IDEAL TOPOLOGIES IN HIGHER DESCRIPTIVE SET THEORY

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Abstract. We investigate generalizations of the topology of the higher Cantor space on 2^κ, based on arbitrary ideals rather than the bounded ideal on κ. Our main focus will be on the topology induced by the nonstationary ideal, and we will call this topology the *non-stationary topology*, or also the *Edinburgh topology* on 2^κ.

It may be of independent interest that as a side result, we show κ-Silver forcing to satisfy a strong form of Axiom A not only if κ is inaccessible (which is well-known), but also under the assumption $\diamondsuit_\kappa$.

Contents

1. Introduction

Let κ be a regular and uncountable cardinal, and let bd_κ denote the bounded ideal on κ. Let ub_κ denote the collection of all unbounded subsets of κ, i.e., letting

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$\mathcal{I}^+$ denote the $\mathcal{I}$-positive sets w.r.t. any ideal $\mathcal{I}$, $\mathsf{ub}_\kappa = (\mathsf{bd}_\kappa)^+$. Let $\mathcal{I}^* = \{\kappa \setminus x \mid x \in \mathcal{I}\}$ denote the filter dual to $\mathcal{I}$. Let NS_κ denote the nonstationary ideal on κ. We want to consider different topologies on 2^κ, induced by ideals other than the bounded ideal on κ. Let $\mathcal{I}$ be a $<\kappa$-complete proper ideal on κ that extends bd_κ throughout our paper.

Definition 1.1. For any set I, let $\mathsf{Fn}_I = \{f \mid f\colon D \to 2 \text{ is a function with } D \in I\}$. For an ideal $\mathcal{I}$ as above, the *$\mathcal{I}$-topology $\tau_\mathcal{I}$* of $\mathcal{I}$-open sets is provided by the basis $\{[f] \mid f \in \mathsf{Fn}_\mathcal{I}\}$ of $\mathcal{I}$-clopen subsets of 2^κ (we also call those $\mathcal{I}$-clopen sets *$\mathcal{I}$-cones*), where, for any partial function $f\colon \kappa \to 2$, $[f] = \{g \in 2^\kappa \mid f \subseteq g\}$, and where we say that a set is *$\mathcal{I}$-clopen* /... if it is clopen /... in the $\mathcal{I}$-topology. An *ideal topology* is an $\mathcal{I}$-topology for an ideal $\mathcal{I}$ as above. If $\mathcal{I} = \mathsf{NS}_\kappa$, we also say that a set is *Edinburgh open* / ... in case it is $\mathcal{I}$-open / ..., we let $\mathsf{Fn} = \mathsf{Fn}_{\mathsf{NS}_\kappa}$, and we refer to the $\mathcal{I}$-topology on 2^κ as the *Edinburgh topology* on 2^κ.

The topology of the higher Cantor space on 2^κ is the bd_κ-topology. Clearly, if $\mathcal{I}_1 \supseteq \mathcal{I}_0$, then $\tau_{\mathcal{I}_1}$ refines $\tau_{\mathcal{I}_0}$. In particular, the Edinburgh topology on 2^κ thus refines the topology of the higher Cantor space 2^κ. Since the bd_κ-topology on 2^κ is Hausdorff, the same holds for every $\mathcal{I}$-topology. A simple observation shows that as soon as we allow our ideals to contain unbounded sets, they yield topologies with the maximal possible number of open sets:

Observation 1.2. *If $\mathcal{I}$ contains an unbounded subset of κ, then:*

(1) There are 2^κ-many disjoint basic open sets whose union is 2^κ.
(2) $|\tau_\mathcal{I}| = 2^{2^\kappa}$.

Proof. (1) Let A be an unbounded subset of κ in $\mathcal{I}$, and let $F = \{f\colon A \to 2\}$. Then, the 2^κ-many $\mathcal{I}$-cones $[f]$ for $f \in F$ are pairwise disjoint, and their union is all of 2^κ, as desired.
(2) For any $X \subseteq F$, let $\mathcal{O}_X = \bigcup_{f \in X}[f]$. Then, $X \neq Y$ implies $\mathcal{O}_X \neq \mathcal{O}_Y$. □

Note that (1) implies in particular a strong failure of (generalized) compactness for $\mathcal{I}$-topologies whenever $\mathcal{I}$ contains an unbounded subset of κ.

One of the most basic topological results holds outright for arbitrary ideal topologies, by the usual argument, which we would nevertheless like to present for the convenience of our readers.

Proposition 1.3. *(Baire category theorem for ideal topologies) The intersection of κ-many $\mathcal{I}$-open dense sets is $\mathcal{I}$-dense.* [1]

Note that this statement is equivalent to the fact that for every $f \in \mathsf{Fn}_\mathcal{I}$, the $\mathcal{I}$-cone $[f]$ is not $\mathcal{I}$-meager.

Proof of Proposition 1.3. Let $(D_\alpha)_{\alpha<\kappa}$ be a sequence of $\mathcal{I}$-open dense sets. For every $\alpha < \kappa$, there exists a set I_α and a sequence $\langle f_i^\alpha \mid i \in I_\alpha\rangle$, with each $f_i^\alpha \in \mathsf{Fn}_\mathcal{I}$, such that $D_\alpha = \bigcup_{i \in I_\alpha}[f_i^\alpha]$. Let $f_0 \in \mathsf{Fn}_\mathcal{I}$. We construct a $\subseteq$-increasing sequence of functions $\langle f_\alpha \mid \alpha < \kappa\rangle$ in $\mathsf{Fn}_\mathcal{I}$ such that $f_\kappa := \bigcup_{\alpha<\kappa} f_\alpha$ satisfies $[f_\kappa] \subseteq \bigcap_{\alpha<\kappa} D_\alpha$. Given an ordinal $\alpha < \kappa$ and $f_\alpha \in \mathsf{Fn}_\mathcal{I}$, since D_α is dense, $D_\alpha \cap [f_\alpha] \neq \emptyset$. Therefore, there exists $i \in I_\alpha$ such that $[f_i^\alpha] \cap [f_\alpha] \neq \emptyset$, yielding that $f_{\alpha+1} := f_i^\alpha \cup f_\alpha$ is a function. Since $\mathcal{I}$ is an ideal, $f_{\alpha+1} \in \mathsf{Fn}_\mathcal{I}$. Assume now that $\alpha < \kappa$ is a limit ordinal, and that $\langle f_\beta \mid \beta < \alpha\rangle$ has been constructed. Then, $\bigcup_{\beta<\alpha} f_\beta$ is a function, and, using that $\mathcal{I}$ is $<\kappa$-complete, $\bigcup_{\beta<\alpha} f_\beta \in \mathsf{Fn}_\mathcal{I}$. In the end, f_κ is clearly as desired. □

[1] We write $\mathcal{I}$-open dense to mean $\mathcal{I}$-open and $\mathcal{I}$-dense. Furthermore, using our above convention, a set being $\mathcal{I}$-dense means that it intersects every $\mathcal{I}$-open set.

The above proof essentially also shows the following, slightly stronger result.

Corollary 1.4. *(Mycielski's theorem for ideal topologies) The intersection of κ-many $\mathcal{I}$-open dense sets contains a perfect set (in the sense of the bounded topology on 2^κ), that is, a closed set that is homeomorphic to the higher Cantor space 2^κ.*

Proof. As for Proposition 1.3, but extending f_0 to two incompatible functions, and then extending those to some f_1 and f_1' in the same way that we extended f_0 to f_1 in the proof of Proposition 1.3. Continuing now with these extensions, and carrying on like this throughout all κ-many stages of our construction, we eventually obtain our desired perfect set in the intersection of our κ-many $\mathcal{I}$-open dense sets. □

We will later show that in many situations, the nonstationary topology satisfies an even stronger form of Mycielski's theorem (see Theorem 6.1).

For the nonstationary topology, we also have the property that cones are isomorphic to the whole space (as is the case for the standard bounded topology).

Lemma 1.5. *If $\mathcal{I} = \mathsf{NS}_\kappa$, then the space 2^κ with the $\mathcal{I}$-topology is homeomorphic to any $\mathcal{I}$-cone with the induced topology.*

More precisely, if $f \in \mathsf{Fn}_\mathcal{I}$ then there exists $\rho : [f] \to 2^\kappa$ which is a homeomorphism both with respect to the bounded topology and with respect to the Edinburgh topology (taking the respective induced topologies on $[f]$).

Proof. Let $f \in \mathsf{Fn}_\mathcal{I}$ be given, so that $A := \operatorname{dom}(f)$ is nonstationary, and let us consider the $\mathcal{I}$-cone $[f]$. We want to construct a homeomorphism between $[f]$ and 2^κ based on a bijection π between $B := \kappa \setminus A$ and κ. Let $C \subseteq B$ be a club subset of κ, such that, in order to simplify the argument to follow, $B \setminus C$ is an unbounded subset of κ. Let $\langle c_\alpha \mid \alpha < \kappa\rangle$ be the increasing enumeration of C, and let $\langle b_\alpha \mid \alpha < \kappa\rangle$ be the increasing enumeration of $B \setminus C$. We define $\pi\colon B \to \kappa$ by setting $\pi(c_\alpha) = 2 \cdot \alpha$, and letting $\pi(b_\alpha) = 2 \cdot \alpha + 1$.

The point of our construction is now that π is a bijection between B and κ such that if $D \subseteq B$, then D is stationary if and only if $\pi[D]$ is stationary: This follows because if $E \subseteq B$ contains a club subset of κ, then also $E \cap C$ contains a club subset of κ, and since π is continuous on C, it follows that $\pi[E]$ contains a club subset of κ. On the other hand, if $E \subseteq \kappa$ contains a club subset of κ, then its restriction to the even ordinals contains a club subset of κ, and its pointwise preimage under π contains a club subset of $C \subseteq B$.

Next, we use π to induce a bijection $\rho\colon [f] \to 2^\kappa$ in a natural way: For each $x \in [f]$, we simply define $\rho(x)$ by letting $\rho(x)(\alpha) := x(\pi^{-1}(\alpha))$ for each $\alpha \in \kappa$. It is easy to see that ρ is a homeomorphism between 2^κ with the bounded topology and $[f]$ (with the induced topology). We will finish our argument by showing that ρ is also a homeomorphism between 2^κ with the nonstationary topology and $[f]$ (with the induced topology).

It suffices to show that both ρ and its inverse map preserve basic open sets. A basic open subset of $[f]$ in its induced topology is of the form $[g]$ for $g \supseteq f$ in $\mathsf{Fn}_\mathcal{I}$. Then, $\rho[[g]] = [h]$, where $\operatorname{dom}(h) = \pi[\operatorname{dom}(g) \setminus A]$ and $h(\pi(\alpha)) = g(\alpha)$ for every $\alpha \in \operatorname{dom}(g) \setminus A$. Since $\operatorname{dom}(h)$ is nonstationary by our above arguments, this shows that $[h]$ is a basic open set in the nonstationary topology.

For the other direction, assume that $[h]$ is a basic open subset of 2^κ in the nonstationary topology. Then, $\rho^{-1}[[h]] = [g]$ where $g \supseteq f$ is such that $\operatorname{dom}(g) = A \cup \pi^{-1}[\operatorname{dom}(h)]$ and for $\alpha \in \operatorname{dom}(g) \setminus A$, $g(\alpha) = h(\pi(\alpha))$. Again by our above arguments, $\operatorname{dom}(g)$ is nonstationary, hence $[g]$ is a basic open set in the induced topology on $[f]$, as desired. □

There are counterexamples to the above homogeneity property for other $\mathcal{I}$-topologies:

Example 1.6. Assume $2^{<\kappa} = \kappa$. Let X be an unbounded subset of κ which also has an unbounded complement, and let $\mathcal{I}$ be the ideal generated by bd_κ together with the set X – that is, $Y \in \mathcal{I}$ if and only if $Y \setminus X$ is a bounded subset of κ. Then, for any $f : X \to 2$, 2^κ with the $\mathcal{I}$-topology is not homeomorphic to $[f]$ with its induced topology.

Proof. Homeomorphic topological spaces need to have the same number of open sets, however by Observation 1.2, $|\tau_\mathcal{I}| = 2^{2^\kappa}$, while there are only 2^κ-many open sets in the induced topology on $[f]$, for it is clearly homeomorphic to the bounded topology on 2^κ (which has a basis of size $2^{<\kappa} = \kappa$). $\square$

Let us show that every $\mathcal{I}$-topology is *homogeneous*, in the sense that for any two elements $x, y \in 2^\kappa$, there is a homeomorphism of 2^κ with respect to $\tau_\mathcal{I}$ that maps x to y. Let $\mathbf{0}$ denote the function with domain κ and constant value 0, and let $\mathbf{1}$ denote the function with domain κ and constant value 1.

Claim 1.7. *The $\mathcal{I}$-topology is homogeneous for any ideal $\mathcal{I}$.*

Proof. It suffices to provide, for each $x \in 2^\kappa$, a homeomorphism $H : 2^\kappa \to 2^\kappa$ with $H(\mathbf{0}) = x$. For any subset x of κ, we define the function $H_x : 2^\kappa \to 2^\kappa$ by $H_x(y)(i) = 1 - y(i)$ for all $i \in x$ and $H_x(y)(i) = y(i)$ otherwise. H_x is a homeomorphism, since $H_x[[f]] = [g]$ for any $f, g \in \mathsf{Fn}_\mathcal{I}$, where $g(i) = 1 - y(i)$ for $i \in \mathrm{dom}(f) \cap x$ and $g(i) = y(i)$ for $i \in \mathrm{dom}(f) \setminus x$. Clearly, $H_x(\mathbf{0}) = x$. $\square$

The following trivial observation will be useful later on:

Observation 1.8. *Let $s \in 2^{<\kappa}$. Then 2^κ is homeomorphic to $[s]$ with respect to the $\mathcal{I}$-topology.*

In particular, there are κ-many disjoint $\mathcal{I}$-cones that are homeomorphic to 2^κ with the $\mathcal{I}$-topology.

Proof. We show that 2^κ is homeomorphic to $[s]$ with respect to the $\mathcal{I}$-topology, using the bijection $\pi : 2^\kappa \to [s]$ which maps x to $s^\frown x$ (where we are thinking of s and of x as sequences of 0's and 1's, and $s^\frown x \in 2^\kappa$ denotes their concatenation). To see that π is an $\mathcal{I}$-homeomorphism, it is enough to show that

$$A \in \mathcal{I} \text{ if and only if } \{|s| + \beta \mid \beta \in A\} \in \mathcal{I}. \tag{1}$$

Observe that there exists $\gamma < \kappa$ such that $|s| + \beta = \beta$ for every $\beta \geq \gamma$, hence $\gamma \cup A = \gamma \cup \{|s| + \beta \mid \beta \in A\}$. Since $\mathsf{bd}_\kappa \subseteq \mathcal{I}$, it follows that (1) holds.

The second statement of the observation easily follows by picking κ-many incompatible functions in $2^{<\kappa}$. $\square$

2. On the Borel hierarchy in ideal topologies

2.1. **A normal form for closed sets.** In this short section, we provide a normal form for closed sets in ideal topologies, that generalizes the usual normal form w.r.t. the bounded topology, and which will be very useful later on.

For $x \in 2^\kappa$ and $J \subseteq \mathcal{P}(\kappa)$, let

$$x \restriction\restriction J = \{x \restriction A \mid A \in J\},$$

and, for $P \subseteq \mathsf{Fn}_{\mathcal{P}(\kappa)}$, let

$$[P]_J := \{x \in 2^\kappa \mid x \restriction\restriction J \subseteq P\}.$$

Note that if $J = \kappa \subseteq \mathcal{P}(\kappa)$ and $T \subseteq \mathsf{Fn}_J$ is a tree, then $[T]_J = [T]$ is exactly the body of T, i.e. the set of branches (of length κ) through T. Our next result shows that the usual normal form for closed sets as sets of branches through trees generalizes to the context of arbitrary ideal topologies.

Proposition 2.1. *If $P \subseteq \mathsf{Fn}_{\mathcal{P}(\kappa)}$, then*

$$[P]_{\mathcal{I}} := \{x \in 2^\kappa \mid x \upharpoonright\upharpoonright \mathcal{I} \subseteq P\}$$

is an $\mathcal{I}$-closed subset of 2^κ.

Conversely, if $X \subseteq 2^\kappa$ is $\mathcal{I}$-closed, then there is $P \subseteq \mathsf{Fn}_{\mathcal{I}}$ such that $X = [P]_{\mathcal{I}}$. Moreover, we may assume that P is closed under restrictions, and that P is pruned *– that is, for every $p \in P$, there is $x \in 2^\kappa$ with $p \in x \upharpoonright\upharpoonright \mathcal{I} \subseteq P$.*

Proof. Let $P \subseteq \mathsf{Fn}_{\mathcal{P}(\kappa)}$, and let $X = [P]_{\mathcal{I}}$. If $x \in 2^\kappa$ is not an element of X, then there is $A \in \mathcal{I}$ with $x \upharpoonright A \notin P$. But then $[x \upharpoonright A]_{\mathcal{I}}$ is disjoint from X, hence the complement of X is open, and thus X is closed.

Conversely, assume now that $X \subseteq 2^\kappa$ is $\mathcal{I}$-closed, and let $P = \{x \upharpoonright A \mid x \in X \wedge A \in \mathcal{I}\}$. Now if $x \in X$, then clearly $x \upharpoonright\upharpoonright \mathcal{I} \subseteq P$ by definition of P. If $x \notin X$, then since X is $\mathcal{I}$-closed, there is $A \in \mathcal{I}$ with $X \cap [x \upharpoonright A]_{\mathcal{I}} = \emptyset$. But then, $x \upharpoonright A \notin P$, hence also $x \upharpoonright\upharpoonright \mathcal{I} \not\subseteq P$. Moreover, observe that P is clearly closed under restrictions and pruned. □

2.2. **Tallness, and related properties of ideals.** Tallness is a well-known notion in case $\kappa = \omega$. Generalizations of this concept will turn out to be very useful and natural in the context of ideal topologies.

Definition 2.2. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on κ. We say that $\mathcal{I}$ is *$\mathcal{J}$-tall* if for every $X \in \mathcal{J}^+$, there is $Y \subseteq X$ in $\mathcal{J}^+ \cap \mathcal{I}$. We say that $\mathcal{I}$ is *tall* if it is bd_κ-tall. We say that $\mathcal{I}$ is *stationary tall* if it is NS_κ-tall.

The following is very easy to prove (compare with Proposition 3.9):

Observation 2.3. *NS_κ is tall, and hence every $\mathcal{I} \supseteq \mathsf{NS}_\kappa$ is tall.*

Note that every normal ideal [2] contains the nonstationary ideal, and is thus tall by the above. Clearly, any $\mathcal{J}$-tall ideal contains a set in $\mathcal{J}^+$, but this is not sufficient for being $\mathcal{J}$-tall. However, there is an easy property which implies $\mathcal{J}$-tallness:

Proposition 2.4. *If $\mathcal{I}$ contains a set in $\mathcal{J}^*$ then $\mathcal{I}$ is $\mathcal{J}$-tall.*

Proof. If $C \in \mathcal{I} \cap \mathcal{J}^*$ and $X \in \mathcal{J}^+$, then $C \cap X \subseteq X$ is in $\mathcal{J}^+ \cap \mathcal{I}$. □

For $x \subseteq \kappa$, let $\chi_x \colon \kappa \to 2$ denote the characteristic function of x. We will always identify x and χ_x. We may thus view a collection $\mathcal{K}$ of subsets of κ as a subset of 2^κ via the identification

$$\mathcal{K} = \{\chi_x \mid x \in \mathcal{K}\}.$$

Observation 2.5. *Let $H_{\mathbf{1}} \colon 2^\kappa \to 2^\kappa$ be the homeomorphism defined as in the proof of Claim 1.7. Since for any ideal $\mathcal{J}$ on κ, $H_{\mathbf{1}}[\mathcal{J}] = \mathcal{J}^*$, it follows that $\mathcal{J}$ and $\mathcal{J}^*$ have the same topological properties in the $\mathcal{I}$-topology, in particular $\mathcal{J}$ is $\mathcal{I}$-open iff $\mathcal{J}^*$ is $\mathcal{I}$-open, and $\mathcal{J}$ is $\mathcal{I}$-closed iff $\mathcal{J}^*$ is $\mathcal{I}$-closed.*

Given arbitrary sets A and i, let c_i^A denote the function with domain A and constant value i. Our next proposition provides a characterization of $\mathcal{J}$-tallness of an ideal $\mathcal{I}$ in terms of whether certain sets lie low down in the $\mathcal{I}$-Borel hierarchy.

Proposition 2.6. *The following are equivalent:*

(1) $\mathcal{I}$ is $\mathcal{J}$-tall.
(2) $\mathcal{J}^+$ is $\mathcal{I}$-open.
(3) $\mathcal{J}$ is $\mathcal{I}$-closed.
(4) $\mathcal{J}^*$ is $\mathcal{I}$-closed.

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[2]Remember that an ideal is *normal* if it is closed under the taking of diagonal unions.

Proof. (1) implies (2): Assume that $\mathcal{I}$ is a $\mathcal{J}$-tall ideal. Then,

$$\mathcal{J}^+ = \bigcup\{[c_1^A] \mid A \in \mathcal{I} \cap \mathcal{J}^+\}$$

is clearly $\mathcal{I}$-open.

(2) implies (1): Assume that $\mathcal{J}^+$ is $\mathcal{I}$-open, and let $A \in \mathcal{J}^+$. Then, $\mathcal{J}^+$ contains an $\mathcal{I}$-cone $[f]$ with $A \in [f]$. Since $[f]$ must only contain elements of $\mathcal{J}^+$, f has to take value 1 on some $B \in \mathcal{J}^+$ with $B \subseteq \operatorname{dom}(f) \in \mathcal{I}$. This shows that every element of $\mathcal{J}^+$ contains an element of $\mathcal{J}^+ \cap \mathcal{I}$, which means that $\mathcal{I}$ is $\mathcal{J}$-tall, as desired.

(2) and (3) are equivalent, because $\mathcal{J}^+$ is the complement of $\mathcal{J}$. The *equivalence of (3) and (4)* follows from Observation 2.5. □

The next proposition provides a similar characterization of a property stronger than $\mathcal{J}$-tallness (see Proposition 2.4).

Proposition 2.7. *The following are equivalent:*

1. *$\mathcal{I}$ contains a set in $\mathcal{J}^*$.*
2. *$\mathcal{J}^+$ is $\mathcal{I}$-closed.*
3. *$\mathcal{J}$ is $\mathcal{I}$-open.*
4. *$\mathcal{J}^*$ is $\mathcal{I}$-open.*

Proof. (1) implies (4): Assume that there exists C in $\mathcal{J}^* \cap \mathcal{I}$. Then, $\mathcal{J}^* = \bigcup_{x \in \mathcal{J}^*}[c_1^x \cap C]$.

(4) implies (1): Now assume that $\mathcal{J}^*$ is $\mathcal{I}$-open, and hence contains an $\mathcal{I}$-cone $[f]$. If the domain of f were not in $\mathcal{J}^*$, then, since $f^{-1}[\{1\}] \in [f]$, we obtain a contradiction. But this implies that $\operatorname{dom}(f) \in \mathcal{J}^* \cap \mathcal{I} \neq \emptyset$, as desired.

Again, *(2) and (3) are equivalent*, because $\mathcal{J}^+$ is the complement of $\mathcal{J}$, and the *equivalence of (3) and (4)* follows from Observation 2.5. □

Note that the two propositions above show that for an ideal $\mathcal{J}$ the following implication holds: [3]

$$\mathcal{J} \text{ is } \mathcal{I}\text{-open} \Rightarrow \mathcal{J} \text{ is } \mathcal{I}\text{-closed}.$$

Of course this can also be shown directly (the proof makes essential use of the property of an ideal to be closed under unions).

Let us shed light on a few relationships between (stationary) tallness and other properties of ideals that we are making use of in this paper:

Observation 2.8.

1. *If $\mathcal{I}$ contains a club subset of κ, then $\mathcal{I}$ is stationary tall.*
2. *If $\mathcal{I}$ is a maximal ideal, then $\mathcal{I}$ is both tall and stationary tall.*
3. *Ideals which contain a stationary subset of κ are not necessarily tall or stationary tall.*
4. *Stationary tall ideals are not necessarily tall.*

Proof.

1. Immediately follows from Proposition 2.4, letting $\mathcal{J} = \mathsf{NS}_\kappa$.
2. Let $\mathcal{I}$ be a maximal ideal, and let A be an unbounded subset of κ that is not in $\mathcal{I}$. Partition A into two disjoint unbounded subsets A_0 and A_1 of κ. By the maximality of $\mathcal{I}$, either A_0 or A_1 is an element of $\mathcal{I}$, yielding $\mathcal{I}$ to be tall.

   More generally, the same proof yields that $\mathcal{I}$ is $\mathcal{J}$-tall, provided that every set in $\mathcal{J}^+$ can be partitioned into two disjoint sets in $\mathcal{J}^+$. Since

[3] By Observation 2.5, an analogous remark applies to filters rather than ideals.

every stationary set can be partitioned into two disjoint stationary sets, $\mathcal{I}$ is also stationary tall.

(3) Let $\mathcal{I}$ be the ideal generated by the bounded ideal and a single stationary and co-stationary subset S of κ. Then, $\mathcal{I}$ is neither tall nor stationary tall, for the complement of S contains no unbounded subset of κ in $\mathcal{I}$.

(4) Let $\mathcal{I}$ be the ideal generated by the bounded ideal and a single club subset C of κ, the complement of which is unbounded in κ. Then, $\mathcal{I}$ is stationary tall by (1). However, the complement of C has no unbounded subset in $\mathcal{I}$, showing that $\mathcal{I}$ is not tall. □

2.3. **On the collection of unbounded sets.** Using that $\mathcal{I}$ is $<\kappa$-complete, the $\mathcal{I}$-open sets are closed under $<\kappa$-intersections, and the $\mathcal{I}$-closed sets are closed under $<\kappa$-unions. We may define an $\mathcal{I}$-Borel hierarchy as usual in higher descriptive set theory, through using κ-intersections, κ-unions and complements. For example, on the second level of this hierarchy, we have the $\mathcal{I}$-F_κ-sets, which are the κ-unions of $\mathcal{I}$-closed sets, and the $\mathcal{I}$-G_κ-sets, which are the κ-intersections of $\mathcal{I}$-open sets, etc.[4]

Let
$$\mathsf{ub}_\kappa = \{\chi_x \mid x \text{ is an unbounded subset of } \kappa\}$$
be the collection of unbounded subsets of κ. The following is an immediate consequence of Proposition 2.6, letting $\mathcal{J} = \mathsf{bd}_\kappa$:

Corollary 2.9. *$\mathcal{I}$ is tall if and only if ub_κ is $\mathcal{I}$-open.*

However, for any choice of ideal $\mathcal{I}$, we will show that ub_κ can never be an $\mathcal{I}$-F_κ set. This in particular yields an example that whenever $\mathcal{I}$ is tall, there is an $\mathcal{I}$-open set that is not $\mathcal{I}$-F_κ. For $X \subseteq \kappa$, let $\mathsf{ub}(X)$ be the collection of subsets of κ which have unbounded intersection with X.

Proposition 2.10. *ub_κ is not $\mathcal{I}$-F_κ. In fact, for every unbounded subset X of κ, the collection $\mathsf{ub}(X)$ is not $\mathcal{I}$-F_κ.*

Proof. Let X be an unbounded subset of κ, and assume for a contradiction that $\mathsf{ub}(X)$ is $\mathcal{I}$-F_κ, i.e., that $\mathsf{ub}(X) = \bigcup_{\alpha<\kappa}[P_\alpha]_\mathcal{I}$, with each $P_\alpha \subseteq \mathsf{Fn}_\mathcal{I}$ closed under restrictions. We want to inductively construct a set in $\mathsf{ub}(X)$ which is not in the above union, and thus reach a contradiction. The key ingredient will be the following claim. If $A \subseteq \kappa$, we say that $f\colon A \to 2$ is *bounded* (in κ) if $\{\alpha < \kappa \mid f(\alpha) = 1\}$ is bounded in κ.[5]

Claim. *If $f \in \mathsf{Fn}_\mathcal{I}$ is bounded, $\alpha < \kappa$, and $[P]_\mathcal{I}$ is $\mathcal{I}$-closed with $P \subseteq \mathsf{Fn}_\mathcal{I}$ closed under restrictions, and such that $[P]_\mathcal{I}$ contains only unbounded subsets of κ, then there is an extension $g \supseteq f$ of f in $\mathsf{Fn}_\mathcal{I}$ which is bounded, such that $g \notin P$, and such that $g(\gamma) = 1$ for some $\gamma \geq \alpha$ in X.*

Proof. Let $f^* \in \mathsf{Fn}_\mathcal{I}$ be bounded and extending f such that $f^*(\gamma) = 1$ for some $\gamma \geq \alpha$ in X. For $A \subseteq \kappa \setminus \operatorname{dom} f^*$ in $\mathcal{I}$, let $f^*_A \in \mathsf{Fn}_\mathcal{I}$ denote the extension of f^* with $\operatorname{dom} f^*_A = \operatorname{dom} f^* \cup A \in \mathcal{I}$ and with $f^*_A(\alpha) = 0$ for every $\alpha \in A$. Now, assume for a contradiction that every such f^*_A were an element of P. But then, letting $x \in 2^\kappa$ be the extension of f^* with $x(\alpha) = 0$ for every $\alpha \in \kappa \setminus \operatorname{dom} f^*$, it follows, using that P is closed under restrictions, that $x \upharpoonright\upharpoonright \mathcal{I} \subseteq P$, and hence that $x \in [P]_\mathcal{I}$. But x is a bounded subset of κ, contradicting our assumption on P. Hence, we may pick $g = f^*_A$, for some A as above, for which $f^*_A \notin P$. □

[4]It is easy to see that if $\kappa^\lambda = \kappa$, then the $\mathcal{I}$-G_κ-sets are closed under λ-unions, and correspondingly, the $\mathcal{I}$-F_κ-sets are closed under λ-intersections. In particular, if $\kappa^{<\kappa} = \kappa$, then these classes are closed under $<\kappa$-unions and $<\kappa$-intersections respectively.

[5]Clearly, if $A = \kappa$, then $f\colon A \to 2$ is bounded in κ if and only if f is bounded in κ in the usual sense when identified with a subset of κ.

Let $f_0 = \emptyset$. Given f_α, let $f_{\alpha+1}$ be a bounded extension of f_α in $\mathsf{Fn}_\mathcal{I}$ with $f_{\alpha+1}(\gamma) = 1$ for some $\gamma \geq \alpha$ in X and with $f_{\alpha+1} \notin P_\alpha$, by an application of the claim. At limit stages $\alpha < \kappa$, let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta \in \mathsf{Fn}_\mathcal{I}$. Then, $f_\kappa = \bigcup_{\alpha<\kappa} f_\alpha \in \mathsf{ub}(X)$, however $f_\kappa \notin \bigcup_{\alpha<\kappa}[P_\alpha]_\mathcal{I}$, which yields our desired contradiction. $\square$

The above allows us to show that any $\mathcal{I}$-topology other than the bounded topology has an $\mathcal{I}$-open set that is not $\mathcal{I}$-F_κ.

Corollary 2.11. *If $X \in \mathcal{I}$ is an unbounded subset of κ, then $\mathsf{ub}(X)$ is $\mathcal{I}$-open, however not $\mathcal{I}$-F_κ.*

Proof.
$$\mathsf{ub}(X) = \bigcup\{[c_1^A] \mid A \text{ is an unbounded subset of } X\}.$$
By Proposition 2.10, $\mathsf{ub}(X)$ is not $\mathcal{I}$-F_κ, as desired. $\square$

2.4. **The lowest levels of the ideal Borel hierarchies.** Corollary 2.11 clearly motivates the question on how the lowest levels of the $\mathcal{I}$-Borel hierarchy are related when $\mathcal{I}$ contains an unbounded set X, given the unusual non-implication from being $\mathcal{I}$-open to being $\mathcal{I}$-F_κ. We will consider some of the lowest level natural classes of $\mathcal{I}$-Borel sets: $\mathcal{I}$-open sets, $\mathcal{I}$-closed sets, $\mathcal{I}$-G_κ-sets, and $\mathcal{I}$-F_κ-sets, as well as the class
$$\mathcal{I}\text{-}\cap = \{A \cap B \mid A \text{ is } \mathcal{I}\text{-open and } B \text{ is } \mathcal{I}\text{-closed}\}$$
of intersections of $\mathcal{I}$-open and $\mathcal{I}$-closed sets, and the class
$$\mathcal{I}\text{-}\cup = \{A \cup B \mid A \text{ is } \mathcal{I}\text{-open and } B \text{ is } \mathcal{I}\text{-closed}\}$$
of unions of $\mathcal{I}$-open and $\mathcal{I}$-closed sets, which correspond to the classes on the second level of the difference hierarchy over the $\mathcal{I}$-open and $\mathcal{I}$-closed sets.

We will show that, assuming that $\mathcal{I}$ is tall, only the following six trivial implications between the above hold, by the very definition of the classes involved:

- $\mathcal{I}$-open sets are $\mathcal{I}$-G_κ, $\mathcal{I}$-closed sets are $\mathcal{I}$-F_κ.
- $\mathcal{I}$-open sets are both $\mathcal{I}$-$\cap$ and $\mathcal{I}$-$\cup$.
- $\mathcal{I}$-closed sets are both $\mathcal{I}$-$\cap$ and $\mathcal{I}$-$\cup$.

Let $\mathsf{I}_\kappa = \{\chi_{\{\alpha\}} \mid \alpha < \kappa\}$. We start by determining exactly to which of the above basic classes some of our major examples belong to – these are marked with a + in the following table. The entries of this table are immediate for 2^κ, and they follow for $\mathsf{ub}(X)$ by Corollary 2.11. We will provide the easy verifications of the other entries in Lemma 2.12 below.

	$\mathcal{I}$-open	$\mathcal{I}$-closed	$\mathcal{I}$-$\cap$	$\mathcal{I}$-$\cup$	$\mathcal{I}$-G_κ	$\mathcal{I}$-F_κ
2^κ	+	+	+	+	+	+
$\{\mathbf{0}\}$		+	+	+	+	+
I_κ			+		+	+
$\mathsf{ub}(X)$	+		+	+	+	

TABLE 1. Borel properties of some basic sets

Lemma 2.12. *(1) $\{\mathbf{0}\}$ is not $\mathcal{I}$-open, however $\mathcal{I}$-closed and $\mathcal{I}$-G_κ.*
(2) I_κ is neither $\mathcal{I}$-open nor $\mathcal{I}$-closed, however $\mathcal{I}$-$\cap$. Moreover, it is not $\mathcal{I}$-$\cup$, however it is both $\mathcal{I}$-G_κ and $\mathcal{I}$-F_κ.

Proof. The properties of $\{\mathbf{0}\}$ and of I_κ marked with a $+$ in the above table correspond exactly to the respective properties that those sets have in the bounded topology. Whenever a set has such a property in the bounded topology, it inherits to the $\mathcal{I}$-topology: for example, $\{\mathbf{0}\}$ is closed in the bounded topology, and hence it is $\mathcal{I}$-closed.

For (1), it remains to show that $\{\mathbf{0}\}$ is not $\mathcal{I}$-open, which follows as every nonempty $\mathcal{I}$-open set has size 2^κ. The same argument shows that I_κ is not $\mathcal{I}$-open. To finish the proof of (2), it remains to show that I_κ is neither $\mathcal{I}$-closed nor $\mathcal{I}$-$\cup$. If I_κ were $\mathcal{I}$-closed, its complement would be $\mathcal{I}$-open, hence would contain an $\mathcal{I}$-cone around $\mathbf{0}$; but each such $\mathcal{I}$-cone contains an element $\chi_{\{\alpha\}}$ of I_κ, a contradiction. Moreover, if I_κ was in $\mathcal{I}$-$\cup$, i.e., the union of an $\mathcal{I}$-open and an $\mathcal{I}$-closed set, the above cardinality argument yields that the $\mathcal{I}$-open part is in fact empty, which is impossible by the above. □

Note that sets with further patterns with respect to the properties in Table 1 can simply be generated by the taking of complements: if $A \subseteq 2^\kappa$ and $B = 2^\kappa \setminus A$ is the complement of A, then B is $\mathcal{I}$-closed if and only if A is $\mathcal{I}$-open, and vice versa, and correspondingly for the properties $\mathcal{I}$-$\cap$ and $\mathcal{I}$-$\cup$, as well as the properties $\mathcal{I}$-G_κ and $\mathcal{I}$-F_κ. In particular, this allows us to obtain all the five possible patterns for sets which are either $\mathcal{I}$-open or $\mathcal{I}$-closed, by also considering the complements of $\{\mathbf{0}\}$ and of ub_κ.

We will show that for sets which are neither $\mathcal{I}$-open nor $\mathcal{I}$-closed, all 16 combinations of the remaining four properties occur (see Table 2), very much unlike for the case of the bounded topology. The combination of the above then shows that there are no implications between the six properties in Table 1 other than the trivial ones listed above. All of our examples below will be based on the basic sets 2^κ, $\{\mathbf{0}\}$, I_κ and $\mathsf{ub}(X)$. We will use the method of taking *unions on disjoint cones*:

Definition 2.13. Given $X, Y \subseteq 2^\kappa$ we say that $Z = X \dot\cup Y$ is *a union of X and Y on disjoint $\mathcal{I}$-cones* in case $[f]$ and $[g]$ are two disjoint $\mathcal{I}$-cones, which are both, using the respective induced topologies, homeomorphic to 2^κ with the $\mathcal{I}$-topology, via bijections $\pi\colon 2^\kappa \to [f]$ and $\rho\colon 2^\kappa \to [g]$, and such that $Z = \pi[X] \cup \rho[Y]$. Unions $X \dot\cup Y \dot\cup Z$ of three (or more) sets are defined analogously. [6]

Lemma 2.14. *Assume that $Z = X \dot\cup Y$ is a union of X and Y on disjoint $\mathcal{I}$-cones. Then, for each in the following list of classes, Z is a member if and only if both X and Y are members – these classes are: $\mathcal{I}$-open, $\mathcal{I}$-closed, $\mathcal{I}$-$\cap$, $\mathcal{I}$-$\cup$, $\mathcal{I}$-G_κ, $\mathcal{I}$-F_κ. An analogous result holds for unions on disjoint $\mathcal{I}$-cones of a larger (finite will be sufficient for our purposes) number of sets.*

Proof. An easy check that we would like to leave to our readers. □

Armed with the above lemma, we may now easily construct sets which are neither $\mathcal{I}$-open nor $\mathcal{I}$-closed, and satisfy arbitrary combinations of the remaining four properties of being $\mathcal{I}$-$\cap$, $\mathcal{I}$-$\cup$, $\mathcal{I}$-G_κ and $\mathcal{I}$-F_κ, by simply combining the four basic sets 2^κ, $\{\mathbf{0}\}$, I_κ and $\mathsf{ub}(X)$ from above via taking unions on disjoint $\mathcal{I}$-cones, and via taking complements of such sets. We illustrate those results in Table 2 below. For a set A, we let $\overline{A}$ denote the complement of A: for example, $\overline{\{\mathbf{0}\}}$ denotes $2^\kappa \setminus \{\mathbf{0}\}$ in the table below. When we put the symbol $\sim$ in our table, this means that a set with exactly the properties indicated by the combination of $+$'s in its row can simply be obtained by considering the complement of one of the other sets used in the table (which might only appear further down in the table) – note that, as we

[6] Note that by Observation 1.8, we always have at least κ-many disjoint $\mathcal{I}$-cones available that are each homeomorphic to 2^κ with the $\mathcal{I}$-topology.

mentioned above, for any set A and any of the classes we consider in the below, A belongs to one such class if and only if its complement doesn't, i.e., taking of complements corresponds to inverting all the corresponding entries (+'s and blank spaces). We will leave the completely straightforward task of verifying any of the entries in Table 2 to our interested readers.

	I-$\cap$	I-$\cup$	I-G_κ	I-F_κ
$\{\mathbf{0}\}\dot\cup\overline{\{\mathbf{0}\}}$	+	+	+	+
$\mathsf{ub}(X)\dot\cup\{\mathbf{0}\}$	+	+	+	
$\sim$	+	+		+
$\mathsf{ub}(X)\,\dot\cup\,\overline{\mathsf{ub}(X)}$	+	+		
I_κ	+		+	+
$\mathsf{I}_\kappa\,\dot\cup\,\mathsf{ub}(X)$	+		+	
$\sim$	+			+
$\mathsf{I}_\kappa\,\dot\cup\,\mathsf{ub}(X)\,\dot\cup\,\overline{\mathsf{ub}(X)}$	+			
$\sim$		+	+	+
$\mathsf{ub}(X)\,\dot\cup\,\overline{\mathsf{I}_\kappa}$		+	+	
$\sim$		+		+
$\sim$		+		
$\mathsf{I}_\kappa\,\dot\cup\,\overline{\mathsf{I}_\kappa}$			+	+
$\mathsf{ub}(X)\,\dot\cup\,\mathsf{I}_\kappa\,\dot\cup\,\overline{\mathsf{I}_\kappa}$			+	
$\sim$				+
$\mathsf{ub}(X)\,\dot\cup\,\overline{\mathsf{ub}(X)}\,\dot\cup\,\mathsf{I}_\kappa\,\dot\cup\,\overline{\mathsf{I}_\kappa}$				

TABLE 2. All 16 properties

We close this section with two questions.

Question 1. *Does the difference hierarchy over the $\mathcal{I}$-open and $\mathcal{I}$-closed sets have length ω_1? Does it union up to $\mathcal{I}$-$F_\kappa \cap \mathcal{I}$-$G_\kappa$?*

Question 2. *Given that $\mathcal{I}$ contains an unbounded subset of κ, is there an $\mathcal{I}$-Borel hierarchy that is reminiscent of the classical Borel hierarchy, at least if $\mathcal{I} = \mathsf{NS}_\kappa$? Are there $\mathcal{I}$-Borel sets which are substantially more complicated than $\mathcal{I}$-F_κ or $\mathcal{I}$-G_κ sets?*

2.5. **On the collection of closed and unbounded sets.** Let

$$\mathrm{Closed}_\kappa = \{\chi_x \mid x \text{ is a closed subset of } \kappa\}$$

be the collection of closed (possibly bounded) subsets of κ, which is a closed set in the bounded topology, and therefore also in any ideal topology. Let

$$\mathrm{Club}_\kappa = \{\chi_x \mid x \text{ is a club subset of } \kappa\} = \mathrm{Closed}_\kappa \cap \mathsf{ub}_\kappa$$

be the collection of club subsets of κ. If $\mathcal{I}$ is tall, it thus follows that Club_κ is, unlike in the bounded topology, an intersection of an $\mathcal{I}$-open and a closed (and hence also $\mathcal{I}$-closed) set. Therefore, by Proposition 2.6, Club_κ is $\mathcal{I}$-G_κ, for it is the intersection of the $\mathcal{I}$-open set ub_κ with the closed set Closed_κ, and every closed set (in the bounded topology) is G_κ (again in the bounded topology), hence also $\mathcal{I}$-G_κ. We now want to deal with the question when Club_κ can be on any of the other low levels of $\mathcal{I}$-Borel hierarchies. We first characterize exactly when Club_κ is $\mathcal{I}$-closed:

Observation 2.15. Club_κ *is* $\mathcal{I}$*-closed if and only if* $\mathcal{I}$ *contains a stationary subset of* κ*.*

Proof. Fix $S \in \mathcal{I}$ stationary. Let $x \subseteq \kappa$ not be in Club_κ, i.e., x not closed unbounded. In case x is not closed, let $\alpha < \kappa$ be such that $x \restriction \alpha$ is not closed; then $[x \restriction \alpha] \cap \mathrm{Club}_\kappa = \emptyset$.

If x is bounded, fix $\alpha < \kappa$ such that $x \subseteq \alpha$, and let $S' := S \setminus \alpha \in \mathcal{I}$. Since S' is stationary, it intersects each closed unbounded subset of κ, and hence $[x \restriction S'] = [\mathbf{0} \restriction S']$ has empty intersection with Club_κ.

The above shows that in each case, x is in the $\mathcal{I}$-interior of the complement of Club_κ, and hence that Club_κ is $\mathcal{I}$-closed, as desired.

For the reverse direction, assume that Club_κ is $\mathcal{I}$-closed. Then, its complement contains an $\mathcal{I}$-cone $[f]$ with $\emptyset \in [f]$. Hence, f has to have constant value 0. Since $[f]$ must not contain a club subset of κ, $\mathrm{dom}\,(f)$ has to be stationary, and $\mathrm{dom}\,(f) \in \mathcal{I}$, which finishes the proof. □

Next, we investigate the possibility of Club_κ being $\mathcal{I}$-open. Let *Lim* denote the club set of all limit ordinals in κ.

Proposition 2.16. Club_κ *is* $\mathcal{I}$*-open if and only if the following Property (*) holds:* $Lim \in \mathcal{I}$*, and for every nonstationary subset* X *of* Lim*, there is a regressive function* $r\colon X \to \kappa$ *such that* $\bigcup_{\alpha \in X}[r(\alpha), \alpha) \in \mathcal{I}$*.*

Proof. Assume that Club_κ is $\mathcal{I}$-open. Let $[f]$ be an $\mathcal{I}$-cone such that $\kappa \in [f] \subseteq \mathrm{Club}_\kappa$. Then, $f = \mathbf{1} \restriction A$ for some $A \in \mathcal{I}$. If $\alpha \in Lim \setminus A$, then $[f]$ contains a subset of κ that does not contain α as an element, however is unbounded below α, contradicting that $[f] \subseteq \mathrm{Club}_\kappa$, and thus showing that $A \supseteq Lim \in \mathcal{I}$.

Let X be a nonstationary subset of *Lim*. Let $C \subseteq Lim$ be a club that is disjoint from X. There is some $[f] \subseteq \mathrm{Club}_\kappa$ with $C \in [f]$, hence $f = C \restriction A$ for some $A \in \mathcal{I}$. If for some $\alpha \in X$, the complement of A were unbounded in α, then $[f]$ again contains a subset of κ which doesn't contain α as an element (due to $C \cap X = \emptyset$), however is unbounded in α, which is again a contradiction. However, this now allows us to construct a regressive function r on X that is as desired.

Assume now that $\mathcal{I}$ satisfies Property (*). We want to show that Club_κ is $\mathcal{I}$-open. Let $C \subseteq \kappa$ be any club subset of κ. It suffices to find a function $f\colon A \to 2$ in $\mathsf{Fn}_\mathcal{I}$ such that $C \in [f] \subseteq \mathrm{Club}_\kappa$.

Let N be the non-stationary set $Lim \setminus C$, let $r\colon N \to \kappa$ be regressive such that $A' := \bigcup_{\alpha \in N}[r(\alpha), \alpha) \in \mathcal{I}$, and let $A = Lim \cup A'$. Let $f = C \restriction A$. Then, clearly, $C \in [f]$. We have to show that $[f] \subseteq \mathrm{Club}_\kappa$. Let $x \in [f]$.

Since $C \cap A$ is unbounded, x is clearly unbounded. It remains to show that x is closed. Take any strictly increasing sequence $\langle \alpha_i \mid i < \mathsf{cof}(\alpha)\rangle$ with limit α. The only problematic case that we have to consider is if $\alpha \notin x$, however for all $i < \mathsf{cof}(\alpha)$, $\alpha_i \in x$. Since x and C agree on $A \supseteq Lim$ and $\alpha \in Lim$, we have $\alpha \notin C$, hence $\alpha \in N$. So by the definition of A, $[r(\alpha), \alpha) \subseteq A$, hence all but boundedly many α_i are in C, contradicting C being closed. □

It remains to observe that ideals satisfying Property (*) actually exist:

Observation 2.17. *There is an ideal* $\mathcal{I}$ *such that* $\mathcal{I}$ *satisfies Property (*).*

Proof. Given $X \subseteq \kappa$, let $X^\oplus = \{\alpha + 1 \mid \alpha \in X\}$, and let $X^\ominus = \{\alpha - 1 \mid \alpha \in X$ is a successor ordinal$\}$. Let $\mathcal{I}$ be the ($<\kappa$-complete) ideal generated by *Lim* together with $\{X^\oplus \mid X \subseteq \kappa$ non-stationary$\}$. It is easy to see that by the $<\kappa$-completeness of NS_κ, $\mathcal{I}$ is a proper ideal on κ. Let N be a nonstationary subset of *Lim*, let

$C \subseteq Lim$ be a club that is disjoint from N, and let

$$A = \bigcup\{[\alpha+2,\beta) \mid \beta \in N \wedge \alpha = \max(C \cap \beta)\}.$$

Let $B = A^{\ominus}$. Then, B is disjoint from C, and therefore nonstationary, yielding that $A \in \mathcal{I}$, and showing that $\mathcal{I}$ satisfies Property (*). □

Club_κ provides a natural example of an Edinburgh G_κ set which is not Edinburgh F_κ: We already observed in the above that Club_κ is Edinburgh G_κ, and also that it is an intersection of an Edinburgh open and an Edinburgh closed set. It is easy to see that Club_κ cannot contain a nonempty Edinburgh open set, implying that Club_κ is neither Edinburgh open nor a union of an Edinburgh open and an Edinburgh closed set. Club_κ is not Edinburgh closed by Observation 2.15. Let us finally verify that Club_κ is not Edinburgh F_κ, by an argument that builds on the argument for Proposition 2.10. We will need the following.

Definition 2.18. If $A \subseteq \kappa$ and $f\colon A \to 2$, we say that f is *closed* if whenever $\lambda < \kappa$ and $\langle \alpha_i \mid i < \lambda\rangle \subseteq \operatorname{dom} f$, is an increasing sequence with $f(\alpha_i) = 1$ for every $i < \lambda$, and with $\alpha = \bigcup_{i<\lambda} \alpha_i \in \operatorname{dom} f$, then $f(\alpha) = 1$.

We say that $\bar{f}\colon \kappa \to 2$ is the *closure* of f in case that $\bar{f} \supseteq f$ and if $\alpha \notin \operatorname{dom} f$, then $\alpha \in \operatorname{dom} \bar{f}$ if and only if there exists $\lambda < \kappa$ and an increasing sequence $\langle \alpha_i \mid i < \lambda\rangle \subseteq \operatorname{dom} f$ with $f(\alpha_i) = 1$ for every $i < \lambda$ and with $\alpha = \bigcup_{i<\lambda} \alpha_i$, and we require that $\bar{f}(\alpha) = 1$ for all such α.

Theorem 2.19. *Club_κ is not Edinburgh F_κ.*

Proof. Assume for a contradiction that it is, i.e. that $\mathrm{Club}_\kappa = \bigcup_{\alpha<\kappa}[P_\alpha]_{\mathsf{NS}_\kappa}$, with each $P_\alpha \subseteq \mathsf{Fn}$ closed under restrictions. We want to inductively construct a club subset of κ which is not in the above union, and thus reach a contradiction. The key ingredient will be the following claim:

Claim. *If $f = \bar{f} \in \mathsf{Fn}$ is closed and bounded, $C \subseteq \kappa$ is a club susbset of κ that is disjoint from $\operatorname{dom} f$, and $[P]_{\mathsf{NS}_\kappa}$ is Edinburgh closed with $P \subseteq \mathsf{Fn}$ closed under initial segments, and such that $[P]_{\mathsf{NS}_\kappa}$ contains only closed subsets of κ, then there is an extension $g \supseteq f$ of f in Fn which is closed and bounded, such that $g \notin P$, and such that for some $\gamma \in C$,*

$$\forall \delta \in dom\, g \setminus dom\, f \quad g(\delta) = 1 \iff \delta = \gamma.$$

In particular, this implies that $\bar{g} = g$.

Proof. Let $C^* = C \setminus \{\min C\}$. Let $f^* \in \mathsf{Fn}$ be the extension of $\bar{f}$ with $\operatorname{dom} f^* = \kappa \setminus C^*$, with $f(\min C) = 1$, and with $f^*(\alpha) = 0$ whenever $\alpha \in \kappa \setminus (\operatorname{dom} f \cup C)$. For $A \subseteq C^*$ in NS_κ, let $f_A \in \mathsf{Fn}$ denote the extension of f^* with $\operatorname{dom} f_A = \operatorname{dom} f^* \cup A$ and with $f_A(\alpha) = 0$ for every $\alpha \in A$. Assume for a contradiction that every such f_A were an element of P. But then, letting $x \in 2^\kappa$ be the extension of f^* with $x(\alpha) = 0$ for every $\alpha \in C^*$, it follows, since P is closed under initial segments, that $x \upharpoonright\upharpoonright \mathsf{NS}_\kappa \subseteq P$, and hence that $x \in [P]_{\mathsf{NS}_\kappa}$. But $x \in \mathsf{NS}_\kappa$, contradicting our assumption on P. □

Let $f_0 = \emptyset$, and let $C_0 = \kappa$. Then $f_0 = \bar{f}_0 \in \mathsf{Fn}$ is closed and bounded. Given $f_\alpha = \bar{f}_\alpha$ and C_α, let $f_{\alpha+1}$ and γ_α be obtained by an application of the claim w.r.t. C_α and P_α, that is, $\gamma_\alpha \in C_\alpha$, and $f_{\alpha+1} = \bar{f}_{\alpha+1}$ is an extension of f_α in Fn which is closed and bounded, with

$$\forall \delta \in \operatorname{dom} f_{\alpha+1} \setminus f_\alpha \quad f_{\alpha+1}(\delta) = 1 \iff \delta = \gamma_\alpha,$$

and such that $f_{\alpha+1} \notin P_\alpha$. Let $C_{\alpha+1} \subseteq C_\alpha \setminus (\gamma_\alpha + 1)$ be a club subset of κ that is disjoint from $\operatorname{dom} f_{\alpha+1}$. At limit stages $\alpha < \kappa$, let $\gamma_\alpha = \bigcup_{\beta<\alpha} \gamma_\beta$, and let

$f_\alpha = \bar{f}_\alpha = \bigcup_{\beta<\alpha} f_\beta \cup \{(\gamma_\alpha, 1)\}$. Note that $\gamma_\alpha \in C_\alpha := \bigcap_{\beta<\alpha} C_\beta$, and therefore $\gamma_\alpha \notin \operatorname{dom} f_\beta$ for any $\beta < \alpha$. Hence, using that the γ_β's are strictly increasing, $f_\alpha = \bar{f}_\alpha$ is indeed a function, and thus an element of Fn, which is easily seen to be closed and bounded. Then $f := \bigcup_{\alpha<\kappa} f_\alpha$ is closed and unbounded, however $f \notin \bigcup_{\alpha<\kappa} [P_\alpha]_{\mathsf{NS}_\kappa}$. □

This yields yet another characterization of when $\mathcal{I}$ contains a stationary subset of κ:

Corollary 2.20. Club_κ *is* $\mathcal{I}$-F_κ *if and only if* $\mathcal{I}$ *contains a stationary subset of* κ.

Proof. If $\mathcal{I}$ contains a stationary subset of κ, then Club_κ is $\mathcal{I}$-closed by Observation 2.15, and hence it is trivially also $\mathcal{I}$-F_κ. On the other hand, Club_κ is not Edinburgh F_κ by Theorem 2.19, and hence if $\mathcal{I} \subseteq \mathsf{NS}_\kappa$, it is not $\mathcal{I}$-F_κ, for the Edinburgh topology then refines the $\mathcal{I}$-topology. □

2.6. **The club filter.** Let $\mathcal{C}_\kappa$ denote the club filter on κ, i.e., the collection of all subsets of κ that contain a club subset of κ. In the bounded topologies on higher cardinals, the club filter is usually the standard example for a non-Borel set. The situation is somewhat different for $\mathcal{I}$-topologies. The following is an immediate consequence of Proposition 2.6 and of Proposition 2.7, letting $\mathcal{J} = \mathsf{NS}_\kappa$:

Corollary 2.21.

- $\mathcal{I}$ *is stationary tall if and only if* $\mathcal{C}_\kappa$ *is* $\mathcal{I}$*-closed.*
- $\mathcal{I}$ *contains a club subset of* κ *if and only if* $\mathcal{C}_\kappa$ *is* $\mathcal{I}$*-open.*

Compare the first item with Observation 2.15, which gives a similar characterization of Club_κ being $\mathcal{I}$-closed. In particular, it follows that

$$\mathcal{C}_\kappa \text{ is } \mathcal{I}\text{-closed} \Rightarrow \mathrm{Club}_\kappa \text{ is } \mathcal{I}\text{-closed.}$$

However, if $\mathcal{I}$ is not stationary tall, then the situation is somewhat less unusual, for then we will show that the club filter is not an $\mathcal{I}$-Borel set. The following argument extends and generalizes [5, Theorem 4.2], and also builds on the proof of that theorem.[7]

Let us first recall some basic topological concepts for $\mathcal{I}$-topologies. A set A is *$\mathcal{I}$-nowhere dense* if for any $\mathcal{I}$-cone $[f]$ there is an $\mathcal{I}$-cone $[g] \subseteq [f]$ with $A \cap [g] = \emptyset$. A set is *$\mathcal{I}$-meager* if it is a κ-union of $\mathcal{I}$-nowhere dense sets. A set is *$\mathcal{I}$-comeager* if its complement is $\mathcal{I}$-meager. A set X has the *$\mathcal{I}$-Baire property*, if there is an $\mathcal{I}$-open set U such that $X \Delta U$ is $\mathcal{I}$-meager. Let us observe that, as usual, every set with the $\mathcal{I}$-Baire property is either $\mathcal{I}$-meager, or is $\mathcal{I}$-comeager in an $\mathcal{I}$-cone. Furthermore, by the usual argument, every $\mathcal{I}$-Borel set has the $\mathcal{I}$-Baire property.

Given a stationary $S \subseteq \kappa$, let

$$\mathcal{C}_\kappa^S = \{A \mid \exists C \subseteq \kappa \text{ club } A \supseteq C \cap S\}.$$

Lemma 2.22. *Assume that* S *is a stationary subset of* κ*, and that* $\mathcal{I}$ *contains no stationary subset of* S*. Then,* $\mathcal{C}_\kappa^S$ *doesn't have the* $\mathcal{I}$*-Baire property.*

Proof. Towards a contradiction, suppose that $\mathcal{C}_\kappa^S$ has the $\mathcal{I}$-Baire property. First assume that $\mathcal{C}_\kappa^S$ is $\mathcal{I}$-meager. Let $\vec{U} = \langle U_i \mid i < \kappa\rangle$ be a sequence of $\mathcal{I}$-open dense sets whose intersection $U := \bigcap_{i<\kappa} U_i$ is disjoint from $\mathcal{C}_\kappa^S$. We construct sequences $\vec{f} = \langle f_j \mid j < \kappa\rangle$ in $\mathsf{Fn}_\mathcal{I}$, $\vec{C} = \langle C_j \mid j < \kappa\rangle$ in Club_κ, and $\vec{\alpha} = \langle \alpha_j \mid j < \kappa\rangle$ in κ with the following properties:

[7] In retrospect, we realized that also the arguments for the proofs of Proposition 2.10 and of Theorem 2.19 are somewhat similar to the arguments in the proof of [5, Theorem 4.2].

(1) $f_i \subseteq f_j$, $C_i \supseteq C_j$ and $\alpha_i < \alpha_j$ for all $i < j < \kappa$,
(2) (a) $\alpha_j = \min(C_j)$ for all $j < \kappa$,
 (b) $\alpha_\lambda = \sup_{i<\lambda} \alpha_i$ for limits $\lambda < \kappa$,
(3) (a) $[f_{j+1}] \subseteq U_j$ for all $j < \kappa$,
 (b) $\operatorname{dom}(f_j) \cap C_j = \emptyset$ for all $j < \kappa$, and
 (c) $f_j(\alpha_i) = 1$ for all $i < j < \kappa$ with $\alpha_j \in S$.

The construction proceeds as follows.

(i) Choose $f_0 \in \mathsf{Fn}_\mathcal{I}$ arbitrary, let C_0 be a club disjoint from $\operatorname{dom}(f_0) \cap S$, and let $\alpha_0 := \min(C_0)$.
(ii) For successors $j+1$, assume that f_j has been constructed. If $\alpha_j \in S$, let $f'_j = f_j \cup \{(\alpha_j, 1)\}$, and let $f'_j = f_j$ otherwise. Since U_j is $\mathcal{I}$-open dense, there is some f_{j+1} extending f'_j with $[f_{j+1}] \subseteq U_j$. Find a club $C_{j+1} \subseteq C_j \setminus (\alpha_j + 1)$ disjoint from $\operatorname{dom}(f_{j+1}) \cap S$ and let $\alpha_{j+1} = \min(C_{j+1})$.
(iii) For limits $\lambda < \kappa$, let $f_\lambda = \bigcup_{i<\lambda} f_i$, $C_\lambda = \bigcap_{i<\lambda} C_i$ and $\alpha_\lambda = \min(C_\lambda)$.

Then, $f := \bigcup_{i<\kappa} f_i$ is constant with value 1 on the intersection of the club $C := \{\alpha_i \mid i < \kappa\}$ with S by (3)(c). Since $[f_{j+1}] \subseteq U_j$ for all $j < \kappa$ by (3)(a), any total extension $g \supseteq f$ contradicts that $\mathcal{C}_\kappa^S$ is disjoint from U.

Finally, assume that $\mathcal{C}_\kappa$ is $\mathcal{I}$-comeager in $[h]$ for some $h \in \mathsf{Fn}_\mathcal{I}$. But then, virtually the same construction, letting $f_0 = h$ and setting $f(\alpha_j) = 1$ instead of 0 in case $\alpha_j \in S$, yields a contradiction just as in the previous case. □

Corollary 2.23.

(1) $\mathcal{C}_\kappa$ is not Edinburgh Borel.
(2) $\mathcal{I}$ is stationary tall if and only if for every stationary $S \subseteq \kappa$, $\mathcal{C}_\kappa^S$ is $\mathcal{I}$-Borel.
(3) If $\mathcal{I}$ is not stationary tall, then there is a set without the $\mathcal{I}$-Baire property.
(4) If $\mathcal{I}$ is not stationary tall, then there is a set which is not $\mathcal{I}$-Borel.

Proof. The above are immediate from Lemma 2.22 and the comment preceding it that $\mathcal{I}$-Borel sets have the $\mathcal{I}$-Baire property, except that we still have to verify for (2) that if $\mathcal{I}$ is stationary tall, then for every stationary $S \subseteq \kappa$, $\mathcal{C}_\kappa^S$ is $\mathcal{I}$-Borel (in fact: $\mathcal{I}$-closed): Let $x \notin \mathcal{C}_\kappa^S$, so for all clubs $C \subseteq \kappa$, $\kappa \setminus x \cap C \cap S \neq \emptyset$. That means, $\kappa \setminus x \cap S$ is stationary. Using that $\mathcal{I}$ is stationary-tall, we can fix a stationary $S' \in \mathcal{I}$ such that $S' \subseteq \kappa \setminus x \cap S$. Let $y \in [x \restriction S']$. Since S' is disjoint from x, y is disjoint from S'. Assume there exists a club B such that $y \supseteq B \cap S$, then also $B \cap S \cap S' = \emptyset$, but $S \cap S' = S'$ is stationary, this contradicts the fact that B is club. Hence $y \notin \mathcal{C}_\kappa^S$ and therefore $\mathcal{C}_\kappa^S$ is disjoint from $[x \restriction S']$, as desired. □

Let us finally remark the following, which was brought to our attention by Vincenzo Dimonte.

Observation 2.24. *For any ideal $\mathcal{I}$ on κ, if $2^{<\kappa} = \kappa$, then there is a set that is not $\mathcal{I}$-Borel.*

Proof. Since $2^{<\kappa} = \kappa$, there exists a Bernstein subset of 2^κ (in the sense of the higher Cantor space 2^κ), that is a set which intersects every perfect subset of 2^κ and also its complement, simply because by our assumption, there are only 2^κ-many perfect subsets of 2^κ. But, unlike $\mathcal{I}$-Borel sets, a Bernstein set cannot have the $\mathcal{I}$-Baire property: If it were $\mathcal{I}$-meager, its complement would contain a perfect set by Corollary 1.4, a contradiction. But otherwise, our Bernstein set would have to be comeager in an $\mathcal{I}$-cone, but then it would contain a perfect set by a relativized version of Corollary 1.4, which is also a contradiction. □

3. Sequences in ideal topologies

3.1. **Convergence and accumulation points.** A prominent notion in analysis is that of *$\mathcal{I}$-convergence*, a generalized notion of convergence with respect to an ideal $\mathcal{I}$ on the set of natural numbers. The idea is that for a sequence to $\mathcal{I}$-converge, it only needs to enter every neighbourhood (in the bounded topology) of its limit on a set in $\mathcal{I}^*$, thus yielding a weakening of the standard notion of convergence. Such a generalized notion of convergence – namely statistical convergence – was first considered in [3], and the generalized notion of $\mathcal{I}$-convergence was introduced only much later in [7]. If we consider topologies other than the bounded topology, it seems most natural to generalize the concept of convergence in two respects, to that of *$(\mathcal{I}, \mathcal{J})$-convergence.*

Definition 3.1. Given ideals $\mathcal{I}$ and $\mathcal{J}$ on κ, and a sequence $\vec{x} = \langle x_\alpha \mid \alpha < \kappa\rangle$ of elements of 2^κ, we say that

- $\vec{x}$ *$(\mathcal{I}, \mathcal{J})$-converges* to $x \in 2^\kappa$ if for every $\mathcal{I}$-open set $\mathcal{O}$ containing x, $\{\alpha < \kappa \mid x_\alpha \in \mathcal{O}\} \in \mathcal{J}^*$; we call x the *$(\mathcal{I}, \mathcal{J})$-limit* of $\vec{x}$;
- $\vec{x}$ *$\mathcal{I}^2$-converges* to x if $\vec{x}$ $(\mathcal{I}, \mathcal{I})$-converges to x; we call x the *$\mathcal{I}^2$-limit* of $\vec{x}$; [8]
- as usual, of course, to $(\mathcal{I}, \mathcal{J})$-converge means to $(\mathcal{I}, \mathcal{J})$-converge to some $x \in 2^\kappa$; similarly for $\mathcal{I}^2$-convergence.

If we only change the ideal that induces our topology, however leave the condition for convergence as usual, then in many cases, we do not obtain an interesting notion:

Proposition 3.2. *Assume that $\mathcal{I}$ is tall, and that $\vec{x} = \langle x_i \mid i < \kappa\rangle$ does $(\mathcal{I}, \mathsf{bd}_\kappa)$-converge to $x \in 2^\kappa$. Then, $\vec{x}$ is eventually constant.*

Proof. Assume for a contradiction that $\vec{x}$ is not eventually constant. Then there are strictly increasing sequences $\vec{\alpha} = \langle \alpha_i \mid i < \kappa\rangle$ and $\vec{\beta} = \langle \beta_i \mid i < \kappa\rangle$ with $x_{\beta_i}(\alpha_i) \neq x(\alpha_i)$ for all $i < \kappa$. Let $A = \{\alpha_i \mid i < \kappa\}$. Since $\mathcal{I}$ is tall, there's an unbounded subset B of A with $B \in \mathcal{I}$. But then, $x_{\beta_i} \notin [x{\restriction}B]$ for all $i \in B$, contradicting the assumption of the proposition. □

Let us provide examples showing that bd^2_κ-convergence and $\mathcal{I}^2$-convergence for $\mathcal{I} \supsetneq \mathsf{bd}_\kappa$ are independent of each other:

Observation 3.3. *Let $\mathcal{I}$ be an ideal on κ that contains an unbounded subset A of κ. Then, the following hold true.*

(1) There are $\mathcal{I}^2$-convergent sequences that are not bd^2_κ-convergent.
(2) There are bd^2_κ-convergent sequences that are not $\mathcal{I}^2$-convergent.

Proof. (1) Any sequence such that $x_\alpha = \mathbf{0}$ for all α in $\kappa\setminus A$ is $\mathcal{I}^2$-convergent with limit $\mathbf{0}$, however it will not be bd^2_κ-convergent for example if we additionally let $x_\alpha = \mathbf{1}$ for all α in A.

(2) Let $\vec{x} = \langle x_\alpha \mid \alpha < \kappa\rangle$ enumerate A in increasing order. Then, $\vec{y} = \langle \{x_\alpha\} \mid \alpha < \kappa\rangle$ is clearly bd^2_κ-convergent with limit $\mathbf{0}$, and it is easy to see that $\mathbf{0}$ is the only possible $\mathcal{I}^2$-limit for $\vec{y}$. However, the cone $[\mathbf{0} \restriction A]$ contains no $\{x_\alpha\}$, yielding that $\vec{y}$ does not $\mathcal{I}^2$-converge.[9] □

A notion closely connected to convergence is that of an accumulation point of a sequence. Given the above, it is pretty obvious what should be the right definition of this notion in our generalized context.

[8] When context makes this obvious, we may sometimes talk about *limits* when we actually mean $(\mathcal{I}, \mathcal{J})$-limits or $\mathcal{I}^2$-limits.

[9] The proof clearly shows that $\vec{y}$ has not even an $\mathcal{I}^2$-accumulation point (see Definition 3.4).

Definition 3.4. With $\mathcal{I}$, $\mathcal{J}$ and $\vec{x}$ as above, we say that $x \in 2^\kappa$ is an $(\mathcal{I}, \mathcal{J})$-*accumulation point* of $\vec{x}$ in case that for every $\mathcal{I}$-open set $\mathcal{O}$ containing x, $\{\alpha < \kappa \mid x_\alpha \in \mathcal{O}\} \in \mathcal{J}^+$. $\mathcal{I}^2$-*accumulation points* are $(\mathcal{I}, \mathcal{I})$-accumulation points.

We make some simple observations:

Observation 3.5. *(1) If $\vec{x}$ is $(\mathcal{I}, \mathcal{J})$-convergent with limit $x \in 2^\kappa$, then x is the unique $(\mathcal{I}, \mathcal{J})$-accumulation point of $\vec{x}$ – this is easily shown as usual (using that $\mathcal{J}^* \subseteq \mathcal{J}^+$). It is then also a bd^2_κ-accumulation point of $\vec{x}$ (using that $\mathcal{J}^* \subseteq \mathsf{ub}_\kappa$).*

(2) Any $(\mathcal{I}, \mathcal{J})$-convergent sequence has a unique $(\mathcal{I}, \mathcal{J})$-limit, and if $\mathcal{J}$ is a maximal ideal, then every sequence has at most one $(\mathcal{I}, \mathcal{J})$-accumulation point. (This uses that the $\mathcal{I}$-topology is Hausdorff and $\mathcal{J}^*$ is a filter.)

Our next observation is trivial to verify.

Observation 3.6. *Let $\mathcal{I} \subseteq \mathcal{I}'$ and $\mathcal{J} \subseteq \mathcal{J}'$ be ideals on κ and let $\vec{x}$ be a sequence. The following implications hold:*

(1) z is the $(\mathcal{I}', \mathcal{J})$-limit of $\vec{x} \Rightarrow z$ is the $(\mathcal{I}, \mathcal{J}')$-limit of $\vec{x}$.
*(2) z is an $(\mathcal{I}', \mathcal{J}')$-accumulation point of $\vec{x} \Rightarrow$
z is an $(\mathcal{I}, \mathcal{J})$-accumulation point of $\vec{x}$.*

Let us observe the following.

Observation 3.7. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on κ, such that $\mathcal{I}$ contains an unbounded subset of κ. Then, there is a sequence $\vec{x}$ without an $(\mathcal{I}, \mathcal{J})$-accumulation point.*

Proof. Using Observation 1.2, let $\{[f_\alpha] \mid \alpha < \kappa\}$ be a family of disjoint $\mathcal{I}$-cones. For every $\alpha < \kappa$, let $x_\alpha \in [f_\alpha]$. Then, $\vec{x} = \langle x_\alpha \mid \alpha < \kappa \rangle$ is clearly as desired. □

When $\mathcal{I} \supseteq \mathsf{NS}_\kappa$, then there are strong restrictions on what $\mathcal{I}^2$-convergent sequences (and also sequences with $\mathcal{I}^2$-accumulation points) can look like.

Definition 3.8. We say that an ideal $\mathcal{I}$ on κ is *sequentially tall* if for each sequence $\{y_i \mid i < \kappa\}$ of unbounded subsets of κ, there exists a set $y \in \mathcal{I}$ such that y has nonempty intersection with every y_i.

It is straightforward to check that every sequentially tall ideal is tall. However, we do not know whether the converse holds true:

Question 3. *Is there a tall ideal which is not sequentially tall?*

Note that if $\mathcal{I}$ sequentially tall, then any $\mathcal{I}' \supseteq \mathcal{I}$ is sequentially tall.

Proposition 3.9. NS_κ *is sequentially tall.*

Proof. Given $\{y_i \mid i < \kappa\}$ with y_i unbounded for each i, define $y = \{\gamma_i \mid i < \kappa\}$ recursively as follows. Let $\delta_0 \in \kappa$ be arbitrary and pick $\gamma_0 > \delta_0$ such that $\gamma_0 \in y_0$. Assume that δ_i and γ_i have been defined. Pick some $\delta_{i+1} > \gamma_i$ and some $\gamma_{i+1} > \delta_{i+1}$, such that $\gamma_{i+1} \in y_{i+1}$. If i is a limit ordinal, and δ_j and γ_j have been defined for each $j < i$, let $\delta_i := \sup_{j<i} \delta_j$ and pick some $\gamma_i > \delta_i$ such that $\gamma_i \in y_i$.

Note that $\{\delta_i \mid i < \kappa\}$ is a club subset of κ which is disjoint from y, hence $y \in \mathsf{NS}_\kappa$, and $\gamma_i \in y \cap y_i$ for each $i < \kappa$. □

Lemma 3.10. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals, let $\mathcal{I}$ be sequentially tall and let $\vec{x} = \langle x_\alpha \mid \alpha < \kappa \rangle$ be a sequence that $(\mathcal{I}, \mathcal{J})$-converges to $x \in 2^\kappa$. Then, there is a set $C \in \mathcal{J}^*$ such that the symmetric difference $x_\alpha \Delta x$ is bounded whenever $\alpha \in C$. If x is only an $(\mathcal{I}, \mathcal{J})$-accumulation point of $\vec{x}$, then we only get this for a set of α's in $\mathcal{J}^+$ (rather than $\mathcal{J}^*$).*

Proof. Considering $x_\alpha \Delta x$ rather than x_α for every $\alpha < \kappa$, we may as well assume that $x = \mathbf{0}$. Since $\mathcal{I}$ is sequentially tall, there exists $y \in \mathcal{I}$ which intersects every x_α which is unbounded, hence $[\mathbf{0} \upharpoonright y]$ contains no unbounded x_α, however it is supposed to contain $\mathcal{J}^*$-many (or, $\mathcal{J}^+$-many if we consider the case of x being only an $(\mathcal{I}, \mathcal{J})$-accumulation point) x_α's. Thus, $\mathcal{J}^*$-many (or, $\mathcal{J}^+$-many) x_α's have to be bounded, i.e., after translating back via taking symmetric differences with x once again, $x_\alpha \Delta x$ is bounded, as desired. □

Let us say that for a set $X \subseteq 2^\kappa$, the $\mathcal{I}$-*closure* of X is the $\subseteq$-minimal $\mathcal{I}$-closed set $Y \supseteq X$, which exists because $\mathcal{I}$-closed sets are closed under the taking of arbitrary intersections.

Observation 3.11. *The $\mathcal{I}$-closure of $\mathcal{I}^*$ is all of 2^κ.*

Proof. Since every non-empty $\mathcal{I}$-open set contains an element of $\mathcal{I}^*$, it follows that the complement of the $\mathcal{I}$-closure of $\mathcal{I}^*$ has to be empty, i.e., that the $\mathcal{I}$-closure of $\mathcal{I}^*$ is all of 2^κ. □

Our next result shows that for many interesting ideals $\mathcal{I}$, $\mathcal{I}$-closure cannot be characterized through $(\mathcal{I}, \mathcal{J})$-limits or through $(\mathcal{I}, \mathcal{J})$-accumulation points of sequences.

Corollary 3.12. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals, where $\mathcal{I}$ is sequentially tall. Then, the following hold true:*

(1) *If $\vec{x}$ is a sequence of elements of $\mathcal{I}^+$, then all $(\mathcal{I}, \mathcal{J})$-accumulation points of $\vec{x}$ are in $\mathcal{I}^+$. In particular, if $\vec{x}$ is $(\mathcal{I}, \mathcal{J})$-convergent, its $(\mathcal{I}, \mathcal{J})$-limit is in $\mathcal{I}^+$.*[10]

(2) *$\mathcal{I}^+$ is closed under $(\mathcal{I}, \mathcal{J})$-accumulation points of sequences, and under $(\mathcal{I}, \mathcal{J})$-limits of sequences. In particular, this shows that $\mathcal{I}$-closed sets cannot be characterized as being closed under $(\mathcal{I}, \mathcal{J})$-accumulation points or closed under $(\mathcal{I}, \mathcal{J})$-limits.*

Proof. (1) By Lemma 3.10, if $\vec{x}$ had an $(\mathcal{I}, \mathcal{J})$-accumulation point outside of $\mathcal{I}^+$, then, using that $\mathsf{bd}_\kappa \subseteq \mathcal{I}$, $\mathcal{J}^+$-many x_α's would have to be outside of $\mathcal{I}^+$, contradicting our assumption.

(2) Immediate by (1) and by Observation 3.11, for $\mathcal{I}^+ \supseteq \mathcal{I}^*$. □

We have so far only seen one trivial example of an $\mathcal{I}^2$-convergent sequence in Observation 3.3, (1). Let us provide an example of an $\mathcal{I}^2$-convergent sequence which is slightly less trivial, in case $\mathcal{I} = \mathsf{NS}_\kappa$:

Observation 3.13. *Let $\mathcal{I} = \mathsf{NS}_\kappa$, and let $x \in \mathcal{I}^*$ be enumerated in increasing order by $\langle x_\alpha \mid \alpha < \kappa \rangle$. Then, $\vec{y} = \langle \{x_\alpha\} \mid \alpha < \kappa \rangle$ $\mathcal{I}^2$-converges to $\mathbf{0}$.*

Proof. Let $A \subseteq \kappa$ be nonstationary. Let $C \subseteq x$ be a club subset of κ that is disjoint from A, and let D be the set of indices α such that $x_\alpha \in C$. Note that D is again a club subset of κ. Now, $[\mathbf{0} \upharpoonright A]$ contains $\{x_\alpha\}$ for all $\alpha \in D$, yielding that $\vec{y}$ does indeed $\mathcal{I}^2$-converge to $\mathbf{0}$. □

3.2. **Subsequences.** It should not come as a surprise that the usual concept of subsequence is not of much use in generalized $\mathcal{I}$-topologies.[11] Let us demonstrate this with the following trivial observation:

Observation 3.14. *Assume that $\mathcal{I}$ contains an unbounded subset A of κ. Then, the following hold true:*

[10] The same holds true for $\mathcal{I}^*$ in place of $\mathcal{I}^+$.

[11] With $\vec{y}$ being a *subsequence* of $\vec{x}$, we mean (as usual) that there is a strictly increasing sequence of ordinals $\langle \beta_\alpha \mid \alpha < \kappa \rangle$, and $y_\alpha = x_{\beta_\alpha}$ for every $\alpha < \kappa$.

(1) *There is a sequence $\vec{x}$ with no $\mathcal{I}^2$-accumulation points which has an $\mathcal{I}^2$-convergent subsequence $\vec{y}$.*
(2) *There is an $\mathcal{I}^2$-convergent sequence (that is also bd^2_κ-convergent) with a subsequence that has no $\mathcal{I}^2$-accumulation points.*
(3) *There is an $\mathcal{I}^2$-convergent sequence with an $\mathcal{I}^2$-convergent subsequence that has a different $\mathcal{I}^2$-limit.*

Proof. (1) Simply take x_α to have constant value for an unbounded subset of α's in $\mathcal{I}$, and for the other α's, pick x_α in disjoint $\mathcal{I}$-cones $[f_\alpha]$, as provided in Observation 1.2. Then, $\vec{x}$ is clearly as desired, with the subsequence of y_α's being of the above constant value.
(2) Let $x_\alpha = \{\alpha\}$ for $\alpha < \kappa$. Then $\vec{x} = \langle x_\alpha \mid \alpha < \kappa\rangle$ is easily seen to be $\mathcal{I}^2$-convergent, however the sequence of $\{\alpha\}$'s for $\alpha \in A$ is a subsequence of $\vec{x}$ that does not have an $\mathcal{I}^2$-accumulation point (see Observation 3.3 (2) and Footnote 9).
(3) Easy, and essentially the same as the proof of [7, Proposition 3.1 (ii)]. $\square$

The following easy observation provides a positive result for certain ideals $\mathcal{I}$:

Observation 3.15. *If $\mathcal{I}$ has a base of size κ, then every sequence with an $(\mathcal{I},\mathcal{J})$-accumulation point x has a subsequence with $(\mathcal{I},\mathcal{J})$-limit x.*

Let us propose the following generalized notion of subsequence, which corresponds to the usual notion of subsequence being based on an *unbounded* set of indices when working with the *bounded* topology:

Definition 3.16. Let $\vec{x} = \langle x_\alpha \mid \alpha < \kappa\rangle$ be a sequence in 2^κ. We say that $\vec{y} = \langle y_\alpha \mid \alpha < \kappa\rangle$ is a $\mathcal{J}$-subsequence of $\vec{x}$, and denote this property as $\vec{y} \hookrightarrow_{\mathcal{J}} \vec{x}$, if there is a strictly increasing sequence $\langle \beta_\alpha \mid \alpha < \kappa\rangle$ of ordinals below κ such that $y_\alpha = x_{\beta_\alpha}$ for every $\alpha < \kappa$, and such that $\{\beta_\alpha \mid \alpha < \kappa\} \in \mathcal{J}^+$.

The following definition provides two properties of ideals which will be shown to be equivalent to some natural properties of the $\mathcal{J}$-subsequence relation. If S is a set of ordinals, let π_S denote the transitive collapsing map of S.

Definition 3.17. Let $\mathcal{J}$ be an ideal on κ. We say that $\mathcal{J}$ is

- *closed under re-enumerations* if for each $S \in \mathcal{J}^+$ and each $A \subseteq S$ with $A \in \mathcal{J}$, we have $\pi_S'' A \in \mathcal{J}$.
- *almost closed under re-enumerations* if for each $S \in \mathcal{J}^+$ and each $A \subseteq S$ with $A \in \mathcal{J}$, we have $\pi_S'' A \notin \mathcal{J}^*$.

Observation 3.18. *If $\mathcal{J}$ is a maximal ideal which is almost closed under re-enumerations, then it is closed under re-enumerations, just because $\mathcal{J}^+ = \mathcal{J}^*$.*

Proposition 3.19. bd_κ *is closed under re-enumerations.*

Proof. This holds because being in bd_κ is just a matter of cardinality: Assume S is in bd_κ^+, i.e. it is has size κ and $A \subseteq S$ is bounded, i.e. it has size $< \kappa$. Then clearly $\pi_S'' A$ has size $< \kappa$, so $\pi_S'' A \in \mathsf{bd}_\kappa$. $\square$

Proposition 3.20. *If $\kappa > \omega_1$, then NS_κ is not closed under re-enumerations.*

Proof. Let $S = \mathsf{cof}\,\omega$, and let $A = \{\alpha + \omega \mid \mathsf{cof}(\alpha) = \omega_1\}$. Then, A is clearly nonstationary, as witnessed by the club that is the closure of $\mathsf{cof}\,\omega_1$, however $\pi_S'' A = \mathsf{cof}\,\omega_1$ is stationary. $\square$

Question 4. *Is NS_{ω_1} closed under re-enumerations?*

However, the weaker of the above properties does hold true for NS_κ:

Proposition 3.21. *Let $\mathcal{J} \supseteq \mathsf{NS}_\kappa$. Then $\mathcal{J}$ is almost closed under re-enumerations.*

This implies, together with Observation 3.18, that any maximal ideal $\mathcal{J} \supseteq \mathsf{NS}_\kappa$ is closed under re-enumerations.

Proof of Proposition 3.21. Let $S \in \mathcal{J}^+$ and $A \subseteq S$ with $A \in \mathcal{J}$.

Let us first prove the following claim:

Claim 3.22. *There is a club D such that $\pi_S \restriction (D \cap S) = id$.*

Proof. Assume not, i.e., for every club D, $\pi_S \restriction (D \cap S) \neq id$. Hence the set $S' := \{\alpha \in S \mid \pi_S(\alpha) \neq \alpha\}$ is stationary. Since π_S is a transitive collapsing map, $\pi_S(\alpha) < \alpha$ whenever $\pi(\alpha) \neq \alpha$. So $\pi_S \restriction S'$ is a regressive function, and hence by Fodor, there is a stationary subset on which it is constant, contradicting the fact that π_S is injective. $\square$

Let $C \in \mathcal{J}^*$ be the complement of A. Use the claim to obtain D, and observe that $D \in \mathcal{J}^*$ and let $D' := D \cap C \in \mathcal{J}^*$; then $\pi_S''(D' \cap S) = D' \cap S \in \mathcal{J}^+$ is disjoint from $\pi_S''A$, showing that $\pi_S''A \notin \mathcal{J}^*$. $\square$

Proposition 3.23. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on κ. Then, the following are equivalent:*

(1) $\mathcal{J}$ is closed under re-enumerations.

(2) Whenever $\vec{y}$ is a $\mathcal{J}$-subsequence of $\vec{x}$, then

$$z \text{ is the } (\mathcal{I},\mathcal{J})\text{-limit of } \vec{x} \implies z \text{ is the } (\mathcal{I},\mathcal{J})\text{-limit of } \vec{y}.$$

(3) Whenever $\vec{y}$ is a $\mathcal{J}$-subsequence of $\vec{x}$, then

$$z \text{ is an } (\mathcal{I},\mathcal{J})\text{-accumulation point of } \vec{y} \implies$$
$$z \text{ is an } (\mathcal{I},\mathcal{J})\text{-accumulation point of } \vec{x}.$$

(4) Being a $\mathcal{J}$-subsequence is a transitive relation, i.e.,

$$\vec{z} \hookrightarrow_{\mathcal{J}} \vec{y} \hookrightarrow_{\mathcal{J}} \vec{x} \implies \vec{z} \hookrightarrow_{\mathcal{J}} \vec{x}.$$

Proof. *(1) $\implies$ (2)*: Let z be the $(\mathcal{I},\mathcal{J})$-limit of $\vec{x}$, so for every $\mathcal{I}$-neighbourhood $\mathcal{O}$ of z there is a $\mathcal{J}^*$-set C such that $x_\alpha \in \mathcal{O}$ for every $\alpha \in C$. Let $S \in \mathcal{J}^+$ be the index set of $\vec{y}$. Since $\mathcal{J}$ is closed under re-enumerations, $\pi_S''(C \cap S) \in \mathcal{J}^*$, and therefore, z is the $(\mathcal{I},\mathcal{J})$-limit of $\vec{y}$.

(1) $\implies$ (3): Let z be an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{y}$, so for every $\mathcal{I}$-neighbourhood $\mathcal{O}$ of z there is a $\mathcal{J}^+$-set S' such that $y_\alpha \in \mathcal{O}$ for every $\alpha \in S'$. Let $S \in \mathcal{J}^+$ be the index set of $\vec{y}$. Since $\mathcal{J}$ is closed under re-enumerations, $\pi_S^{-1}S' \in \mathcal{J}^+$, and therefore, z is an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{x}$.

(1) $\implies$ (4): Let $S \in \mathcal{J}^+$ be the index set of $\vec{y}$ as a subsequence of $\vec{x}$, and let $S' \in \mathcal{J}^+$ be the index set of $\vec{z}$ as a subsequence of $\vec{y}$. Since $\mathcal{J}$ is closed under re-enumerations, we have $\pi_S^{-1}S' \in \mathcal{J}^+$, hence $\vec{z}$ is a $\mathcal{J}$-subsequence of $\vec{x}$.

(2) $\implies$ (1): Assume that $\mathcal{J}$ is not closed under re-enumerations, as witnessed by S and by A. Let $x_\alpha = \mathbf{1}$ for $\alpha \in A$, and let $x_\alpha = \mathbf{0}$ for $\alpha \notin A$. Clearly, $\mathbf{0}$ is the $(\mathcal{I},\mathcal{J})$-limit of $\vec{x}$. Now $\vec{y} := \{x_\alpha \mid \alpha \in S\}$ is a $\mathcal{J}$-subsequence, and $\mathbf{0}$ is not the $(\mathcal{I},\mathcal{J})$-limit of $\vec{y}$, because $\pi_S''A \in \mathcal{J}^+$.

(3) $\implies$ (1): Assume that $\mathcal{J}$ is not closed under re-enumerations, as witnessed by S and by A. Let $x_\alpha = \mathbf{1}$ for $\alpha \in A$ and $x_\alpha = \mathbf{0}$ for $\alpha \notin A$. Clearly $\mathbf{1}$ is not an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{x}$. Now $\vec{y} := \{x_\alpha \mid \alpha \in S\}$ is a $\mathcal{J}$-subsequence, and $\mathbf{1}$ is an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{y}$, because $\pi_S''A \in \mathcal{J}^+$.

(4) $\implies$ (1): Assume that $\mathcal{J}$ is not closed under re-enumerations, as witnessed by S and by A. Let $\vec{x}$ be a sequence, let $\vec{y} := \{x_\alpha \mid \alpha \in S\}$, and let $\vec{z} := \{x_\alpha \mid \alpha \in A\}$. Then, $\vec{z} \hookrightarrow_{\mathcal{J}} \vec{y} \hookrightarrow_{\mathcal{J}} \vec{x}$, however $\vec{z} \hookrightarrow_{\mathcal{J}} \vec{x}$ does not hold. $\square$

It follows that bd_κ-subsequences have all these nice properties, while NS_κ-subsequences do not have them (at least if $\kappa > \omega_1$). But since NS_κ is almost closed under re-enumerations, the following proposition gives weaker properties which hold for NS_κ-subsequences.

Proposition 3.24. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on κ. Then, the following are equivalent:*

(1) $\mathcal{J}$ almost closed under re-enumerations.

(2) Whenever $\vec{y}$ is a $\mathcal{J}$-subsequence of $\vec{x}$, then
$$z \text{ is } (\mathcal{I},\mathcal{J})\text{-limit of } \vec{x} \implies z \text{ is } (\mathcal{I},\mathcal{J})\text{-accumulation point of } \vec{y}.$$

(3) Whenever $\vec{y}$ is a $\mathcal{J}$-subsequence of $\vec{x}$, then
$$z \text{ is } (\mathcal{I},\mathcal{J})\text{-limit of } \vec{y} \implies z \text{ is } (\mathcal{I},\mathcal{J})\text{-accumulation point of } \vec{x}.$$

Proof. *(1) $\Longrightarrow$ (2)*: Let $\mathcal{O}$ be an $\mathcal{I}$-open set containing x. Since $\vec{x}$ $(\mathcal{I},\mathcal{J})$-converges to x, x_α is an element of $\mathcal{O}$ for a $\mathcal{J}^*$-set of α's. But then, since $\mathcal{J}$ is almost closed under re-enumerations this implies that y_α is an element of $\mathcal{O}$ for a $\mathcal{J}^+$-set of α's, yielding that x is an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{y}$, as desired.

(1) $\Longrightarrow$ (3): Let z be an $(\mathcal{I},\mathcal{J})$-limit of $\vec{y}$, so for every $\mathcal{I}$-neighbourhood $\mathcal{O}$ of z there is a $\mathcal{J}^*$-set C such that $y_\alpha \in \mathcal{O}$ for every $\alpha \in C$. Let $S \in \mathcal{J}^+$ be the index set of $\vec{y}$. Since $\mathcal{J}$ is closed under re-enumerations, $\pi_S^{-1} C \in \mathcal{J}^+$, and therefore z is an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{x}$.

(2) $\Longrightarrow$ (1): Assume that $\mathcal{J}$ is not almost closed under re-enumerations and let S and A be witnesses for that. Let $x_\alpha = \mathbf{1}$ for $\alpha \in A$ and $x_\alpha = \mathbf{0}$ for $\alpha \notin A$. Clearly, $\mathbf{0}$ is the $(\mathcal{I},\mathcal{J})$-limit of $\vec{x}$. Now $\vec{y} := \{x_\alpha \mid \alpha \in S\}$ is a $\mathcal{J}$-subsequence, and $\mathbf{0}$ is not an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{y}$, because $\pi_S'' A \in \mathcal{J}^*$. In fact, $\mathbf{1}$ is the $(\mathcal{I},\mathcal{J})$-limit of $\vec{y}$.

(3) $\Longrightarrow$ (1): Assume that $\mathcal{J}$ is not almost closed under re-enumerations and let S and A be witnesses for that. Let $x_\alpha = \mathbf{1}$ for $\alpha \in A$ and $x_\alpha = \mathbf{0}$ for $\alpha \notin A$. Clearly $\mathbf{1}$ is not an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{x}$. Now $\vec{y} := \{x_\alpha \mid \alpha \in S\}$ is a $\mathcal{J}$-subsequence of $\vec{x}$, and $\mathbf{1}$ is the $(\mathcal{I},\mathcal{J})$-limit of $\vec{y}$, because $\pi_S'' A \in \mathcal{J}^*$. □

Remark 3.25. If $\mathcal{J}$ is almost closed under re-enumerations, but not closed under re-enumerations, then the example from the proof of *(2) $\Longrightarrow$ (1)* of Proposition 3.23 yields sequences $\vec{y} \hookrightarrow_{\mathcal{J}} \vec{x}$ such that x is the $(\mathcal{I},\mathcal{J})$-limit of $\vec{x}$, such that x is an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{y}$ (by (2) of Proposition 3.24), and such that x is not the unique $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{y}$.

Let us provide an example of a strong failure of Proposition 3.23.

Proposition 3.26. *Let $\mathcal{I}$ be an ideal on κ that contains an unbounded subset of κ, and let $\mathcal{J}$ be an ideal that is not closed under re-enumerations. Then, there is a sequence $\vec{x}$ with no $(\mathcal{I},\mathcal{J})$-accumulation points, which has a $\mathcal{J}$-subsequence $\vec{y}$ with a unique $(\mathcal{I},\mathcal{J})$-accumulation point, such that $\vec{y}$ has an $(\mathcal{I},\mathcal{J})$-convergent $\mathcal{J}$-subsequence $\vec{z}$.* [12]

Proof. Let S and A be witnesses for $\mathcal{J}$ not being closed under re-enumerations. By Observation 1.2, we may pick disjoint $\mathcal{I}$-cones $[f_\alpha]$ for $\alpha < \kappa$. Define a sequence $\vec{x}$ by taking x_α to have constant value on A, and for $\alpha \notin A$, pick some $x_\alpha \in [f_\alpha]$. Then, $\vec{x}$ has no $(\mathcal{I},\mathcal{J})$-accumulation points. However, letting $\langle \beta_\alpha \mid \alpha < \kappa \rangle$ enumerate S in increasing order, and letting $y_\alpha = x_{\beta_\alpha}$ for $\alpha < \kappa$, we obtain a sequence $\vec{y}$ with unique $(\mathcal{I},\mathcal{J})$-accumulation point, which is the constant value on the $\mathcal{J}^+$-set of

[12] Note that, since NS_κ is almost closed under re-enumerations, this $\mathcal{J}$-subsequence $\vec{y}$ cannot be $(\mathcal{I},\mathcal{J})$-convergent, for otherwise its $(\mathcal{I},\mathcal{J})$-limit were an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{x}$. This also shows again that being a $\mathcal{J}$-subsequence is not a transitive relation, because if $\vec{z}$ were a $\mathcal{J}$-subsequence of $\vec{x}$, its $(\mathcal{I},\mathcal{J})$-limit would be an $(\mathcal{I},\mathcal{J})$-accumulation point of $\vec{x}$ by Proposition 3.24.

indices $\pi_S'' A$. The final statement of the proposition now follows by considering the $\mathcal{J}$-subsequence of $\vec{y}$ with domain $\pi_S'' A$. □

We give yet another example of a sequence that does not $\mathcal{I}^2$-converge, for $\mathcal{I} = \mathsf{NS}_\kappa$. This should be compared to Observation 3.3, (1) and to Observation 3.13.

Corollary 3.27. *If $\kappa > \omega_1$ is regular and $\mathcal{I} = \mathsf{NS}_\kappa$, then if $\langle s_\alpha \mid \alpha < \kappa \rangle$ is the increasing enumeration of $\mathsf{cof}\,\omega$, the sequence $\vec{s} = \langle \{s_\alpha\} \mid \alpha < \kappa \rangle$ does not $\mathcal{I}^2$-converge.*

Proof. The sequence $\vec{s}$ could only $\mathcal{I}^2$-converge to $\mathbf{0}$. Let π denote the transitive collapsing map of $\mathsf{cof}\,\omega$, and let A be a nonstationary subset of $\mathsf{cof}\,\omega$ for which $\pi'' A$ is stationary, as provided by Proposition 3.20. This means that there is a stationary set T of indices α for which $s_\alpha \in A$. Assuming that $\vec{s}$ does indeed $\mathcal{I}^2$-converge to $\mathbf{0}$, there is a club C of indices α for which $\{s_\alpha\} \in [\mathbf{0} \restriction A]$, i.e., $s_\alpha \notin A$. Since $T \cap C \neq \emptyset$ however, this yields a contradiction. □

4. Connections with topologies generated by forcing partial orders

Ideal topologies can be seen as a special case of topologies connected to *tree-like* forcing notions, that we will describe below, following [4]. [13] While this connection may be interesting enough to be mentioned here in its own right, we will also make use of this connection later on. We start with the following definition from [4], slightly adapted to our present purposes.

Definition 4.1. [4, Definition 3.1] A forcing notion $\mathbb{P}$ is called *κ-tree-like* if

(1) conditions of $\mathbb{P}$ are pruned and $<\kappa$-closed trees on $2^{<\kappa}$, ordered by inclusion,
(2) $2^{<\kappa} \in \mathbb{P}$, and whenever $T \in \mathbb{P}$ and $s \in T$, then $\{t \in T \mid s \subseteq t \text{ or } t \subseteq s\} \in \mathbb{P}$, and
(3) if $\langle T_\alpha \mid \alpha < \lambda \rangle$ is a decreasing sequence of conditions of length $\lambda < \kappa$, then $\bigcap_{\alpha<\lambda} T_\alpha \in \mathbb{P}$.

In many cases, κ-tree-like forcing notions induce natural topologies on 2^κ. The following is a minor modification of [4, Definition 3.6, 1], as the definition used in that paper seems to be slightly too weak in order to yield a topology base.

Definition 4.2. A κ-tree-like notion of forcing $\mathbb{P}$ is *topological* if $\{[T] \mid T \in \mathbb{P}\}$ forms a topology base for 2^κ, that is whenever $S, T \in \mathbb{P}$, and $x \in [S] \cap [T]$, then there is $R \in \mathbb{P}$ such that $x \in [R] \subseteq [S] \cap [T]$. In this case we call the topology generated by the basic open sets of the form $[T]$ for $T \in \mathbb{P}$ the *topology generated by* $\mathbb{P}$, or the $\mathbb{P}$*-topology.*

Ideal topologies are generated by generalizations of Grigorieff forcing to uncountable cardinals:

Definition 4.3. Let κ be an infinite cardinal and let $\mathcal{I}$ be an ideal on κ. *Grigorieff forcing with the ideal* $\mathcal{I}$ is the notion of forcing consisting of conditions from $\mathsf{Fn}_\mathcal{I}$, ordered by reverse inclusion.

At first sight, Grigorieff forcing may not seem to be a κ-tree-like notion of forcing, however it can be represented as one: We identify a condition $f \in \mathsf{Fn}_\mathcal{I}$ with a tree T on $2^{<\kappa}$, which we construct by induction on α as follows: Given $t \in T$ of order-type α, let $t^\frown 0 \in T$ if and only if $f(\alpha) \neq 1$, and let $t^\frown 1 \in T$ if and only if $f(\alpha) \neq 0$ (these are both supposed to include the cases when α is not in the domain of f, i.e., t is splitting if and only if α is not in the domain of f). At limit levels α, we extend

[13] Even more generally, this may be seen as a special case of the natural topology that can be constructed on the Stone space of any partial order, see for example [6, Section 3].

every branch through the tree constructed so far. It is straightforward to check that the resulting forcing is indeed a κ-tree-like forcing, using that $\mathcal{I} \supseteq \mathsf{bd}_\kappa$ and that $\mathcal{I}$ is $<\kappa$-complete. Note moreover that if T is the tree on $2^{<\kappa}$ corresponding to the condition $f \in \mathsf{Fn}_\mathcal{I}$, then $[T] = [f]$.

For any ideal $\mathcal{I}$, Grigorieff forcing with the ideal $\mathcal{I}$ is topological: Given $f, g \in \mathsf{Fn}_\mathcal{I}$, assuming that $[f] \cap [g] \neq \emptyset$, it follows that $f \cup g \in \mathsf{Fn}_\mathcal{I}$, and that $[f \cup g] = [f] \cap [g]$.

The following is now immediate by comparing the basic open sets (which are the same) used to generate the respective topologies:

Observation 4.4. *The topology on 2^κ generated by Grigorieff forcing with the ideal $\mathcal{I}$ is exactly the topology $\tau_\mathcal{I}$.* □

Another notion of forcing that is closely connected to $\mathcal{I}$-topologies is *κ-Silver forcing*, which is sometimes also called *κ-club-Silver forcing*:

Definition 4.5. Given a regular cardinal κ, κ-Silver forcing is the notion of forcing consisting of all conditions $f \in \mathsf{Fn}_\mathcal{I}$ for which $\kappa \setminus \operatorname{dom} f$ is a club subset of κ, ordered by reverse inclusion.

Note that clearly, κ-Silver forcing is a dense subset of Grigorieff forcing with the ideal NS_κ. In fact, whenever p is a condition in the latter forcing and $x \in 2^\kappa$ is such that $p \subseteq x$, then p can be extended to a condition $q \subseteq x$ in κ-Silver forcing. This implies both that κ-Silver forcing can be represented as a κ-tree-like notion of forcing (see also [4, Example 3.2, 6]), that κ-Silver forcing is topological, [14] and that it generates the same topology as does Grigorieff forcing with NS_κ, namely the Edinburgh topology on 2^κ.

Let us finally remark that it is straightforward to formulate and verify an analogue of Proposition 2.1 for the more general setting of topologies that are generated by κ-tree like forcing notions. We will however leave this to the interested reader, for we will not need it in the remainder of our paper.

5. On κ-Silver forcing and Axiom A^*

In [4, Definition 3.6, 2], a strengthening of Axiom A is defined as follows:

Definition 5.1. We say that a κ-tree-like notion of forcing $\mathbb{P}$ satisfies Axiom A^* if there are orderings $\{\leq_\alpha \mid \alpha < \kappa\}$ with $\leq_0 = \leq$, satisfying:[15]

(1) $g \leq_\beta f$ implies $g \leq_\alpha f$ for all $\alpha \leq \beta$.
(2) If $\langle f_\alpha \mid \alpha < \lambda \rangle$ is a sequence of conditions with $\lambda \leq \kappa$ satisfying that $f_\beta \leq_\alpha f_\alpha$ for all $\alpha \leq \beta$, then there is $f \in \mathbb{P}$ such that $f \leq_\alpha f_\alpha$ for all $\alpha < \lambda$.
(3) For all $f \in \mathbb{P}$, D dense below f in $\mathbb{P}$, and $\alpha < \kappa$, there exists $E \subseteq D$ and $g \leq_\alpha f$ such that $|E| \leq \kappa$ and E is predense below g, such that[16]additionally $[g] \subseteq \bigcup\{[h] \mid h \in E\}$.

Many of our subsequent proofs are going to work either under the assumption that κ is inaccessible, or that $\diamondsuit_\kappa$ holds. Recall that, for κ being regular uncountable, $\diamondsuit_\kappa$ holds if there exists a *$\diamondsuit_\kappa$-sequence* $\langle A_\alpha \mid \alpha < \kappa \rangle$, that is, for every $A \subseteq \kappa$, there is a stationary set of α's such that $A_\alpha = A \cap \alpha$.

Definition 5.2. We say that κ is *simple*, if κ is inaccessible or $\diamondsuit_\kappa$ holds.

[14]The opposite is wrongly claimed in [4] without justification, see for example [4, Table 1].

[15]Since this definition is about κ-tree-like notions of forcing, one would usually use S and T to denote the elements of the forcings. We use f and g instead, for we will only be concerned with the special case of Grigorieff forcing.

[16]Without this additional clause, this would be the usual Axiom A.

The assumption $2^{<\kappa} = \kappa$ is very common in higher descriptive set theory. It is neccessary (but not quite sufficient) for κ being simple.

Observation 5.3. *If κ is simple, then $2^{<\kappa} = \kappa$.*

On the other hand, by results of Shelah (see [8]), if κ is a successor cardinal and $\kappa > \omega_1$, then $2^{<\kappa} = \kappa$ implies that $\diamondsuit_\kappa$ holds. For $\kappa = \omega_1$, however, it is consistent that $2^{<\kappa} = \kappa$ (i.e., CH holds) and $\diamondsuit_\kappa$ (i.e., $\diamondsuit$) fails. The case $\kappa = \omega_1$ is therefore particularly interesting for potential counterexamples.

It is well-known that the standard proof to verify that Silver forcing (on ω) satisfies Axiom A can be adapted to show that κ-Silver forcing satisfies Axiom A in case κ is inaccessible, and it is easy to see from the proof that in fact it even yields Axiom A^*. We want to show that the same conclusion also holds under the assumption $\diamondsuit_\kappa$. We will then apply this result in the next section.

Theorem 5.4. *If $\diamondsuit_\kappa$ holds, then κ-Silver forcing satisfies Axiom A^*. (So κ-Silver forcing satisfies Axiom A^* whenever κ is simple.)*

Proof. Let $\langle \mathbb{P}, \leq \rangle$ denote κ-Silver forcing. For any $\alpha < \kappa$, let $g \leq_\alpha f$ if $g \leq f$ and the first α-many elements of the complements of the domains of f and of g are the same. It is clear that Items (1) and (2) in Definition 5.1 are thus satisfied, and we only have to verify Item (3).

(If κ is inaccessible, this follows from standard arguments, as mentioned above; also compare with the proofs of Theorem 6.2 and Lemma 7.3 in which we provide details for both the case that κ is inaccessible and the case that $\diamondsuit_\kappa$ holds.)

Fix a $\diamondsuit_\kappa$-sequence $\langle A_i \mid i < \kappa \rangle$. Let $f \in \mathbb{P}$, let $\alpha < \kappa$, and let $D \subseteq \mathbb{P}$ be dense below f. We inductively construct a decreasing sequence $\langle f_i \mid i \leq \kappa \rangle$ of conditions in $\mathbb{P}$ with $f_i = f$ for $i \leq \alpha$, and a sequence $\langle \alpha_i \mid i < \kappa \rangle$ of ordinals with the property that $\langle \alpha_j \mid j \leq i \rangle$ enumerates the first $(i+1)$-many elements of $\kappa \setminus \operatorname{dom}(f_i)$ for every $i \leq \kappa$, as follows. Let $\langle \alpha_i \mid i \leq \alpha \rangle$ enumerate the first $\alpha + 1$-many elements of the complement of the domain of f.

Assume that we have constructed f_i, and also α_j for $j \leq i$.

Using that D is dense below f, let $g_i^0 \leq f_i$ be such that

- $g_i^0(\alpha_j) = A_i(j)$ for all $j < i$,
- $g_i^0(\alpha_i) = 0$, and
- $g_i^0 \in D$,

and let $g_i^1 \leq g_i^0 \restriction (\operatorname{dom}(g_i^0) \setminus \{\alpha_i\})$ be such that

- $g_i^1(\alpha_i) = 1$, and
- $g_i^1 \in D$.

Let $f_{i+1} = g_i^1 \restriction (\operatorname{dom}(g_i^1) \setminus \{\alpha_j \mid j \leq i\})$, and note that $f_{i+1} \leq_i f_i$, for $\{\alpha_j \mid j < i\}$ is contained in the complement of $\operatorname{dom}(f_{i+1})$. Let α_{i+1} be the least element of $\kappa \setminus \operatorname{dom}(f_{i+1})$ above α_i.

For limit ordinals $i \leq \kappa$, let $f_i = \bigcup_{j<i} f_j$, and if $i < \kappa$, let $\alpha_i = \bigcup_{j<i} \alpha_j$ be the least element of $\kappa \setminus \operatorname{dom}(f_i)$, using that the intersection of $<\kappa$ many club subsets of κ is again a club subset of κ. Let $E = \{g_i^0 \mid i < \kappa\} \cup \{g_i^1 \mid i < \kappa\}$.

In order to verify Axiom A, first note that $f_\kappa \leq_\alpha f$. Now we want to show that E is predense below f_κ. Thus, let $h \leq f_\kappa$ be given. Using the properties of our diamond sequence, pick $i < \kappa$ such that $i \geq \alpha$, and such that for all $j < i$ with $\alpha_j \in \operatorname{dom}(h)$, $A_i(j) = h(\alpha_j)$. Pick $\delta \in \{0, 1\}$ such that $h(\alpha_i) = \delta$ in case $\alpha_i \in \operatorname{dom}(h)$. Then, g_i^δ is compatible to h, as desired.

In order to check the additional property for Axiom A^*, note that any extension x of f_κ to a total function from κ to 2 can be treated in the same way as h above, yielding some $i < \kappa$ and $\delta \in \{0, 1\}$ such that $x \in [g_i^\delta]$. $\square$

6. Edinburgh cones and $\mathcal{I}$-meagerness

In this section, we provide two results on properties of simple cardinals with respect to Edinburgh cones, and two consequences on Edinburgh meager sets.

The first result is due to S. Friedman, Khomskii and Kulikov ([4, Section 3.2]) in case κ is inaccessible, however making use of Theorem 5.4, it follows to hold true also under the assumption of $\diamondsuit_\kappa$:

Theorem 6.1. *If κ is simple, then every κ-intersection of Edinburgh open dense subsets of 2^κ contains an Edinburgh open dense set,* [17] *i.e., the Edinburgh meager sets are the same as the Edinburgh nowhere dense sets.*

Proof. Let us first observe that the two conclusions of the theorem are simply equivalent, using that the Edinburgh closure of an Edinburgh nowhere dense set (that is, the smallest Edinburgh closed set containing it) is still Edinburgh nowhere dense. [18] It follows from Theorem 5.4, that κ-Silver forcing satisfies Axiom A^*, and we have observed in Section 4 that κ-Silver forcing is topological and generates the Edinburgh topology on 2^κ. It is straightforward to derive from Axiom A^* that the Edinburgh meager sets are the same as the Edinburgh nowhere dense sets, as desired (see also [4, Lemma 3.8]). [19] □

Note that the above theorem is a strengthening (of a special case) of Corollary 1.4.

We do not know the answer to the following question however.

Question 5. *Is it consistent that the Edinburgh meager sets are not the same as the Edinburgh nowhere dense sets? (By the above theorem, κ would necessarily be not simple.)*

Note that the proof of Theorem 6.1 actually yields the following more general fact: If Grigorieff forcing with $\mathcal{I}$ satisfies Axiom A^*, then $\mathcal{I}$-meager is the same as $\mathcal{I}$-nowhere dense.

Question 6. *Are there ideals $\mathcal{I}$ other than NS_κ such that Grigorieff forcing with $\mathcal{I}$ satisfies Axiom A^* (or such that $\mathcal{I}$-meager implies $\mathcal{I}$-nowhere dense)?*

The arguments for the next result are essentially the same as the argument in the proof of [4, Lemma 4.9, 6] in case κ is inaccessible, and they can again be generalized to include the case when $\diamondsuit_\kappa$ holds. Since the context and notation in [4] are somewhat different to ours, we would like to include a proof of this theorem in both cases, for the benefit of our readers.

Theorem 6.2. *Let κ be simple. If $s \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$, then every κ-intersection of open dense subsets of 2^κ contains an Edinburgh cone $[f]$ with $[f] \subseteq [s]$. In other words, every comeager subset of 2^κ contains a dense set that is Edinburgh open.*

Proof. If κ is not inaccessible, $\diamondsuit_\kappa$ holds by the assumption that κ is simple; in this case, let us fix a $\diamondsuit_\kappa$-sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$.

Let $s \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$, and let $\langle D_i \mid i < \kappa \rangle$ be a sequence of open dense subsets of 2^κ. Since the intersection of less than κ many open dense subsets of 2^κ is open

[17] In order to avoid any possible confusion, let us remark that by a convention made earlier on in this paper, *Edinburgh open dense* means Edinburgh open and Edinburgh dense.

[18] We mostly provide the formulation via Edinburgh open dense sets here as well in order to relate this result more obviously to the statement of Theorem 6.2 below.

[19] It would be possible, however less informative, to verify Theorem 6.1 without making use of Axiom A^*, by extending on and making appropriate modifications to the proof of Theorem 6.2 below.

dense, we can assume that $D_j \subseteq D_i$ for $j > i$. By induction on $i < \kappa$, we define a $\subseteq$-increasing sequence of functions $\langle f_i \mid i < \kappa\rangle$ in $\mathsf{Fn}_{\mathsf{bd}_\kappa}$ with $f_0 = s$, as well as a club subset $C = \{\alpha_j \mid j < \kappa\}$ of κ that is disjoint from $\operatorname{dom} f_i$ for each $i < \kappa$. Then, letting $f := \bigcup_{i<\kappa} f_i$, we show that $[f] \subseteq \bigcap_{i<\kappa} D_i$, and since $[f] \subseteq [s]$, this finishes the proof.

Let $f_0 = s$, and pick $\alpha_0 > \sup\operatorname{dom} s$. Let $i < \kappa$, and assume that $\langle \alpha_j \mid j \leq i\rangle$ and $f_i \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$ are defined, such that $\{\alpha_j \mid j \leq i\}$ is a closed subset of κ that is disjoint from $\operatorname{dom} f_i$. For the successor step we consider two cases:

Case 1: κ is inaccessible – Let $\prec$ be a wellorder of 2^{i+1}. For $t \in 2^{i+1}$, by induction on $\prec$, we pick $g_i^t \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$ such that

(1) g_i^t extends f_i,
(2) $g_i^t(\alpha_j) = t(j)$ for $j \leq i$,
(3) g_i^t extends g_i^u on $\operatorname{dom}(g_i^u) \setminus \{\alpha_j \mid j \leq i\}$ whenever $u \in 2^{i+1}$ and $u \prec t$, and
(4) $[g_i^t] \subseteq D_i$.

This is possible because κ is inaccessible, hence in particular $|2^{i+1}| < \kappa$, and because D_i is open dense. Let

$$f_{i+1} := \bigcup_{t \in 2^{i+1}} (g_i^t \restriction (\operatorname{dom} g_i^t \setminus \{\alpha_j \mid j \leq i\})),$$

and let $\alpha_{i+1} > \alpha_i$ such that $\alpha_{i+1} > \sup\operatorname{dom}(f_{i+1})$. Note that our above construction ensures that $f_{i+1} \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$, and that $[f_{i+1}] \subseteq D_i$, since $[f_{i+1}] \subseteq \bigcup_{t\in 2^{i+1}} [g_i^t] \subseteq D_i$.

Case 2: $\diamondsuit_\kappa$ holds – Using that D_i is open dense, pick $h_i^0 \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$ such that

(1) h_i^0 extends f_i,
(2) $h_i^0(\alpha_j) = A_i(j)$ for $j < i$,
(3) $h_i^0(\alpha_i) = 0$, and
(4) $[h_i^0] \subseteq D_i$.

Now, using again that D_i is open dense, pick $h_i^1 \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$ such that

(1) h_i^1 extends h_i^0 on $\operatorname{dom} h_i^0 \setminus \{\alpha_i\}$,
(2) $h_i^1(\alpha_i) = 1$, and
(3) $[h_i^1] \subseteq D_i$.

Let $f_{i+1} = h_i^1 \restriction (\operatorname{dom} h_i^1 \setminus \{\alpha_j \mid j \leq i\})$, and pick some $\alpha_{i+1} > \sup\operatorname{dom} f_{i+1}$.

Now again in both cases, for limit ordinals $i \leq \kappa$, let $f_i = \bigcup_{j<i} f_j$ and, in case $i < \kappa$, let $\alpha_i = \sup_{j<i} \alpha_j < \kappa$, by the regularity of κ. Thus, $C = \{\alpha_i \mid i < \kappa\}$ is a club subset of κ, and, letting $f = f_\kappa$, $\operatorname{dom}(f) \subseteq \kappa \setminus C$ is nonstationary, yielding that $f \in \mathsf{Fn}_{\mathsf{NS}_\kappa}$.

Our above construction ensured that $[f] \subseteq \bigcap_{i<\kappa} D_i$: In the case of κ inaccessible, this holds by construction, as mentioned above. Assume thus that κ is a successor cardinal for which $\diamondsuit_\kappa$ holds. Given $x \in [f]$, let $A = \{i < \kappa \mid x(\alpha_i) = 1\}$. Since $\langle A_\alpha \mid \alpha < \kappa\rangle$ is a $\diamondsuit_\kappa$-sequence there exists a stationary set S such that $A \cap i = A_i$ for $i \in S$. For any such i, by our above construction of f_{i+1}, we have $x \in [h_i^0] \subseteq D_i$ if $x(\alpha_i) = 0$ and $x \in [h_i^1] \subseteq D_i$ if $x(\alpha_i) = 1$. In both cases, $x \in D_i$. Since we assumed the sequence $\langle D_i \mid i < \kappa\rangle$ to be $\subseteq$-decreasing, and S is unbounded, this yields that x is in the intersection of all the D_i, as desired. □

The above now allows us to easily infer the following:

Corollary 6.3. *If κ is simple, $X \subseteq 2^\kappa$ has the Baire property and is $\mathcal{I}$-meager for $\mathcal{I} \supseteq \mathsf{NS}_\kappa$, then X is meager.*

Proof. Assume that X has the Baire property and is not meager. We show that X is not $\mathcal{I}$-meager. By our assumptions on X, there is an $s \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$ such that $X \cap [s]$ is comeager in $[s]$. In other words: there exists a sequence $\langle D_\alpha \mid \alpha < \kappa\rangle$ of open dense sets such that $\bigcap_{\alpha<\kappa} D_\alpha \cap [s] \subseteq X$. Applying Theorem 6.2, there exists $f \supseteq s$, $f \in \mathsf{Fn}_{\mathsf{NS}_\kappa} \subseteq \mathsf{Fn}_{\mathcal{I}}$ with $[f] \subseteq \bigcap_{\alpha<\kappa} D_\alpha \cap [s] \subseteq X$, but $[f]$ is not $\mathcal{I}$-meager by Baire Category for the $\mathcal{I}$-topology (see Proposition 1.3), thus X is not $\mathcal{I}$-meager. □

Again, we do not know the following:

Question 7. *Is it consistent that there is $X \subseteq 2^\kappa$ with the Baire property which is Edinburgh meager, but not meager? In particular: Is the above consistent for $\kappa = \omega_1$ together with $2^{<\kappa} = \kappa$ (by Corollary 6.3, $\diamondsuit$ has to fail)?*

We will also apply Theorem 6.2 once again in Section 9 below.

7. The reaping number and some of its variants

In this section, we will take a small detour in order to investigate some cardinal invariants of the higher Cantor space 2^κ (with our results however also applying to the classical Cantor space 2^ω), and we will apply these results in our later sections. We thus assume that κ is a regular infinite cardinal. In particular, we also allow for $\kappa = \omega$. Remember that for $a, b \in \mathsf{ub}_\kappa$, *$a$ splits b* if $a \cap b$ and $b \setminus a$ are both of size κ.

Definition 7.1.

- An *unsplit family* at κ is a set $F \subseteq \mathsf{ub}_\kappa$ for which there is no $a \subseteq \kappa$ such that for all $b \in F$, a splits b.
- The *reaping number* $\mathfrak{r}(\kappa)$ is the smallest size of an unsplit family at κ.
- A *strongly unsplit family* at κ is a set $F \subseteq \mathsf{ub}_\kappa$ such that for every $a \subseteq \kappa$, there is $b \in F$ for which either $a \cap b = \emptyset$ or $b \setminus a = \emptyset$.
- We let $\mathfrak{R}(\kappa)$ denote the smallest size of a strongly unsplit family at κ.
- A *cone covering family* at κ is a set $\mathcal{F} \subseteq \mathsf{Fn}_{\mathsf{ub}_\kappa}$ such that
$$\bigcup_{f\in\mathcal{F}} [f] = 2^\kappa.$$
- $R(\kappa)$ is the smallest size of a cone covering family at κ.
- We let $R^*(\kappa)$ denote the smallest size of a family $\mathcal{F} \subseteq \mathsf{Fn}_{\mathsf{ub}_\kappa}$ such that there exists a comeager set $X \subseteq 2^\kappa$ with $X \subseteq \bigcup_{f\in\mathcal{F}}[f]$.

The next lemma collects some easy basic facts about these cardinal invariants, together with the result that $\mathfrak{R}(\kappa) = R(\kappa)$.

Lemma 7.2. *$\kappa^+ \leq \mathfrak{r}(\kappa) \leq \mathfrak{R}(\kappa) = R(\kappa) \leq 2^{<\kappa} \cdot \mathfrak{r}(\kappa) \leq 2^\kappa$, and $\kappa^+ \leq R^*(\kappa) \leq R(\kappa)$.*

Proof. It is well-known (and very easy to check) that $\kappa^+ \leq \mathfrak{r}(\kappa) \leq 2^\kappa$. Clearly, every strongly unsplit family at κ is an unsplit family at κ, and this directly yields that $\mathfrak{r}(\kappa) \leq \mathfrak{R}(\kappa)$. If F is an unsplit family at κ, then
$$F' = \{a \subseteq \kappa \mid \exists b \in F\ a \subseteq b \wedge |b \setminus a| < \kappa\}$$
is clearly a strongly unsplit family at κ, of size $2^{<\kappa} \cdot |F|$, yielding that $\mathfrak{R}(\kappa) \leq 2^{<\kappa} \cdot \mathfrak{r}(\kappa)$. That $R^*(\kappa) \leq R(\kappa)$ is immediate from the definitions. To see that $\kappa^+ \leq R^*(\kappa)$ note that $[f]$ is nowhere dense for every $f \in \mathsf{Fn}_{\mathsf{ub}_\kappa}$ (see Proposition 8.1), hence the union of κ many such cones is meager. Therefore it cannot cover any comeager set, because 2^κ is not meager by the Baire Category Theorem for the bounded topology. It remains to show that $\mathfrak{R}(\kappa) = R(\kappa)$.

$R(\kappa) \leq \mathfrak{R}(\kappa)$: Let F be a strongly unsplit family at κ. Remember that for any sets A and x, c_x^A denotes the function with domain A and constant value x. We show that $\{c_i^b \mid b \in F, i \in 2\}$ is a cone covering family at κ. Since F is a strongly unsplit family, for every $x \in 2^\kappa$, there is $b \in F$ and $i \in 2$ such that $x^{-1}(i) \cap b = \emptyset$. Therefore, $x \in [c_{1-i}^b]$.

$\mathfrak{R}(\kappa) \leq R(\kappa)$: Let $\mathcal{F}$ be a cone covering family at κ. Let

$$F := \{f^{-1}(i) \mid f \in \mathcal{F}, i \in 2\} \cap \mathsf{ub}_\kappa.$$

We show that F is a strongly unsplit family at κ. Let $a \subseteq \kappa$. Since $\mathcal{F}$ is a cone covering family, there is $f \in \mathcal{F}$ with $\chi_a \in [f]$. Thus, $f^{-1}(\{1\}) \subseteq a$, and hence $f^{-1}(\{1\}) \setminus a = \emptyset$. On the other hand, $f^{-1}(\{0\}) \subseteq \kappa \setminus a$, hence $f^{-1}(\{0\}) \cap a = \emptyset$. Since $\operatorname{dom} f \in \mathsf{ub}_\kappa$, either $f^{-1}(\{1\})$ or $f^{-1}(\{0\})$ has to be an unbounded subset of κ, and hence an element of F. □

This shows that in particular if $2^{<\kappa} \leq \mathfrak{r}(\kappa)$, then $\mathfrak{r}(\kappa) = \mathfrak{R}(\kappa) = R(\kappa)$. Our next result will show that if κ is simple or $\kappa = \omega$, $R^*(\kappa)$ is equal to the other invariants as well. We first need an auxiliary result:

Lemma 7.3. *Let κ be simple or $\kappa = \omega$, and let X be a comeager subset of 2^κ. Then, there is a tree $T \subseteq 2^{<\kappa}$ of height κ such that $[T] \subseteq X$, and such that the following two properties hold:*

(a) T has uniform splitting (i.e., on each level, either all nodes are splitting nodes, or none of them is) and the set of splitting levels of T form a club[20] subset C of κ, such that $0 \in C$.
For each $i < \kappa$, let i_C^- be the largest $\delta \leq i$ with $\delta \in C$ (such a δ exists since C is closed and contains 0).

(b) For any two $x, y \in [T]$, and any $i < \kappa$, $x(i) = y(i)$ if and only if $x(i_C^-) = y(i_C^-)$. So we can define a function $p : \kappa \times 2 \to 2$, such that for any $x \in [T]$ and any $i < \kappa$, we have

$$x(i_C^-) = p(i, x(i)). \tag{2}$$

Proof. In case $\diamondsuit_\kappa$ holds, fix a $\diamondsuit_\kappa$-sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$.

Since X is a comeager subset of 2^κ, there exists a sequence $\langle D_i \mid i < \kappa \rangle$ of open dense subsets of 2^κ such that $X \supseteq \bigcap_{i<\kappa} D_i$. Since the intersection of less than κ many open dense subsets of 2^κ is again open dense, we can assume that the sequence of D_i's is $\subseteq$-decreasing.

We construct T by induction in κ many steps, by constructing a sequence $\vec{S} = \langle S_i \mid i < \kappa \rangle$ of unboundedly many levels of T; we let δ_i be the length of the elements of S_i (all sequences in S_i will have the same length) for all $i < \kappa$, and we let our club $C = \{\delta_i \mid i < \kappa\}$. Let $S_0 = \{\emptyset\}$. Given S_i, consisting of binary sequences of equal length, we will define S_{i+1} by distinguishing two cases.

Let us fix the following notation: for $u \in 2^{<\kappa}$, let $1 - u$ denote the sequence with all the bits of u flipped, i.e., $1 - u \in 2^{<\kappa}$ is such that $\operatorname{dom}(1 - u) = \operatorname{dom}(u)$ and $(1 - u)(\beta) = 1 - u(\beta)$ for each $\beta \in \operatorname{dom}(u)$.

Case 1: κ is inaccessible or $\kappa = \omega$ – Let $\lambda := |S_i|$, and let $\{t_l \mid l < \lambda\}$ be an enumeration of S_i. Note that $\lambda < \kappa$. We will define an increasing sequence $\{u_l^0 \mid l < \lambda\}$ in $2^{<\kappa}$ by induction on $l < \lambda$, and afterwards, again by induction on $l < \lambda$, we will define another increasing sequence $\{u_l^1 \mid l < \lambda\}$ in $2^{<\kappa}$.

Let $u_0^0 := \emptyset$. For $l < \lambda$, let $u_{l+1}^0 \supseteq u_l^0$ be such that

$$[t_l{}^\frown 0{}^\frown u_{l+1}^0] \subseteq D_i.$$

[20]For $\kappa = \omega$, a club subset is just an unbounded subset.

At limits $l \leq \lambda$, let $u_l^0 := \bigcup_{j<l} u_j^0$. Let $u_0^1 := 1 - u_\lambda^0$. For $l < \lambda$, let $u_{l+1}^1 \supseteq u_l^1$ be such that

$$[t_l^\frown 1^\frown u_{l+1}^1] \subseteq D_i.$$

At limits $l \leq \lambda$, let $u_l^1 := \bigcup_{j<l} u_j^1$.

Let $u^1 := u_\lambda^1$, and let $u^0 := 1 - u^1$. Finally, let

$$S_{i+1} := \{t_l^\frown 0^\frown u^0 \mid l < \lambda\} \cup \{t_l^\frown 1^\frown u^1 \mid l < \lambda\}.$$

This finishes the construction of S_{i+1} in case κ is inaccessible. Note that every node in S_i is a splitting node of T, and the elements of S_{i+1} are sequences of equal length less than κ (which we call δ_{i+1}), using that $\lambda < \kappa$. Finally, it is straightforward to check that $[s] \subseteq D_i$ for each $s \in S_{i+1}$.

Case 2: $\diamondsuit_\kappa$ holds – If $A_{\delta_i} \notin S_i$, we simply let

$$S_{i+1} = \{t^\frown 0 \mid t \in S_i\} \cup \{t^\frown 1 \mid t \in S_i\}.$$

Otherwise, we proceed as follows: Let $u_0 \in 2^{<\kappa}$ be such that $[A_{\delta_i}^\frown 0^\frown u_0] \subseteq D_i$, and let $u_1 := 1 - u_0$. Let $v_1 \in 2^{<\kappa}$ be such that $[A_{\delta_i}^\frown 1^\frown u_1^\frown v_1] \subseteq D_i$, and let $v_0 := 1 - v_1$. Finally, let

$$S_{i+1} = \{t^\frown 0^\frown u_0^\frown v_0 \mid t \in S_i\} \cup \{t^\frown 1^\frown u_1^\frown v_1 \mid t \in S_i\}.$$

This finishes the construction of S_{i+1} in case $\diamondsuit_\kappa$ holds. Note that, again, every node in S_i is a splitting node of T, and the elements of S_{i+1} are sequences of equal length less than κ (which we call δ_{i+1}).

In both cases, for limit ordinals $i < \kappa$, we take unions, i.e., we let

$$S_i = \{t \in 2^{<\kappa} \mid \exists \langle t_j \mid j < i\rangle\ t = \bigcup_{j<i} t_j \wedge t_j \in S_j \text{ for } j < i\}.$$

Note that elements of S_i are again sequences of length less than κ by the regularity of κ. Finally, let $T = \{t \mid \exists i < \kappa\ \exists s \in S_i\ t \subseteq s\}$ be the tree induced by the S_i's. It remains to check that T is as desired. It is straightforward to check from our construction that T satisfies Properties (a) and (b) in either of our two cases. We have to check that moreover, our construction ensures that $[T] \subseteq \bigcap_{i<\kappa} D_i$ (and hence $[T] \subseteq X$).

Case 1: κ is inaccessible or $\kappa = \omega$ – Given $x \in [T]$ and $i < \kappa$, note that $x \restriction \delta_{i+1} \in S_{i+1}$, so, as discussed above, $[x \restriction \delta_{i+1}] \subseteq D_i$, and hence $x \in D_i$, as desired.

Case 2: $\diamondsuit_\kappa$ holds – Given $x \in [T]$ and $i < \kappa$, let $j \geq i$ be such that $A_{\delta_j} = x \restriction \delta_j$, which is possible because $\{\delta_j \mid i \leq j < \kappa\}$ is a club and the $\diamondsuit_\kappa$-sequence guesses correctly on a stationary set. Let $\xi := x(\delta_j)$. Note that, by construction, $x \in [A_{\delta_j}^\frown \xi^\frown u_\xi^\frown v_\xi] \subseteq D_j \subseteq D_i$, as desired. □

We are now ready to show that in many cases, all the cardinal invariants introduced in this section are actually equal.

Theorem 7.4. *If κ is simple or $\kappa = \omega$, then*

$$\mathfrak{r}(\kappa) = \mathfrak{R}(\kappa) = R(\kappa) = R^*(\kappa).$$

Proof. If $\kappa = \omega$, we clearly have $2^{<\kappa} = \kappa$; if κ is simple, the same holds by Observation 5.3. So, by Lemma 7.2, we only need to show that $R^*(\kappa) \geq R(\kappa)$. Let X be a comeager subset of 2^κ and let $\mathcal{F} \subseteq \mathsf{Fn}_{\mathsf{ub}_\kappa}$ be such that $X \subseteq \bigcup_{f \in \mathcal{F}} [f]$. To finish the proof, it is enough to find such a family of the same size which covers the whole space 2^κ.

Let T, C, and p be as provided by[21] Lemma 7.3 with respect to X, and let $\{\delta_i \mid i < \kappa\}$ be an increasing enumeration of C. Since $[T] \subseteq X$ also $[T] \subseteq \bigcup_{f\in\mathcal{F}}[f]$.

By passing to a suitable function $f' \subseteq f$ (with $f' \in \mathsf{Fn}_{\mathsf{ub}_\kappa}$) for each $f \in \mathcal{F}$, we may assume that, for every $f \in \mathcal{F}$ and every $i < \kappa$, $\operatorname{dom} f$ and the ordinal interval $[\delta_i, \delta_{i+1})$ intersect in at most one element. [22]

Now, define $\Omega, \Omega' : \mathcal{F} \to \mathsf{Fn}_{\mathsf{ub}_\kappa}$ as follows: for each $f \in \mathcal{F}$, let

$$\Omega(f) := \{(i_C^-, p(i, f(i))) \mid i \in \operatorname{dom}(f)\}. \tag{3}$$

Let $\Omega'(f)$ be such that $i \in \operatorname{dom}(\Omega'(f))$ if and only if $\delta_i \in \operatorname{dom}(\Omega(f))$, and, for such i, let $\Omega'(f)(i) := \Omega(f)(\delta_i)$. Clearly, for each $f \in \mathcal{F}$, $\Omega'(f) \in \mathsf{Fn}_{\mathsf{ub}_\kappa}$, and we want to finish our argument by showing that

$$2^\kappa \subseteq \bigcup_{f\in\mathcal{F}} [\Omega'(f)].$$

To see this, let $\psi : 2^{<\kappa} \to T$ be the embedding for which for $s \in 2^{<\kappa}$, we have $\psi(s)(\delta_i) = s(i)$. [23] Let ψ also denote its canonical extension to 2^κ, i.e., for $y \in 2^\kappa$, let $\psi(y) = \bigcup_{i<\kappa} \psi(y \restriction i)$. Now let $y \in 2^\kappa$. Let $x := \psi(y)$. Since $x \in [T]$, we can fix $f \in \mathcal{F}$ such that $x \in [f]$, i.e. $x \supseteq f$. By (2) from Lemma 7.3 and (3), it follows that $x \supseteq \Omega(f)$ as well. Consequently, $y \supseteq \Omega'(f)$, i.e., $y \in [\Omega'(f)]$, thus finishing the argument. $\square$

Question 8. *Is it consistent that $R^*(\kappa) < \mathfrak{r}(\kappa)$ for any regular uncountable cardinal κ?*

8. Meager sets in ideal topologies

In this section, we investigate the notion of meagerness in ideal topologies. One point of the results of this section is to highlight some aspects of the complex relationship between the bounded topology and generalized ideal topologies on 2^κ, and in particular the Edinburgh topology. Our first simple proposition shows that meagerness does not imply $\mathcal{I}$-meagerness.

Proposition 8.1. *If $f \in \mathsf{Fn}_{\mathsf{ub}_\kappa}$, then $[f]$ is meager (in fact, closed nowhere dense). Thus, there is a meager subset of 2^κ which is not $\mathcal{I}$-meager whenever $\mathcal{I} \supsetneq \mathsf{bd}_\kappa$.*

Proof. $[f]$ is closed, for

$$2^\kappa \setminus [f] = \bigcup_{\alpha\in\operatorname{dom}(f), f(\alpha)\neq i} [\{(\alpha, i)\}]$$

is open.

Let $s \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$. Since $\operatorname{dom}(f)$ is unbounded in κ, we may pick some $\alpha \in \operatorname{dom} f \setminus \operatorname{dom} s$. Let $t = s \cup \{(\alpha, 1 - f(\alpha))\} \in \mathsf{Fn}_{\mathsf{bd}_\kappa}$. Then, $[t] \cap [f] = \emptyset$, hence $[f]$ is nowhere dense.

Finally, if $\mathcal{I} \supsetneq \mathsf{bd}_\kappa$, there is $f \in \mathsf{Fn}_\mathcal{I} \cap \mathsf{Fn}_{\mathsf{ub}_\kappa}$; then $[f]$ is meager, but $[f]$ is not $\mathcal{I}$-meager by Baire Category for the $\mathcal{I}$-topology (see Proposition 1.3). $\square$

We now show that there cannot be a small basis for the ideal of $\mathcal{I}$-meager sets (provided that $\mathcal{I}$ is not the bounded ideal). The proof uses a similar strategy as corresponding proofs in the context of tree forcings on ω which have large antichains (see [2]).

[21]Note that our assumption on κ is only needed in order to be able to invoke Lemma 7.3 at this point.

[22]This uses that, clearly, if $f' \subseteq f$, then $[f'] \supseteq [f]$.

[23]This is the unique isomorphism between $2^{<\kappa}$ and the set $\operatorname{split}(T)$ of splitting nodes of T that preserves lexicographical order.

Proposition 8.2. *Let $\mathcal{I} \supsetneq \mathsf{bd}_\kappa$. Then $\mathsf{cof}(\mathcal{M}_\mathcal{I}) > 2^\kappa$.*

Proof. Let $\{f_i \mid i < 2^\kappa\}$ with $f_i \in \mathsf{Fn}_\mathcal{I}$ be as in Observation 1.2(1), i.e., $\{[f_i] \mid i < 2^\kappa\}$ is a partition of 2^κ. Fix $\{X_i \mid i < 2^\kappa\}$ with X_i $\mathcal{I}$-meager. We will show that there exists an $\mathcal{I}$-nowhere dense (and hence $\mathcal{I}$-meager) set A, which is not contained in any X_i. This shows that $\mathsf{cof}(\mathcal{M}_\mathcal{I}) > 2^\kappa$.

Since $[f_i]$ is not $\mathcal{I}$-meager by Proposition 1.3, we can pick $a_i \in [f_i] \setminus X_i$ for every $i < 2^\kappa$. Let $A := \{a_i \mid i < 2^\kappa\}$. Clearly $A \nsubseteq X_i$ for every $i < 2^\kappa$. It remains to show that A is $\mathcal{I}$-nowhere dense.

Let $f \in \mathsf{Fn}_\mathcal{I}$. Clearly, there is an i such that $[f] \cap [f_i] \neq \emptyset$ and hence $g := f \cup f_i$ is in $\mathsf{Fn}_\mathcal{I}$ and $[g] \subseteq [f_i]$. Since $[f_i]$ is disjoint from $[f_j]$ for all j with $j \neq i$, there is at most one element in $A \cap [g]$. Therefore, we can extend g to $h \in \mathsf{Fn}_\mathcal{I}$ such that $[h] \cap A = \emptyset$, as desired. □

We now want to look at the question whether $\mathcal{I}$-meagerness could possibly imply meagerness, which we can answer negatively in many cases.

Proposition 8.3. *If $\mathcal{I}$ is tall,*[24] *then every set of size $< 2^\kappa$ is $\mathcal{I}$-nowhere dense.*

Proof. Assume $|X| < 2^\kappa$, and let $f \in \mathsf{Fn}_\mathcal{I}$. Using that $\mathcal{I}$ is tall, let $S \subseteq \kappa \setminus \mathrm{dom}(f)$ with $S \in \mathcal{I} \cap \mathsf{ub}_\kappa$. There are 2^κ many distinct extensions of f to $\mathrm{dom}(f) \cup S$, and the cones of these are disjoint. Since $|X| < 2^\kappa$, there exists $g \supseteq f$ in $\mathsf{Fn}_\mathcal{I}$ with $[g] \cap X = \emptyset$. This shows that X is $\mathcal{I}$-nowhere dense, as desired. □

The following is thus immediate:

Corollary 8.4. *If $\mathcal{I}$ is tall and $\mathsf{non}(\mathcal{M}_\kappa) < 2^\kappa$, then there is an $\mathcal{I}$-nowhere dense set that is not meager.* □

As a sidenote, let us show the following:

Proposition 8.5. *There is a closed*[25] *set X of size 2^κ which is $\mathcal{I}$-nowhere dense for every ideal $\mathcal{I}$. In fact, $X = [T]$ for a perfect subtree T of $2^{<\kappa}$.*

Proof. Let $X := \{x \in 2^\kappa \mid x(2i) = x(2i+1) \text{ for each } i < \kappa\}$.

Let $\mathcal{I}$ be an ideal. To show that X is $\mathcal{I}$-nowhere dense, let $f \in \mathsf{Fn}_\mathcal{I}$. Since $\kappa \notin \mathcal{I}$, $\mathrm{dom}(f) \neq \kappa$, so there exists an $i < \kappa$ such that either $2i \notin \mathrm{dom}(f)$ or $2i+1 \notin \mathrm{dom}(f)$. Since $\mathsf{bd}_\kappa \subseteq \mathcal{I}$, we can extend f to a function $g \in \mathsf{Fn}_\mathcal{I}$ with $g \supseteq f$ such that $g(2i) \neq g(2i+1)$. Consequently, $[g] \subseteq [f]$ and $[g] \cap X = \emptyset$, as desired. □

The following yields another situation in which we obtain $\mathcal{I}$-meager sets that are not meager, and should also be contrasted with Corollary 6.3.

Theorem 8.6. *If κ is*[26] *simple, $\mathfrak{r}(\kappa) = 2^\kappa$, and $\mathcal{I}$ is tall,*[27] *then there exists an $\mathcal{I}$-nowhere dense set A of size 2^κ which does not have the Baire property. In particular, A is not meager.*

Proof. Let $\lambda = |2^\kappa|$. We construct A in λ-many steps. Fix an enumeration $\langle D_i \mid i < \lambda\rangle$ of all κ-intersections of open dense sets (this is possible since $2^{<\kappa} = \kappa$ by Observation 5.3), an enumeration $\langle [s_i] \mid i < \kappa\rangle$ of the basic open sets (of the bounded

[24]In fact, we only need that every $\mathcal{I}$-cone contains 2^κ many disjoint $\mathcal{I}$-cones, which is guaranteed by the following property of $\mathcal{I}$ (which is actually weaker than tallness): for each $Z \in \mathcal{I}$, there exists an unbounded set Z' in $\mathcal{I}$ with $Z' \subseteq \kappa \setminus Z$.

[25]This implies that the set is $\mathcal{I}$-closed for every ideal $\mathcal{I}$.

[26]What this proof actually needs is $R^*(\kappa) = 2^\kappa$ and $2^{(2^{<\kappa})} = 2^\kappa$. So the assumptions of the theorem on κ can also be replaced by GCH at κ and $2^{<\kappa} = \kappa$ (which does not imply the assumptions of the theorem).

[27]We need tallness only in order to be able to apply Proposition 8.3. Therefore, the weaker property from footnote 24 is in fact sufficient.

topology on 2^κ), and an enumeration $\langle f_i \mid i < \lambda\rangle$ of $\mathsf{Fn}_{\mathcal{I}\cap\mathsf{ub}_\kappa}$. Let $\varphi\colon \lambda \to \lambda \times \kappa$ be a bijection.

(1) Let $A_0 = \emptyset$, $B_0 = \emptyset$.
(2) Let $\varphi(i) =: (j, k)$. Let $A_{i+1} = A_i \cup \{a_i\}$, $B_{i+1} = B_i \cup \{b_i\}$ with
  - $a_i \in D_j \setminus B_i$, $a_i \notin [f_l]$ for $l < i$, and
  - $b_i \in [s_k] \cap D_j \setminus A_{i+1}$.

  Such a_i exists since D_j is comeager, B_i has size less than λ, and $\mathfrak{r}(\kappa) = \lambda = R^*(\kappa)$ by assumption and by Theorem 7.4, hence[28] $\{[f_l] \mid l < i\} \cup \{[x] \mid x \in B_i\}$ cannot cover D_j. Such b_i exists, because D_j is comeager, thus $[s_k]\cap D_j$ is comeager in $[s_k]$, and therefore[29] $|[s_k] \cap D_j| = \lambda$ while $|A_{i+1}| < \lambda$.
(3) For limit ordinals $i \le \lambda$, let $A_i = \bigcup_{j<i} A_j$ and $B_i = \bigcup_{j<i} B_j$.

Let $A = A_\lambda$ and $B = B_\lambda$.

A is $\mathcal{I}$-nowhere dense: Let $f \in \mathsf{Fn}_\mathcal{I}$. Since $\mathcal{I}$ contains unbounded sets, we can pick $i < \lambda$ with $f \subseteq f_i$. By construction, $|A \cap [f_i]| < 2^\kappa$, but every set of size $< 2^\kappa$ is $\mathcal{I}$-nowhere dense by Proposition 8.3. So there exists $g \in \mathsf{Fn}_\mathcal{I}$ with $[g] \subseteq [f_i] \subseteq [f]$ such that $A \cap [g] = \emptyset$, as desired.

A does not have the Baire property: Let U be open. If $U = \emptyset$, $A\Delta U = A$, and A is not meager, because $A \cap D \neq \emptyset$ for every comeager set D. If $U \neq \emptyset$, $A\Delta U \supseteq U \setminus A \supseteq B \cap [s_k]$ for some $k < \kappa$. But $B \cap [s_k] \cap D \neq \emptyset$ for every comeager set D. Thus, $A\Delta U$ is not meager. □

Question 9. *Is there always an $\mathcal{I}$-meager set that is not meager, at least if κ is simple? By our above results, this clearly relates to the question whether it is consistent that κ is simple, and $\mathfrak{r}(\kappa) < \mathrm{non}(\mathcal{M}_\kappa) = 2^\kappa$, which also appears to be open.*

We remark that even consequences of $\mathfrak{r}(\kappa) < \mathrm{non}(\mathcal{M}_\kappa) = 2^\kappa$ have appeared as open questions in the literature. For instance, as proved by Raghavan and Shelah, it holds that $\mathfrak{b}_\kappa \le \mathfrak{r}(\kappa)$ for all regular κ. Brendle, Brooke-Taylor, Friedman and Montoya (see [1]) asked whether it is consistent that $\mathfrak{b}_\kappa < \mathrm{non}(\mathcal{M}_\kappa)$ holds for some regular uncountable κ.

9. The Baire property in the nonstationary topology

In this section, we provide two results on the connections between the Baire property in the bounded and in the nonstationary topology.

Proposition 9.1. *Assume that $\mathcal{I} = \mathsf{NS}_\kappa$. There is a subset of 2^κ with the Baire property, but not the $\mathcal{I}$-Baire property.*

Proof. Take any[30] $f \in \mathsf{Fn}_{\mathsf{ub}_\kappa} \cap \mathsf{Fn}_\mathcal{I}$. Since $[f]$ is nowhere dense (see Proposition 8.1), every subset of $[f]$ is meager and thus has the Baire property.

Since NS_κ does not contain any stationary set, it is not stationary tall, hence by Corollary 2.23(3), 2^κ has a subset without the $\mathcal{I}$-Baire property. Since 2^κ and $[f]$ are $\mathcal{I}$-homeomorphic by Lemma 1.5, there exists a subset of $[f]$ without the $\mathcal{I}$-Baire property (which has the Baire property by the above). □

Note that on the other hand, by Theorem 8.6, there is an Edinburgh meager set without the Baire property, assuming that κ is simple and $\mathfrak{r}(\kappa) = 2^\kappa$ (or $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, see footnote 26).

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[28] Note that $\bigcup\{[x] \mid x \in B_i\} = \bigcup\{\{x\} \mid x \in B_i\} = B_i$.

[29] It is well-known that comeager sets have full size, see Corollary 1.4.

[30] In fact, this proof works for every ideal $\mathcal{I}$ with the following two properties: first, there is an $f \in \mathsf{Fn}_{\mathsf{ub}_\kappa} \cap \mathsf{Fn}_\mathcal{I}$ such that $[f]$ is $\mathcal{I}$-homeomorphic to 2^κ, and second, the assumption of Corollary 2.23(3) is satisfied, i.e., $\mathcal{I}$ is not stationary tall.

We will need the following, which may also be of independent interest:

Observation 9.2. *Let κ be simple, let $\mathbb{P}$ denote κ-Silver forcing, and let $\mathcal{I} = \mathsf{NS}_\kappa$. Then, $A \subseteq 2^\kappa$ satisfies the $\mathcal{I}$-Baire property if and only if for every $f \in \mathbb{P}$, there is $g \leq f$ in $\mathbb{P}$ such that either $[g] \subseteq A$ or $[g] \cap A = \emptyset$.*[31]

Proof. This is straightforward to check, using that under our assumptions, every $\mathcal{I}$-meager set is $\mathcal{I}$-nowhere dense (see Theorem 6.1).

Alternatively, it follows from our observation in Section 4 that κ-Silver forcing is topological and generates the Edinburgh topology on 2^κ, together with the combination of [4, Lemma 3.8, 1 and 2] and of Theorem 5.4. □

The following is shown for inaccessible κ in [4, Lemma 4.9, 6], however making use of Theorem 6.2, it can also be shown to hold under the assumption of $\diamondsuit_\kappa$. It also shows that the statement of [4, Corollary 3.14] for κ-Silver forcing can be generalized to include the case that $\diamondsuit_\kappa$ holds: indeed, note that for regular and uncountable cardinals κ, adding κ^+-many Cohen subsets of κ forces that every $\mathbf{\Delta}^1_1$ subset of 2^κ has the Baire property (see for example [4, Theorem 3.13(1)]), and also forces $\diamondsuit_\kappa$ by an easy folklore standard argument.

Theorem 9.3. *If κ is simple and every $\mathbf{\Delta}^1_1$ subset of 2^κ has the Baire property, then every $\mathbf{\Delta}^1_1$ subset of 2^κ has the $\mathcal{I}$-Baire property for $\mathcal{I} = \mathsf{NS}_\kappa$.* [32]

Proof. Let $\mathbb{P}$ denote κ-Silver forcing, and let Γ denote the collection of $\mathbf{\Delta}^1_1$ subsets of 2^κ. All we will actually need in the argument, as in [4, Lemma 4.9], is that Γ is closed under continuous preimages.

Let $A \in \Gamma$, and let $f \in \mathbb{P}$. By Observation 9.2 it is enough to show that there exists a non-empty $\mathcal{I}$-open subset of $[f]$ which is either contained in A or disjoint from A.

By Lemma 1.5, there exists $\varphi : 2^\kappa \to [f]$ which is an homeomorphism with respect to the bounded topology and with respect to the Edinburgh topology (and the respective induced topologies on $[f]$). Let $A' = \varphi^{-1}[A]$, which is again in Γ because φ is continuous, and hence has the Baire property by our assumption. This means that either A' is meager, or it is comeager in some basic open set $[s]$ of the bounded topology on 2^κ. If A' is meager, then $2^\kappa \setminus A'$ is comeager, so Theorem 6.2 yields an $\mathcal{I}$-cone $[g]$ that is disjoint from A'. If it is comeager in $[s]$, then there is a comeager set D such that $D \cap [s] \subseteq A'$, so Theorem 6.2 yields an $\mathcal{I}$-cone $[g] \subseteq D \cap [s] \subseteq A'$.

But then, since φ is an $\mathcal{I}$-homeomorphism, $\varphi[[g]] \subseteq [f]$ is an $\mathcal{I}$-open set that is either disjoint from or contained in A. □

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[31] It suffices to assume that each $\mathcal{I}$-meager set is $\mathcal{I}$-nowhere dense (which is in particular the case if κ is simple and $\mathcal{I} = \mathsf{NS}_\kappa$).

[32] In the language of [4], this means that whenever κ is simple, then $\mathbf{\Delta}^1_1(\mathbb{C}_\kappa) \to \mathbf{\Delta}^1_1(\mathbb{V}_\kappa)$, where $\mathbb{C}_\kappa$ denotes κ-Cohen forcing, and $\mathbb{V}_\kappa$ denotes κ-Silver forcing.

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