of $j(\mathbb{P}) * j(\mathbb{S})$ we also have to show that if two conditions $(p, s), (p', s') \in \mathbb{P}^* * \mathbb{S}^*$ are compatible in $j(\mathbb{P}) * j(\mathbb{S})$, then they are compatible in $\mathbb{P}^* * \mathbb{S}^*$. To see this, note that the set of conditions $(p, s)$ with the following property is dense in $j(\mathbb{P}) * j(\mathbb{S})$: There exists $s^*$ such that

(1) $p \Vdash s \leq s^*$
(2) $p \Vdash s^* \in \mathbb{S}^*$
(3) If $p \Vdash s \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash s^* \leq \bar{s}$.

$(p, s)$ fulfills this property, if $p$ decides $s$: Let $s^*$ be $s$ restricted to the nodes on levels below $\kappa_k$. So $p \Vdash s \leq s^* \wedge s^* \in \mathbb{S}^*$ and if $p \Vdash s \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash s^* \leq \bar{s}$, because in this case $\bar{s} \subseteq s^*$.

So this set is dense.

Suppose now that $(p^*_0, s^*_0)$ and $(p^*_1, s^*_1)$ are in $\mathbb{P}^* * \mathbb{S}^*$ and they are compatible in $j(\mathbb{P}) * j(\mathbb{S})$. Let $(p, s)$ be a witness for the compatibility in the dense set. So $(p, s^*)$ is also below $(p^*_0, s^*_0)$ and $(p^*_1, s^*_1)$. Now $(\pi(p), s^*)$ is in $\mathbb{P}^* * \mathbb{S}^*$

and stronger than $(p_0^*, s_0^*)$ and $(p_1^*, s_1^*)$: Since $p \Vdash s^* \in \mathbb{S}^* \wedge s^* \leq s_0^*, s_1^*$ and that depends only on $\mathbb{P}^*$ the same holds for $\pi(p)$.

**Proof of (2)**: Let $G(\mathbb{P}^*)$ be generic for $\mathbb{P}^*$ and $G(\mathbb{P})$ the corresponding generic for $\mathbb{P}$. Let $j\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ be a lifting of the supercompact embedding.

Let $A^*$ be a maximal antichain in $\mathbb{S}^*$. Since $\mathbb{S}^*$ is the same as $\mathbb{S}$, just with a different name, $A^*$ is also a maximal antichain in $\mathbb{S}$. By elementarity $j(A^*)$ is maximal in $j(\mathbb{S})$. Since $j$ is the identity on $\mathbb{S}$ it follows that $A^* \subseteq j(A^*)$.

Let $G(j(\mathbb{P})/G(\mathbb{P}^*))$ be a generic filter for $j(\mathbb{P})/G(\mathbb{P}^*)$ and assume that

$$M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models s \in j(\mathbb{S}).$$

**Claim.** *$M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models \exists a \in A^*$ which is compatible with $s$.*

*Proof.* Let $p \in G(j(\mathbb{P})/G(\mathbb{P}^*))$ with $p \Vdash s \in j(\mathbb{S})$. The following set is dense in $j(\mathbb{P})/G(\mathbb{P}^*)$: $\{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \perp p\} \cup \{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \Vdash \exists a \in A^*$ which is compatible with $s\}$: Let $\bar{p}$ be compatible with $p$ and $p' \leq \bar{p}, p$, thus $p' \Vdash s \in j(\mathbb{S})$ and there exists a reduction $s'$ of $s$. Since $s' \in \mathbb{S}$ there exists $a \in A^*$ and $q \leq p'$ such that $q \Vdash a$ is compatible with $s'$. Since $s'$ is a reduction of $s$, it follows that $q \Vdash a$ is compatible with $s'$.

Thus there exists a $q \in G(j(\mathbb{P})/G(\mathbb{P}^*)) \cap (\{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \perp p\} \cup \{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \Vdash \exists a \in A*$ which is compatible with $s\})$. And since $p$ is in the generic filter, there exists $q$ in the generic with $q \Vdash \exists a \in A^*$ which is compatible with $s$. □

Thus it follows that $A^*$ is a maximal antichain in $j(\mathbb{S})$.

Since $j(A^*)$ is an antichain and $A^* \subseteq j(A^*)$ it follows that $A^* = j(A^*)$. Thus $|j(A^*)| < j(\kappa_k)$ and by elementarity $|A^*| < \kappa_k$.

**Proof of (3)**: $\mathbb{P}^*$ is forcing equivalent to $\mathbb{P}$ and $\mathbb{S}^*$ in $V[\mathbb{P}^*]$ is the same forcing as $\mathbb{S}$ in $V[\mathbb{P}]$.

**Proof of (4)**: Let $G(j(\mathbb{S}))$ be generic for $j(\mathbb{S})$ over $M[G(j(\mathbb{P}))]$. Since $\mathbb{S}$ is a regular subforcing of $j(\mathbb{S})$, $G(j(\mathbb{S}))$ contains a generic filter $G(\mathbb{S})$ for $\mathbb{S}$. Thus, by the Lifting Lemma (Lemma 3.5), $j$ can be lifted to $V[G(\mathbb{P})][G(\mathbb{S})] \to M[G(j(\mathbb{P}))][G(j(\mathbb{S}))]$.

**Proof of (5)**: $j(\mathbb{P} * \mathbb{S})/(G(\mathbb{P}^* * \mathbb{S}^*)) = j(\mathbb{P})/(G(\mathbb{P}^*)) * j(\mathbb{S})/G(\mathbb{S}^*)$. So we just have to show that $j(\mathbb{S})/G(\mathbb{S}^*)$ is $<\kappa_{k-1}$-closed. Conditions in $j(\mathbb{S})/G(\mathbb{S}^*)$ are conditions in $j(\mathbb{S})$ with domain above $\kappa_k$ which are compatible with $G(\mathbb{S}^*)$. And since the union of $<\kappa_{k-1}$ many conditions in $j(\mathbb{S})$ which are compatible with $G(\mathbb{S}^*)$ is also compatible with $G(\mathbb{S}^*)$, this is $<\kappa_{k-1}$-closed. □

**Lemma 5.13.** *Let $\mathbb{P}$ be a forcing with $V[G(\mathbb{P})] \models \kappa_k = \aleph_k$. Let $\mathbb{P}^*$ be a regular subforcing of $j_k(\mathbb{P})$, forcing equivalent to $\mathbb{P}$ and $\pi^*\colon j_k(\mathbb{P}) \to \mathbb{P}^*$ a reduction map. Let $j_k(\mathbb{P})/G(\mathbb{P}^*)$ be $<\kappa_{k-1}$-closed and $j_k\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ a lifting of the supercompact embedding of $\kappa_k$. Let $\mathbb{S}$ be an iteration of limit* *length $\alpha \leq \delta$ of forcings to specialize $\kappa_k$-Aronszajn trees with $<\kappa_{k-1}$-support in $V[G(\mathbb{P})]$.*

*The following hold in $V[G(\mathbb{P})]$. For every $\beta \leq \alpha$ there is a regular subforcing $\mathbb{P}^* * \mathbb{S}^*_\beta$ of $j_k(\mathbb{P}) * j_k(\mathbb{S} \upharpoonright \beta)$ which is forcing equivalent to $\mathbb{P} * \mathbb{S} \upharpoonright \beta$ with a reduction map $\pi_\beta \colon j_k(\mathbb{P}) * j_k(\mathbb{S} \upharpoonright \beta) \to \mathbb{P}^* * \mathbb{S}^*_\beta$ such that $\pi_\beta(p \upharpoonright \beta) \geq \pi_\gamma(p \upharpoonright \gamma)$ for every $\gamma \geq \beta$ and the supercompact embedding can be lifted to this extension, $|\mathbb{S}^*_\beta| < j_k(\kappa_k)$, $j_k(\mathbb{S} \upharpoonright \beta)/\mathbb{S}^*_\beta$ is $<\kappa_{k-1}$-closed and $V[G(\mathbb{P})] \models \mathbb{S} \upharpoonright \beta$ has the $\kappa_k$-c.c..*

*Moreover for every $j_k(\mathbb{P} * \mathbb{S})$-name $\dot{T}$ for a $\kappa_{k-1}$-tree with level $i$ being $\{i\} \times \kappa_{k-2}$, there exists a $\mathbb{P}^* * \mathbb{S}^*$-name $\dot{T}'$ for the same object.*

*Proof.* Let $\mathbb{S}^*_\alpha$ be the iteration $\mathbb{S}^*_1 * \mathbb{S}^*_2 * \mathbb{S}^*_3 \dots$ with $<\kappa_k$-support. Let $p \in \mathbb{S}$ and let $\{\beta_i \mid i < \kappa_{k-1}\}$ be the indices of the support of $p$. Let $\pi_{\beta_i}$ be the reduction map of the iteration of length $\beta_i + 1$. Since these maps cohere, $\pi_{\beta_0}(p \upharpoonright (\beta_0 + 1)) \geq \pi_{\beta_1}(p \upharpoonright (\beta_1 + 1)) \geq \pi_{\beta_2}(p \upharpoonright (\beta_2 + 1)) \dots$ and since $\mathbb{S}^*_1 * \mathbb{S}^*_2 * \mathbb{S}^*_3 \dots$ is $<\kappa_k$-closed, there exists a lower bound of these reductions, let $\pi(p)$ be such a lower bound. So $\pi$ is a reduction map, which is coherent with the earlier $\pi_\beta$.

If two conditions $(p, \vec{s}), (p', \vec{s'}) \in \mathbb{P}^* * \mathbb{S}^*$ are compatible in $j(\mathbb{P}) * j(\mathbb{S})$, then they are compatible in $\mathbb{P} * \mathbb{S}^*$. To see this, note that the set of conditions $(p, \vec{s})$ with the following property is dense in $j(\mathbb{P}) * \mathbb{S})$: There exists $\vec{s^*}$ such that

(1) $p \Vdash \vec{s} \leq \vec{s^*}$
(2) $p \Vdash \vec{s^*} \in \mathbb{S}^*$
(3) If $p \Vdash \vec{s} \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash \vec{s^*} \leq \bar{s}$.

$(p, \vec{s})$ fulfills this property, if $p$ decides $\vec{s}$: Restrict each coordinate of $\vec{s}$ to nodes on levels below $\kappa_k$, let this be $\vec{s^*}$. So $p \Vdash \vec{s} \leq \vec{s^*} \wedge \vec{s^*} \in \mathbb{S}^*$ and if $p \Vdash \vec{s} \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash \vec{s^*} \leq \bar{s}$, because in this case every coordinate of $\bar{s}$ is forced to be a subset of the corresponding coordinate if $\vec{s^*}$. So this set is dense.

Suppose now that $(p^*_0, \vec{s^*_0})$ and $(p^*_1, \vec{s^*_1})$ are in $\mathbb{P}^* * \mathbb{S}^*$ and they are compatible in $j(\mathbb{P} * \mathbb{S})$. Let $(p, \vec{s})$ be a witness for the compatibility in the dense set. So $(p, \vec{s^*})$ is also below $(p^*_0, \vec{s^*_0})$ and $(p^*_1, \vec{s^*_1})$. Now $(\pi(p), \vec{s^*})$ is in $(\mathbb{P}^*) * \mathbb{S}^*$ and stronger than $(p^*_0, \vec{s^*_0})$ and $(p^*_1, \vec{s^*_1})$: Since $p \Vdash \vec{s^*} \in \mathbb{S}^* \wedge \vec{s^*} \leq \vec{s^*_0}, \vec{s^*_1}$ and that depends only on $\mathbb{P}^*$ the same holds for $\pi(p)$.

Show that $j(\mathbb{P}) \Vdash j(\mathbb{S})$ has the $j(\kappa_k)$-c.c.. This follows by the same argument as (2) of Lemma 4.10:

Let $G(\mathbb{P}^*)$ be generic for $\mathbb{P}^*$ and $G(\mathbb{P})$ the corresponding generic for $\mathbb{P}$.

Let $A^*$ be a maximal antichain in $\mathbb{S}^*$. Since $\mathbb{S}^*$ is the same as $\mathbb{S}$, just with a different name, $A^*$ is also a maximal antichain in $\mathbb{S}$. By elementarity $j(A^*)$ is maximal in $j(\mathbb{S})$. Since $j$ is the identity on $\mathbb{S}$ it follows that $A^* \subseteq j(A^*)$.

Now let $G(j(\mathbb{P})/G(\mathbb{P}*))$ be generic for $j(\mathbb{P})/G(\mathbb{P}^*)$ and assume that $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models \vec{s} \in j(\mathbb{S})$.

**Claim.** *$M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models \exists a \in A^*$ which is compatible with $\vec{s}$.*

*Proof.* Let $p \in G(j(\mathbb{P})/G(\mathbb{P}^*))$ with $p \Vdash \vec{s} \in j(\mathbb{S})$. The following set is dense in $j(\mathbb{P})/G(\mathbb{P}^*)$: $\{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \perp p\} \cup \{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \Vdash \exists a \in A^*$ which is compatible with $\vec{s}\}$: Let $\bar{p}$ be compatible with $p$ and $p' \leq \bar{p}, p$, thus $p' \Vdash$ "$\vec{s} \in j(\mathbb{S})$ and there exists a reduction $\vec{s'}$ of $\vec{s}$". Since $\vec{s'} \in \mathbb{S}$ there exists $a \in A^*$ and $q \leq p'$ such that $q \Vdash$ "$a$ is compatible with $\vec{s'}$". Since $\vec{s'}$ is a reduction of $\vec{s}$, it follows that $q \Vdash a$ is compatible with $\vec{s'}$.

Thus there exists a $q \in G(j(\mathbb{P})/G(\mathbb{P}^*)) \cap (\{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \perp p\} \cup \{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \Vdash$ "$\exists a \in A*$ which is compatible with $\vec{s}\})$". And since $p$ is in the generic filter, there exists $q$ in the generic with $q \Vdash$ "$\exists a \in A^*$ which is compatible with $\vec{s}$". ◻

Thus it follows that $A^*$ is a maximal antichain in $j(\mathbb{S})$. Since $j(A^*)$ is an antichain and $A^* \subseteq j(A^*)$ it follows that $A^* = j(A^*)$. Thus $|j(A^*)| < j(\kappa_k)$ and by elementarity $|A^*| < \kappa_k$.

That $j$ can be lifted, follows by the same proof as (4) of Lemma 4.10: Let $G(j(\mathbb{S}))$ be generic for $j(\mathbb{S})$ over $M[G(j(\mathbb{P}))]$. Since $\mathbb{S}$ is a regular subforcing of $j(\mathbb{S})$ and $j[\mathbb{S}] = \mathbb{S}$, $G(j(\mathbb{S}))$ contains a generic filter $G(\mathbb{S})$ for $\mathbb{S}$. Thus, by the Lifting Lemma (Lemma 3.5), $j$ can be lifted to $V[G(\mathbb{P})][G(\mathbb{S})] \to M[G(j(\mathbb{P}))][G(j(\mathbb{S}))]$.

$j(\mathbb{P} * \mathbb{S})/G(\mathbb{P}^* * \mathbb{S}^*) = j(\mathbb{P})/G(\mathbb{P}^*) * j(\mathbb{S}_0)/G(\mathbb{S}_0^*) * j(\mathbb{S}_1)/G(\mathbb{S}_1^*) \dots$, so by induction this is a $<\kappa_k$-support iteration of forcings which are $<\kappa_k$-closed, so it is $<\kappa_k$-closed itself.

For the moreover part: Let $j_k \colon V[\mathbb{P} * \mathbb{S}] \to M[j_k(\mathbb{P} * \mathbb{S})]$ be a lifting of the supercompact embedding and $\dot{T}$ a $j_k(\mathbb{P} * \mathbb{S})$-name for a $\kappa_{k-1}$-tree. Since $\kappa_{k-1}$ is below the critical point of $j_k$ it follows that $j_k(\kappa_{k-1}) = \kappa_{k-1}$, so $\dot{T} = j(\dot{T})$ is a $j_k(\mathbb{P} * \mathbb{S})$-name for a $j(\kappa_{k-1})$-tree. By the elementarity of $j_k$ it follows that $\dot{T}$ is a $\mathbb{P} * \mathbb{S}$-name for a $\kappa_{k-1}$-tree. Since $\mathbb{P}^* * \mathbb{S}^*$ is forcing equivalent, there exists a $\mathbb{P}^* * \mathbb{S}^*$-name $\dot{T}'$ such that for every generic filter $G(\mathbb{P} * \mathbb{S})$ for $\mathbb{P} * \mathbb{S}$ there exists a generic filter $G(\mathbb{P}^* * \mathbb{S}^*)$ such that $\dot{T}[G(\mathbb{P} * \mathbb{S})] = \dot{T}'[G(\mathbb{P}^* * \mathbb{S}^*)]$. ◻

**Lemma 5.14.** *In $V[\mathbb{L}_{<k}]$ the following holds: For every $\mathbb{L}_{\geq k} * \mathbb{S}_\delta$-name $\dot{T}$ for an $\aleph_{k-1}$-tree with level $\alpha$ being $\{\alpha\} \times \aleph_{k-2}$ there exists a regular subforcing $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$ of $\mathbb{L}_{\geq k} * \mathbb{S}_\delta$ with the following properties:*

(1) *$|\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}| < \kappa_k$,*
(2) *$\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k}$ is $<\kappa_{k-1}$-closed,*
(3) *$\bar{\mathbb{L}}$ is a regular subforcing of $\mathbb{L}_{\geq k}$ and $\mathbb{L}_{\geq k}/\bar{\mathbb{L}}$ is $<\kappa_{k-1}$-closed,*
(4) *$\mathbb{L}_{\geq k}$ forces "$\bar{\mathbb{S}}^{\geq k}$ is a regular subforcing of $\mathbb{S}_\delta^{\geq k}$ and $\mathbb{S}_\delta^{\geq k}/\bar{\mathbb{S}}^{\geq k}$ is $<\kappa_{k-1}$-closed",*
(5) *there exists an $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$-name $\dot{T}'$ such that $\mathbb{L}_{\geq k} * \mathbb{S}_\delta \Vdash \dot{T} = \dot{T}'$.*

*Proof.* $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{>k}$ fulfills the requirements for Lemma 5.13, so there exists a lifting of $j_k$ to $j_k \colon V[\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k}] \to M[j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k})]$.

$\mathbb{S}_\delta^{<k+1}$ is invariant under $j_{k+1}$, so it is easy to lift $j_{k+1}$ further to $j_{k+1} \colon V[\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k+1} * \mathbb{S}_\delta^{<k+1}] \to M[j_{k+1}(\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k+1} * \mathbb{S}_\delta^{<k+1})]$. Let $\dot{T}$ be an $\mathbb{L}_{\geq k} * \mathbb{S}_\delta$-name for an $\aleph_{k-1}$-tree with level $\alpha$ being $\{\alpha\} \times \omega_{k-2}$. Since the critical point of $j_k$ is $\kappa_k$, $j_k(\dot{T})$ is an $j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta)$-name for an $\aleph_{k-1}$-tree.

Let $\mathbb{L}^*_{\geq k}$ be the regular subforcing of $j_k(\mathbb{L}_{\geq k})$ which is equivalent to $\mathbb{L}_{\geq k}$ with $|\mathbb{L}^*_{\geq k}| < j_k(\kappa_k)$ and $\mathbb{S}^*$ the regular subforcing of $j_k(\mathbb{S}_\delta^{\geq k})$ as in Lemma 5.13. So there exists an $\mathbb{L}^* * \mathbb{S}^*$-name $\tilde{\mathbb{S}}^{<k}$ such that $j(\mathbb{L}_{\geq k} * \mathbb{S}^{\geq k}) \Vdash \tilde{\mathbb{S}}^{<k} = \mathbb{S}_\delta^{<k}$.

Moreover $\mathbb{L}^*_{\geq k} * \mathbb{S}^* \Vdash \tilde{\mathbb{S}}_\delta^{<k} < j(\kappa_k)$. So $\mathbb{L}^*_{\geq k} * \mathbb{S}^* * \tilde{\mathbb{S}}_\delta^{<k}$ is a regular subforcing of $j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta * \mathbb{S}_\delta^{<k})$ which is equivalent to $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}$ with $|\mathbb{L}^*_{\geq k} * \mathbb{S}^* * \tilde{\mathbb{S}}_\delta^{<k}| < j_k(\kappa_k)$, so there exists an $\mathbb{L}^*_{\geq k} * \mathbb{S}^* * \tilde{\mathbb{S}}_\delta^{<k}$-name $\dot{T}^*$ such that $j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}) \Vdash j_k(\dot{T}) = \dot{T}^*$.

Thus we have that there exist regular subforcings $\mathbb{L}^*$, $\mathbb{S}^*$, $\tilde{\mathbb{S}}_\delta^{<k}$ of $j_k(\mathbb{L}_{\geq k})$, $j_k(\mathbb{S}_\delta)$, $j_k(\mathbb{S}_\delta^{<k})$ such that $\mathbb{L}^* * \mathbb{S}^*$ is $<\kappa_{k-1}$-closed, $|\mathbb{L}^* * \mathbb{S}^* * \tilde{\mathbb{S}}_\delta^{<k}| < j_k(\kappa_k)$ and an $\mathbb{L}^* * \mathbb{S}^* * \tilde{\mathbb{S}}_\delta^{<k}$-name $\dot{T}^*$ for $j_k(\dot{T})$. And $j_k(\mathbb{L}_{\geq k})/\mathbb{L}^*$ is $<\kappa_{k-1}$-closed and $j_k(\mathbb{L}_{\geq k}) \Vdash$ "$j_k(\mathbb{S}_\delta^{\geq k})/\mathbb{S}^*$ is $<\kappa_{k-1}$-closed".

By elementarity of $j_k$ the same holds for $\mathbb{L}_{\geq k} * \mathbb{S}_\delta$: There exist regular subforcings $\bar{\mathbb{L}}$, $\bar{\mathbb{S}}^{\geq k}$, $\bar{\mathbb{S}}^{<k}$ of $\mathbb{L}_{\geq k}$, $\mathbb{S}_\delta^{\geq k}$, $\mathbb{S}_\delta^{<k}$ such that $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k}$ is $<\kappa_{k-1}$-closed, $|\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}| < \kappa_k$, $\mathbb{L}_{\geq k}/\bar{\mathbb{L}}$ is $<\kappa_{k-1}$-closed, $\bar{\mathbb{L}} \Vdash$ "$\mathbb{S}_\delta^{\geq k}/\bar{\mathbb{S}}^{\geq k}$ is $<\kappa_{k-1}$-closed" and an $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$-name $\dot{T}'$ such that $\mathbb{L}_{\geq k} * \mathbb{S}_\delta \Vdash \dot{T} = \dot{T}'$. □

**Corollary 5.15.** *In $V[\mathbb{L}_\omega * \mathbb{S}_\delta^{>k}]$ the forcing $\mathbb{S}_\delta^k$ has the $\kappa_k$-c.c..*

*Proof.* $\mathbb{L}_\omega * \mathbb{S}_\delta^{>k}$ fulfills the requirements of Lemma 5.13, so it forces that $\mathbb{S}_\delta^k$ has the $\kappa_k$-c.c.. □

**Corollary 5.16.** *In $V[\mathbb{L}_\omega]$ the forcing $\mathbb{S}_\delta$ preserves every $\aleph_k$.*

*Proof.* For every $k < \omega$ the forcing $\mathbb{S}_\delta = \mathbb{S}_\delta^{>k} * \mathbb{S}_\delta^{\leq k}$. This is an iteration of a forcing which is $<\kappa_k$-closed and a forcing which has the $\kappa_k$-c.c., therefore it does not collapse $\aleph_k = \kappa_k$. □

**Lemma 5.17.** *In $V[\mathbb{L}_\omega * \mathbb{S}_\delta^{>k}]$ the forcing $\mathbb{S}_\delta^k \times \mathbb{S}_\delta^k$ has the $\kappa_k$-c.c..*

*Proof.* Let $\varphi$ be a function such that $\varphi(\mathbb{P}_\alpha)$ is a $\mathbb{P}_\alpha$-name for an $\aleph_k$-Aronszajn tree (if there exists one) for every forcing $\mathbb{P}_\alpha$ and $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{S}}(\varphi(\mathbb{P}_\alpha))$ is a forcing to specialize this tree, i.e., $\varphi$ is a bookkeeping function which gives $\mathbb{S}_\delta^k$ as an iteration. Now define $\tilde{\varphi}$ and let $\tilde{\mathbb{S}}_{\aleph_k}$ be the forcing iteration, given by the bookkeeping $\tilde{\varphi}$. For $\alpha < \delta$ let $\tilde{\varphi}(\tilde{\mathbb{P}}_\alpha) = \varphi(\mathbb{P}_\alpha)$. For $\alpha = \delta + \beta$ for some $\beta < \delta$, let $\tilde{\varphi}(\tilde{\mathbb{P}}_\alpha) = \varphi(\mathbb{P}_\beta)$, i.e., we repeat the same iteration which was done up to $\delta$ again between $\delta$ and $\delta + \delta$.

By Corollary 5.15 $\tilde{\mathbb{S}}_{\delta+\delta}^k$ has the $\aleph_k$-c.c., and since no new sets of size $\aleph_{k-2}$ are added by $\mathbb{S}_\delta^k$ it holds true that $\mathbb{S}_\delta^k \times \mathbb{S}_\delta^k = \mathbb{S}_\delta^k * \mathbb{S}_\delta^k = \tilde{\mathbb{S}}_{\delta+\delta}^k$. □

5.3. **The final model.** Now we are ready to finish the proof of the main theorem.

**Theorem 5.18.** *It follows from the consistency of $\omega$ many supercompact cardinals that it is consistent that for all $0 < n < \omega$, all $\aleph_n$-Aronszajn trees are special, there are such, and there is no $\aleph_n$-Kurepa tree.*

To prove the theorem, we analyze the forcing extension by $\mathbb{L}_\omega * \mathbb{S}_\delta$. We show that $V[\mathbb{L}_\omega * \mathbb{S}_\delta] \models$ For all $0 < n \in \omega$

there exists an $\aleph_n$-Aronszajn tree,
all $\aleph_n$-Aronszajn trees are special,
and there exists no $\aleph_n$-Kurepa tree.

We have already shown that all $\aleph_n$-Aronszajn trees are special in this model right after the definition of the forcing. The following two lemmata conclude the proof.

**Lemma 5.19.** $V[\mathbb{L}_\omega * \mathbb{S}_\delta] \models$ *"There exists a special $\aleph_k$-Aronszajn tree" for every $0 < k \in \omega$.*

*Proof.* In the extension by $\mathbb{L}_\omega$ GCH holds, hence by Proposition 2.11 there exists a special $\aleph_k$-Aronszajn tree, for every $0 < k \in \omega$. Since the specializing forcing $\mathbb{S}_\delta$ does not collapse cardinals, these special $\aleph_k$-Aronszajn trees are preserved. □

**Lemma 5.20.** $V[\mathbb{L}_\omega * \mathbb{S}_\delta] \models$ *"There exists no $\aleph_{k-1}$-Kurepa tree" for every $1 < k \in \omega$.*

*Proof.* Assume that $V[\mathbb{L}_\omega * \mathbb{S}_\delta] \models$"There exists an $\aleph_{k-1}$-Kurepa tree".

Remember that $\mathbb{L}_\omega * \mathbb{S}_\delta = \mathbb{L}_{<k} * \mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}$.

Work in $V[\mathbb{L}_{<k}]$, let $\dot{T}$ be an $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}$-name for an $\aleph_{k-1}$-tree with level $\alpha$ equal to $\{\alpha\} \times \aleph_{k-2}$.

By Lemma 5.14, there exists a regular subforcing $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$ of $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}$ with the following properties:

1. $|\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}| < \kappa_k$
2. There exists an $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$-name $\dot{T}'$ which is equivalent to $\dot{T}$.
3. $\mathbb{L}_{\geq k}/\bar{\mathbb{L}} * \mathbb{S}_\delta^{\geq k}/\bar{\mathbb{S}}^{\geq k}$ is $<\kappa_{k-1}$-closed.

Since $|\mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}| < \kappa_k$, and $\kappa_k$ is a Laver indestructible supercompact cardinal, $V[\mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}] \models \kappa_k$ is inaccessible. So $V[\mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}] \models |[\dot{T}]| < \kappa_k$

Since $\mathbb{L}_{\geq k}/\bar{\mathbb{L}} * \mathbb{S}_\delta^{\geq k}/\bar{\mathbb{S}}^{\geq k}$ is $<\kappa_{k-1}$-closed, it does not add branches to $\dot{T}'$.

Let us show that $(\mathbb{S}_\delta^{<k}/\bar{\mathbb{S}}_\delta^{<k}) \times (\mathbb{S}_\delta^{<k}/\bar{\mathbb{S}}_\delta^{<k})$ has the $\kappa_{k-1}$-c.c.. By Lemma 5.17 $\mathbb{S}_\delta^{<k} \times \mathbb{S}_\delta^{<k}$ has the $\kappa_{k-1}$-c.c., hence by Lemma 3.7 $(\mathbb{S}_\delta^{<k}/\bar{\mathbb{S}}_\delta^{<k}) \times (\mathbb{S}_\delta^{<k}/\bar{\mathbb{S}}_\delta^{<k})$ has the $\kappa_{k-1}$-c.c..

Therefore by Lemma 3.9 $\mathbb{S}_\delta^{<k}/\bar{\mathbb{S}}^{<k}$ does not add branches to $\dot{T}$. So also $V[\mathbb{L}_\omega * \mathbb{S}_\delta] = V[\mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k} * \mathbb{L}_{\geq k}/\bar{\mathbb{L}} * \mathbb{S}_\delta^{\geq k}/\bar{\mathbb{S}}^{\geq k} * \mathbb{S}_\delta^{<k}/\bar{\mathbb{S}}^{<k}] \models |[\dot{T}]| < \kappa_k$. □

## 6. Trees for all successors of regular cardinals

In this section we generalize our result and show that it follows from the existence of a proper class of supercompact cardinals that it is consistent that for all successors of regular cardinals, all Aronszajn trees are special, and there exist such, while there exist no Kurepa trees on these cardinals.

**Lemma 6.1.** *Let $\alpha$ be a limit ordinal and $\langle \kappa_n \mid 1 < n < \omega \rangle$ an increasing sequence of Laver indestructible supercompact cardinals. Then there exists a forcing $\mathbb{R}^\alpha$ with the following properties:*

*(1) $\mathbb{R}^\alpha$ is $\aleph_{\alpha+1}$-directed closed,*
*(2) $|\mathbb{R}^\alpha| = (\sup_{1<n<\omega} \kappa_n)^{++} =: \delta^\alpha$,*
*(3) $\mathbb{R}^\alpha \Vdash$ "$\aleph_{\alpha+n} = \kappa_n$ for every $1 < n < \omega$",*
*(4) $\mathbb{R}^\alpha \Vdash$ "$2^{\aleph_{\alpha+n}} = \aleph_{\alpha+\omega+2} = \delta^\alpha$ for every $1 < n < \omega$",*
*(5) $\mathbb{R}^\alpha \Vdash$ "all $\aleph_{\alpha+n}$-Aronszajn trees are special and there exist some for every $1 < n < \omega$",*
*(6) $\mathbb{R}^\alpha \Vdash$ "there exists no $\aleph_{\alpha+n}$-Kurepa tree for all $0 < n < \omega$".*

*Proof.* Let us define the forcing $\mathbb{R}^\alpha$. For every $0 < n < \omega$ let $\mathbb{L}^\alpha_{n+1} := \mathrm{Col}(\aleph_{\alpha+n}, < \kappa_{n+1})$ in $V[\mathbb{L}_2 * \cdots * \mathbb{L}_n]$ and let $\mathbb{L}^\alpha_\omega := \mathbb{L}^\alpha_2 * \mathbb{L}^\alpha_3 * \mathbb{L}^\alpha_4 * \dots$ be the countable support iteration. Let $\mathbb{R}^\alpha_1 := \mathbb{L}^\alpha_\omega$. Let $\{A_n \mid n \in \omega\}$ be a partition of $\delta^\alpha$ such that every $A_n$ is cofinal in $\delta^\alpha$.

As in the case of specializing all $\aleph_n$-Aronszajn trees, continue the iteration such that all $\aleph_{\alpha+n}$-Aronszajn trees for $n > 1$ get specialized: In every step $\beta$ take the forcing to specialize the name of an $\aleph_{\alpha+n}$-Aronszajn tree given by a bookkeeping function (where the $n$ depends on the $A_n$ to which $\beta$ belongs).

Analogously to the case of specializing all $\aleph_n$-Aronszajn trees, the forcing $\mathbb{R}^\alpha$ fulfills items (1)-(6). □

Now we can combine all the forcings $\mathbb{R}^\alpha$ in an Easton support iteration to specialize all Aronszajn trees for all successors of regular cardinals:

**Theorem 6.2.** *If there is a proper class of supercompact cardinals with no inaccessible limit, then there is an extension in which for all successors of regular cardinals, all Aronszajn trees are special, there exist such, and for all regular uncountable cardinals there are no Kurepa trees.*

*Proof.* Let $\mathbb{R}$ be the Easton support iteration of the $\mathbb{R}^\alpha$. The supercompact cardinals get collapsed by $\mathbb{R}$ to the $\aleph_{\alpha+2+n}$, where $\alpha$ is 0 or a limit cardinal. The successors of a limit of supercompact cardinals and $\aleph_1$ are preserved.

The forcing $\mathbb{R}^\alpha$ fulfills Lemma 6.1 in $V[\mathbb{R}^{<\alpha}]$, therefore, as in the case of specializing all $\aleph_n$-Aronszajn trees, $\mathbb{R}^{\leq\alpha} \Vdash$ "all $\aleph_{\alpha+n}$-Aronszajn trees are special for all $n \in \omega$ and there exist such and there exist no $\aleph_{\alpha+n}$-Kurepa trees". Furthermore $\mathbb{R}^{>\alpha}$ is $<\aleph_{\alpha+\omega+1}$-closed so $\mathbb{R}^{>\alpha}$ does not add new subsets of $\aleph_{\alpha+\omega}$, therefore it does not add new $\aleph_{\alpha+n}$-trees for $n \in \omega$ and it does not add new branches to such trees which already exist, therefore there are no $\aleph_{\alpha+n}$-Kurepa trees in the extension by $\mathbb{R}$ and all $\aleph_{\alpha+n}$-Aronszajn trees are special and there exists one. □

## 7. Questions

**Question 7.1.** *Can we specialize trees of height $\aleph_n$ which have no cofinal branches but levels of size $\geq \aleph_n$? Is it possible to specialize these trees and control the existence of $\aleph_n$-Kurepa trees at the same time?*

This question cannot be solved by the same technique as the one in our construction, because we use that the levels are of size $< \aleph_n$ and therefore do not get changed under the supercompact embedding. New ideas are necessary to overcome this issue.

We can also consider models in which Kurepa trees exist and be more precise about the number of branches of Kurepa trees:

**Question 7.2.** *Can we control the exact number of branches of the $\aleph_n$-Kurepa trees in a model in which all $\aleph_n$-Aronszajn trees are special?*

In our model from Theorem 6.2 all the limit cardinals are not strong limits and there are no inaccessible cardinals. Actually $2^{\aleph_{\alpha+n}} = \aleph_{\alpha+\omega+2}$ for each $\alpha$, and the only regular cardinals are $\aleph_0$ and successor cardinals.

**Question 7.3.** *Is it possible to specialize all $\aleph_n$-Aronszajn trees for all $n \in \omega$ while keeping $\aleph_\omega$ a strong limit? Is it possible to specialize all $\kappa^+$-Aronszajn-trees for all regular cardinals $\kappa$ while keeping limit cardinals strong limit?*

**Question 7.4.** *Is it possible to specialize all $\kappa^+$-Aronszajn trees for all regular cardinals $\kappa$ such that there are inaccessibles in the resulting model?*

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