# The non-constructive nature of social welfare relations

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# Background

Utility streams over an infinite time horizon and/or for models with infinitely many individuals have been extensively studied along the years in economics and social choice theory.

F.P. Ramsey, *A mathematical theory of savings*, The Economic Journal, 1928.

P.A. Diamond, *Evaluation of infinite utility streams*, Econometrica, 1965.

P.C. Fishburn, *Arrow's impossibility theorem: Concise proof and infinite voters*, Journal of economic theory, 1970.

# Infinite utility streams

We consider:

- a set of *utility levels* $Y$ with some given topology (e.g., $Y = \{0,1\}, [0,1], \omega$)
- $X := Y^\omega$ *space of infinite utility streams*.

Given $x, y \in X$ we use the following notation:

- $x \leq y$ iff for all $n \in \omega$, $x(n) \leq y(n)$
- $x < y$ iff $x \leq y$ and $\exists n \in \omega$, $x(n) < y(n)$
- $\mathcal{F} := \{\pi : \omega \to \omega : \text{ finite permutation}\}$

# Efficiency principles

In this context pre-orders $\preceq$ (reflexive and transitive relations) on a space of utility streams are usually called *social welfare relations* (SWR).

## Definition

A SWR $\preceq$ is said to be:

- *strong Pareto* (SP) if $\forall x, y \in X(x < y \Rightarrow x \prec y)$
- *intermediate Pareto* (IP) if for all $x, y \in X$, $\exists^\infty n(x(n) < y(n)) \Rightarrow x \prec y$
- *weak Pareto* (WP) if for all $x, y \in X$, $\forall n(x(n) < y(n)) \Rightarrow x \prec y$.

# Intergenerational equity principles

**Definition**

A SWR $\preceq$ on $X$ is called *anonymous* (AN) if whenever for $x, y \in X$ there are $i, j \in \omega$ such that:

- $x(i) = y(j)$ and $y(i) = x(j)$
- for all $k \neq i, j$, $x(k) = y(k)$,

then $x \sim y$.

In some cases, total SWRs can be well-represented

**Definition**

Let $\preceq$ be a SWR. Then $\preceq$ is said to be *represented* by the *utility function* $u : X \to \mathbb{R}$ iff

$$x \preceq y \Leftrightarrow u(x) \leq u(y).$$

**Easy example**: Let $Y = \omega$ and define $u : Y^\omega \to \mathbb{R}$ as:

$$u(x) := \min\{x(n) : n \in \omega\}.$$

The associated $\preceq_u$ satisfies WP and AN.

# Some results

**Proposition (Folklore)**

*Assume there exists a non-principal ultrafilter on $\omega$. Then there exists a total SWR on $2^\omega$ satisfying IP and AN.*

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**Proposition (Folklore)**

*Assume there exists a non-principal ultrafilter on $\omega$. Then there exists a total SWR on $2^\omega$ satisfying IP and AN.*

**Proposition (Zame, 2007)**

*If there is a total SWR on $[0,1]^\omega$ satisfying WP and AN, then there is a non-measurable set.*

**Proposition (Lauwers, 2011)**

*If there is a total SWR on $2^\omega$ satisfying IP and AN, then there is a non-Ramsey set.*

**First General Question**

When we combine two equity and/or efficiency principles, investigate whether it is possible to define a welfare function or at least a total SWR associated without using the axiom of choice, and on the contrary whether it has a non-constructive nature.

**Second General Question**

In case the total SWR reveals a non-constructive nature, investigate the exact fragment of AC associated, in particular compared to other well-known irregular sets (non-Ramsey, non-Lebesgue, non-Baire sets)

# Paretian SWRs and Baire property

**Question** (Bowler, Delommé, Di Prisco, Mathias)

Investigate the interplay between Paretian SWRs and the Baire property.

**Proposition** (L., 2017)

*Assume that there is a total social welfare relation on* $X = 2^\omega$ *satisfying SP and AN. Then there exists a non-Baire set.*

The proof is a sort of application of Kuratowski-Ulam theorem. The details of the proof reveal one cannot easily generalise such a method to IP, or to WP with larger utility domains.

**Proposition (Dubey - L., 2020)**

*Assume there is a total social welfare relation on $X = 2^\omega$ satisfying IP and AN. Then there exists a non-Baire set.*

**Proposition (Dubey - L., 2020)**

*Let $Y \subseteq \mathbb{R}$ be a set with order type $\mathbb{Z}$. Assume that there is a total social welfare relation on $X = Y^\omega$ satisfying WP and AN. Then there exists a non-Baire set in $2^\omega$.*

The key idea is the use of a variant of Silver trees.

**Definition**

A tree $p \subseteq 2^{<\omega}$ is called *Silver* tree if and only if $p$ is perfect and for every $s$, $t \in p$, with $|s| = |t|$ one has $s^\frown 0 \in p \Leftrightarrow t^\frown 0 \in p$ and $s^\frown 1 \in p \Leftrightarrow t^\frown 1 \in p$.

Note that one can associate any Silver tree $p$ with a partial function $f_p : \omega \to 2$ in such a way that

$$N_{f_p} := \{x \in 2^\omega : \forall n \in \text{dom}(f_p)(x(n) = f_p(n))\} = [p]$$

**Notation.** Let $p$ be a Silver tree:

$$S(p) := \omega \setminus \text{dom}(f_p)$$
$$U(p) := \{n \in \text{dom}(f_p) : f_p(n) = 1\}$$

## Definition

Let $p \subseteq 2^{<\omega}$ be a Silver tree with $\{n_k^p : k \geq 1\}$ enumeration of $S(p) \cup U(p)$; $p$ is called a *Mathias-Silver* tree ($p \in \mathbb{MV}$) if and only if there are infinitely many triples $(n_{m_j}^p, n_{m_j+1}^p, n_{m_j+2}^p)$'s such that:

1. for all $j \geq 1$, $m_j$ is even;
2. for all $j \geq 1$, $n_{m_j}^p, n_{m_j+1}^p, n_{m_j+2}^p$ are in $S(p)$ with $n_{m_j}^p + 1 < n_{m_j+1}^p$ and $n_{m_j+1}^p + 1 < n_{m_j+2}^p$;
3. for all $j \geq 1$, $t \in p$, $i < |t|$ $(n_{m_j}^p < i < n_{m_j+1}^p \vee n_{m_j+1}^p < i < n_{m_j+2}^p \Rightarrow t(i) = 0)$.

Such $(n_{m_j}^p, n_{m_j+1}^p, n_{m_j+2}^p)$ are called *Mathias triples*.

**Lemma**

*Given any comeager set $C \subseteq 2^\omega$ there exists $p \in \mathbb{MV}$ such that $[p] \subseteq C$.*

**Definition**

A set $X \subseteq 2^\omega$ is called *Mathias-Silver measurable set* (or $\mathbb{MV}$-measurable set) if and only if there exists $p \in \mathbb{MV}$ such that $[p] \subseteq X$ or $[p] \cap X = \emptyset$.

**Corollary**

*If $A \subseteq 2^\omega$ satisfies the Baire property, then $A$ is a $\mathbb{MV}$-measurable set.*

## Proposition

*Let $\preceq$ denote a total SWR satisfying IP and AN on $X = 2^\omega$. Then there exists a subset of $X$ which is not $\mathbb{MV}$-measurable.*

## Proof-sketch.

Given $x \in 2^\omega$, let $U(x) := \{n \in \omega : x(n) = 1\}$ and $\{n^x_k : k \geq 1\}$ enumerate the numbers in $U(x)$. Define

$$\begin{aligned} o(x) &:= [n^x_1, n^x_2) \cup [n^x_3, n^x_4) \cdots [n^x_{2j+1}, n^x_{2j+2}) \cup \cdots \\ e(x) &:= [n^x_2, n^x_3) \cup [n^x_4, n^x_5) \cdots [n^x_{2j+2}, n^x_{2j+3}) \cup \cdots \end{aligned} \tag{1}$$

Then put $\Gamma := \{x \in 2^\omega : e(x) \prec o(x)\}$. The aim is to show $\Gamma$ is not $\mathbb{MV}$-measurable.

. . . .

Fix $p \in \mathbb{MV}$ and $x \in [p]$ such that $x(n_k) = 1$, for all $k \in \omega$. We have to find $z \in [p]$ such that

$$x \in \Gamma \Leftrightarrow z \notin \Gamma.$$

The proof splits into three cases: $e(x) \prec o(x)$, $e(x) \sim o(x)$ and $o(x) \prec e(x)$. We only sketch the case $e(x) \prec o(x)$.
Let

$$z(n) := \begin{cases} x(n) & \text{if } n \notin \{n_{m_1+1}, n_{m_j}, n_{m_j+1} : j > 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} O(m_1) &:= [n_1, n_2) \cup [n_3, n_4) \cdots [n_{m_1-1}, n_{m_1}), \\ E(m_1) &:= [n_2, n_3) \cup [n_4, n_5) \cdots [n_{m_1}, n_{m_1+1}). \end{aligned}$$

. . . .

Let $\{k_1, k_2, \cdots, k_M\}$ enumerate the elements in $O(m_1)$, and let $\{k^1, \cdots, k^M\}$ enumerate the initial $M$ elements of the infinite set $\bigcup_{j>1} [n_{m_j}, n_{m_j+1})$. We permute $e(z)(k_1)$ with $e(z)(k^1)$, $e(z)(k_2)$ with $e(z)(k^2)$, continuing likewise till $e(z)(k_M)$ with $e(z)(k^M)$ to obtain $e^\pi(z)$. Further, $o^\pi(z)$ is obtained by carrying out identical permutation on $o(z)$. Observe that $e^\pi(z)$ and $o^\pi(z)$ are finite permutations of $e(z)$ and $o(z)$ respectively.

A technical argument allows to show that:

1. AN implies $e^\pi(z) \sim e(z)$ and $o^\pi(z) \sim o(z)$.
2. IP implies $o(x) \prec e^\pi(z)$ and $o^\pi(z) \prec e(x)$.

Combining 1 and 2 (and transitivity) we get
$o(z) \sim o^\pi(z) \prec e(x) \prec o(x) \prec e^\pi(z) \sim e(z) \rightarrow o(z) \prec e(z)$,
which implies $z \notin \Gamma$.

□

## Representation and construction of Asymptotic Pareto SWRs

Given $S \subseteq \omega$

$$\underline{d}(S) = \liminf_{n \to \infty} \frac{|S \cap \{1, 2, \cdots, n\}|}{n}.$$

Similarly, the upper asymptotic density of $S$ is defined as follows with $\limsup$ instead of $\liminf$.

**Definition**

Upper Asymptotic Pareto (UAP henceforth): Given $x, y \in X$, if $x \geq y$ and there exists $S \subseteq \omega$ such that $\overline{d}(S) > 0$ and for all $i \in S$, $x(i) > y(i)$, then $x \succ y$.

Similarly LAP (Lower Asymptotic Pareto) is defined by using $\underline{d}(S)$.

**Theorem (Petri, 2018)**

*There is a social welfare function $W : Y^\omega \to \mathbb{R}$ satisfying LAP and AN iff $Y$ is finite.*

**Question 1**

What happens when we strengthen LAP to UAP?

**Question 2**

Can we define a total SWR satisfying LAP and AN without using the axiom of choice?

**Proposition (Dubey - L - Ruscitti, 2020)**

*There does not exist any social welfare function satisfying UAP and AN on* $X = 2^\omega$.

**Proposition (Dubey - L. - Ruscitti, 2020)**

*Assume that there is a total social welfare relation on* $X = 2^\omega$ *satisfying UAP and AN. Then there exists a non-Ramsey set.*

**Proposition (Dubey - L. - Ruscitti, 2020)**

*Assume that there is a total social welfare relation on* $X = \omega^\omega$ *satisfying LAP and AN. Then there exists a non-Ramsey set.*

**Ongoing project: Generalised distributive equity principles**

# Distributive equity principles

## Definition

A SWR on $X$ satisfies *Hammond equity* (HE) if for every $x, y \in X$ there are $i, j \in \omega$ such that

- $x(i) < y(i) < y(j) < x(j)$
- for all $k \neq i, j$, $x(k) = y(k)$,

then $x \prec y$.

# Ongoing project

- Generalise the Hammond equity principles in order to always rank $\langle 1, 2, 1, 2, 1, 2 \ldots \rangle$ being better-off than $\langle 0, 3, 0, 3, 0, 3 \ldots \rangle$;
- Avoid pathogical cases where the reduction of inequality occurs on a very sparse set, so that such equity principle raises conflict even with basic intuitive weak efficiency principles.

**Question**

Study the representation and constructive nature of SWRs satisfyng these generalised Hammond equity principles.

THANK YOU FOR YOUR ATTENTION!

## Some references

1. R.S. Dubey, G. Laguzzi, F. Ruscitti, *On the representation and construction of social welfare orders*, Mathematical Social Science, Volume 7, 2020.
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3. Luc Lauwers, *Ordering infinite utility streams comes at the cost of a non-Ramsey set*, Journal of Math. Economics, Volume 46, 2009.
4. William R. Zame, *Can intergenerational equity be operationalized?*, Theoretical Economics 2, 2007.