Uncountable trees and Cohen sequences

Giorgio Laguzzi

Universität Freiburg

Bonn Set Theory Workshop - September 2016

Tree-forcing

In *set theory of the reals* some rather important objects are the so-called *tree-forcings*. This kind of forcings plays a relevant role in many applications regarding cardinal characteristics and regularity properties.

Definition

- $T \subseteq 2^{<\kappa}$ is called *perfect tree* iff T is closed under initial segments, T is closed under increasing $< \kappa$-sequences of nodes and $\forall s \in T \exists t \in T$ such that $s \subseteq t$ and t is *splitting*.
- $\mathbb{P}$ is called *tree-forcing* iff every $p \in \mathbb{P}$ is a perfect tree, and we define $q \leq p \Leftrightarrow q \subseteq p$.

Tree-ideals and tree-measurability

Definition

- X is $\mathbb{P}$-*open dense* iff $\forall T \in \mathbb{P} \exists T' \subseteq T (T' \in \mathbb{P} \wedge [T'] \subseteq X)$. The complement of a $\mathbb{P}$-open dense set is called $\mathbb{P}$-*nowhere dense*. X is $\mathbb{P}$-*meager* iff it can be covered by a $\leq \kappa$-size union of $\mathbb{P}$-nowhere dense sets. The ideal of $\mathbb{P}$-meager sets is denoted by $\mathcal{I}_{\mathbb{P}}$.
- X is $\mathbb{P}$-measurable iff for every $T \in \mathbb{P}$ there is $T' \subseteq T$, $T' \in \mathbb{P}$ such that $[T'] \cap X \in \mathcal{I}_{\mathbb{P}}$ or $[T'] \setminus X \in \mathcal{I}_{\mathbb{P}}$.

Some well-known examples in the ω-case

- *Cohen forcing* $\mathbb{C} := \{T_s : s \in 2^{<\omega}\}$, where $T_s := \{t \in 2^{<\omega} : s \subseteq t\}$ --→ Baire property
- *Sacks forcing* $\mathbb{S} := \{T \subseteq 2^{<\omega} : T \text{ perfect tree}\}$ --→ Bernstein property.
- *Random forcing* $\mathbb{B} := \{T \subseteq 2^{<\omega} : T \text{ perfect} \wedge \mu([T]) > 0\}$. --→ Lebesgue measurability.
- *Miller forcing* $\mathbb{M} := \{T \subseteq \omega^{<\omega} :$ $T \text{ perfect tree} \wedge (t \text{ splitting } \Rightarrow |\{i \in \kappa : t^\frown i \in T| = \kappa\})\}$.

These forcings add new reals. **What if we want to add a large set of new reals?**

Remark. Adding large sets of generic reals is very often used for proving results about the additivity numbers and regularity properties with respect to some specific ideal.

Amoeba for Sacks

We start with an example for Sacks forcing.

Definition

We define $\mathbb{AS}$ the poset consisting of pairs (n, T), with T perfect tree and $n \in \omega$. The ordering is given by:

$$(n', T') \leq (n, T) \Leftrightarrow n' \geq n \wedge T' \subseteq T \wedge T' \restriction n = T \restriction n.$$

Given a generic filter G for $\mathbb{AS}$, one may easily check that $T_G := \bigcap\{T : (n, T) \in G\}$ is a perfect tree such that each branch is Sacks generic. From now on we refer to T_G as a **generic tree**.

Remark

Let T_G be the generic tree and $\{t_n : n \in \omega\}$ be the sequence of its leftmost splitting nodes. Define $z \in 2^\omega$ so that $z(n) = 0$ iff $|t_{n+1}| \leq \min\{|t| : t \in \textsc{Split}(T_G) : t \supsetneq t_n^\frown 0\}$.

It easy to check that z is a Cohen real.
However, there is way to define finer versions of Sacks-amoeba in order to *kill* this kind (and all other kinds) of Cohen reals.

Similar situations occurs for Miller and Laver forcing. Mathias forcing is its own amoeba and it is known to satisfy the Laver property (and so not to add Cohen reals).
The situation for Silver forcing in completely different, as proven by Otmar Spinas recently.

But what about the generalized context?

Amoeba for Sacks in 2^κ

Club Sacks forcing

There are several ways to generalize Sacks forcing. For instance one can consider the following.

Definition

Let $T \subseteq 2^{<\kappa}$ be a perfect tree. We say T is club Sacks iff for each $x \in [T]$ one has

$$\{\alpha < \kappa : x{\restriction}\alpha \text{ is splitting}\} \text{ is club.}$$

The forcing consisting of this kind of trees is called **club Sacks forcing** and denoted by $\mathbb{S}^{\mathrm{Club}}$.

Amoeba for club Sacks

Definition

Define $\mathbb{AS}^{\text{Club}}$ as the poset consisting of pairs (p, T), with T club Sacks tree in $2^{<\kappa}$ and $p \subset T$ club subtree with size $< \kappa$. The order is:

$$(p', T') \leq (p, T) \Leftrightarrow p' \supseteq^{\text{end}} p \wedge T' \subseteq T.$$

As in the ω-case, given a generic filter G for $\mathbb{AS}^{\text{Club}}$, one can check that $T_G := \bigcap \{T : (\alpha, T) \in G\}$ is a club Sacks tree of generic branches.

Some important properties:

- $\mathbb{AS}^{\text{Club}}$ satisfies κ-axiom A, for κ inaccessible.
- $\mathbb{AS}^{\text{Club}}$ satisfies **quasi pure decision**: for every $D \subseteq \mathbb{AS}$ dense, $(p, T) \in \mathbb{AS}$, there is T' such that for every $(q, S) \leq (p, T')$,

$$(q, S) \in D \Rightarrow (q, T' \restriction q) \in D.$$

As in the ω-case, $\mathbb{AS}^{\text{Club}}$ adds κ-Cohen reals.
Question. Can we consider *well-sorted* perfect trees in order to get the κ-Laver property? NO
Moreover, the answer is much stronger than that. Indeed we are going to prove that there is a kind of κ-Cohen sequence which is *very robust*, and in fact we prove that **whenever we add a generic club Sacks tree we necessarily add κ-Cohen sequences**. This is in sharp contrast with the standard case.

To get in touch with the main difference from the ω-case consider the following easy example.

Remark

Fix $S \subseteq \kappa$ stationary and co-stationary. Let $\{t_\alpha : \alpha < \kappa\}$ be the sequence of leftmost splitting nodes of T_G. Define $z \in 2^\kappa$ as follows:

$$z(\alpha) = 1 \Leftrightarrow |t_{\alpha+1}| \in S.$$

One can easily check that z is Cohen.

Indeed for $(p, T) \in \mathbb{AS}$, let $\{t_\alpha : \alpha \leq \delta\}$ be the sequence of leftmost splitting nodes of T_G already decided by (p, T). Since there are club many splitting nodes above t_δ, one can freely shrink T in such a way that $t_{\delta+1} \in S$ or not. One can then extends p in order to freeze the $\delta + 1$-st splitting of the "new" subtree of T.

As we mentioned above we can prove a much stronger result.

Theorem

Let $V \subseteq N$ be ZFC-models such that in N there is an absolute $\mathbb{S}^{\text{club}}$-generic tree over V. Then there exists $z \in N \cap 2^\kappa$ that is Cohen over V.

The word *absolute* must be understood as follows: for every $< \kappa$-closed forcing $\mathbb{P}$,

$$N \models \text{"}T \text{ generic tree over } V\text{"} \Leftrightarrow N^{\mathbb{P}} \models \text{"}T \text{ generic tree over } V\text{"}.$$

Coding by stationary sets

Let κ be inaccessible. Fix $\{S_\tau : \tau \in 2^{<\kappa}\}$ family of pairwise disjoint stationary sets.

Lemma (Pruning Lemma)

Let $T \in \mathbb{S}^{\text{club}}$, $\{D_\xi : \xi < \kappa\}$ be a $\subseteq$-decreasing sequence of dense subsets of $\mathbb{C}$. One can find $T' \in \mathbb{S}^{\text{club}}$, $T' \subseteq T$ such that for every $\alpha < \kappa$, there is $\tau_\alpha \in 2^{<\kappa}$ such that

$$\forall t \in \textsc{Split}_{\alpha+1}(T') \forall s \in 2^{\leq\alpha}, |t| \in S_{\tau_\alpha} \text{ and } \in s^\frown\tau_\alpha \in D_\alpha.$$

The key step to prove the Pruning Lemma is the following result.

Lemma (Coding Lemma)

Let $T \in \mathbb{S}^{\text{club}}$, $\alpha \in \kappa$, $\tau \in 2^{<\kappa}$. There exists $T' \leq_\alpha T$, $T' \in \mathbb{S}^{\text{club}}$ such that

$$\forall t \in \textsc{Split}_{\alpha+1}(T'), |t| \in S_\tau.$$

Sketch of proof.

For every $t \in \textsc{Split}_\alpha(T)$, $i \in \{0,1\}$, pick $\sigma(t,i) \in T$ such that $\sigma(t,i) \supseteq t^\frown i$ and $|\sigma(t,i)| \in S_\tau$. Notice that we can do that as S_τ is stationary and there are club many splitting nodes extending $t^\frown i$. Furthermore if $\tau' \neq \tau$, then $|\sigma(t,i)| \notin S_{\tau'}$, since the stationary sets are pairwise disjoint. Finally put
$T' := \bigcup\{T_{\sigma(t,i)} : t \in \textsc{Split}_{\alpha+1}(T), i \in \{0,1\}\}$. □

Proof of Coding Lemma.

We build a fusion sequence $\{T_\alpha : \alpha < \kappa\}$, with $T_{\alpha+1} \leq_\alpha T_\alpha$ as follows (we start with $T_0 = T$):

case $\alpha + 1$: first find $\tau_\alpha \in 2^{<\kappa}$ so that $\forall s \in 2^{\leq\alpha}$, $s^\frown\tau_\alpha \in D_\alpha$; note this is possible since $|2^{\leq\alpha}| < \kappa$ and D_α is open dense in $\mathbb{C}$. Then apply the Coding Lemma for $T = T_\alpha$ and $\tau = \tau_\alpha$ to obtain $T_{\alpha+1} \leq_\alpha T_\alpha$ such that

$$\forall t \in \textsc{Split}_{\alpha+1}(T_\alpha) \forall s \in 2^{\leq\alpha}, |t| \in S_{\tau_\alpha} \text{ and } s^\frown\tau_\alpha \in D_\alpha.$$

case λ limit: put $T_\lambda := \bigcap_{\alpha<\lambda} T_\alpha$.

Finally put $T' := \bigcap_\alpha T_\alpha$. Note that, $\textsc{Split}_\alpha(T') = \textsc{Split}_\alpha(T_\alpha)$. Hence, by construction T' has the desired properties.

□

Proof of the theorem

Theorem

Let $V \subseteq N$ be ZFC-models such that in N there is an absolute $\mathbb{S}^{\text{club}}$-generic tree over V. Then there exists $z \in N \cap 2^\kappa$ that is Cohen over V.

Sketch of proof.

By an appropriate use of Cohen forcing, one can read off a sequence $\{T^i : i < \kappa\}$ of Sacks tree in such a way that for every $D \subseteq \mathbb{C}$ open dense there is T^i satisfying the Pruning Lemma for such D. For every T^i, consider the corresponding $\{\tau^i_\alpha : \alpha < \kappa\}$ defined by the Pruning Lemma. Finally put $\mathbf{z} := \bigcup_{i<\kappa} \sigma_\mathbf{i}$, where the σ_i's are recursively defined as follows: $\sigma_\mathbf{0} := \emptyset$; $\sigma_{\mathbf{i+1}} := \sigma_i^\frown \tau^i_{\alpha_{i+1}}$, where α_{i+1} satisfies $2^{\leq \alpha_{i+1}} \ni \sigma_i$; $\sigma_\mathbf{i} := \bigcup_{j<i} \sigma_j$, for i limit ordinal. □

Other generalized Sacks forcings

As we mentioned at the beginning $\mathbb{S}^{\text{Club}}$ is not the only possible choice for generalazing the Sacks forcing.
For instance another version might be to consider "measure one" splitting levels instead of club. More precisely, for κ measurable, one could define $\mathbb{S}^{\mathcal{U}}$ as the poset consisting of perfect trees T so that $\forall x \in T, \{\alpha < \kappa : x{\restriction}\alpha \text{ splits }\} \in \mathcal{U}$.
The main difference between $\mathbb{S}^{\text{Club}}$ and $\mathbb{S}^{\mathcal{U}}$ is that the latter satisfies the following.

Fact (Basic partition property)

Let $T \in \mathbb{S}^{\mathcal{U}}$ and $C : \textsc{Split}(T) \to \{0,1\}$ be a 2-coloring. Then there exists $T' \subseteq T$, $T' \in \mathbb{S}^{\mathcal{U}}$ such that $C{\restriction}\textsc{Split}(T')$ is constant, i.e., there is $i \in \{0,1\}$ such that $\forall t \in \textsc{Split}(T')$, $C(t) = i$.

What fails even for $\mathbb{S}^{\mathcal{U}}$ is the following generalization of that basic partition property.

Partition property. Let $\{T_i : i < \delta\}$, with $\delta < \kappa$ and $T_i \in \mathbb{S}^{\mathcal{U}}$. Let $C : \prod_{i<\delta} \textsc{Split}(T_i) \to \{0,1\}$ be a 2-coloring. Then there exist $T'_i \subseteq T_i$, $T'_i \in \mathbb{S}^{\mathcal{U}}$ such that $C \restriction \prod_{i<\delta} \textsc{Split}(T'_i)$ is constant, i.e., there is $k \in \{0,1\}$ such that $\forall (t_i : i < \delta) \in \prod_{i<\delta} \textsc{Split}(T'_i)$, $C((t_i : i < \delta)) = k$.

We are going to provide a countexample to such a partition property and this will give rise to a "new" type of Cohen κ-sequence.

Definition

Given $T \subseteq 2^{<\kappa}$ perfect tree, we say that T is *ω-perfect* iff there is an $\subseteq$-isomorphism $h : 2^\omega \to T$, i.e., for every $s, t \in 2^\omega$ one has $s \subseteq t \Leftrightarrow h(s) \subseteq h(t)$ and $s \perp t \Leftrightarrow h(s) \perp h(t)$ (roughly speaking, T is ω-perfect if it is an isomorphic copy of 2^ω inside $2^{<\kappa}$). Then we define

$$\Omega := \{T \subseteq 2^{<\kappa} : T \text{ is } \omega\text{-perfect}\}.$$

Given $T \in \Omega$, x_T denotes the leftmost ω-branch in T. For every $T, T' \in \Omega$ we define

$$T \sim T' \Leftrightarrow x_T = x_{T'} \wedge \exists t \subseteq x_T (T_t = T'_t).$$

Define the following coloring $C : \Omega \to \{0, 1\}$.

Given $T \in \Omega$, pick the corresponding representative $T^* \sim T$. Then find $S \subseteq T^* \cap T$ such that $S = T_\sigma = T^*_\sigma$, where $\sigma = \mathrm{S}\textsc{tem}(S)$. Moreover let $\tau = \mathrm{S}\textsc{tem}(T)$ and $\tau^* = \mathrm{S}\textsc{tem}(T^*)$.
For every $t, t' \in T$, with $t \subseteq t' \subseteq x_T$, let

$$\Delta(t, t') := \begin{cases} 0 & \text{iff } |\{s \in \mathrm{S}\textsc{plit}(T) : t \subseteq s \subsetneq t'\}| \text{ is even,} \\ 1 & \text{else} \end{cases}$$

Then define $C(S) := \Delta(\tau^*, \sigma)$ and

$$C(T) := \begin{cases} 0 & \text{iff } C(S) = \Delta(\tau, \sigma) \\ 1 & \text{else} \end{cases}$$

Fact

There is no $T \in \mathbb{S}$ homogeneous for C w.r.t. ω-perfect trees, i.e., there is no $T \in \mathbb{S}$ and $i \in \{0, 1\}$ such that $\forall T' \subseteq T$, $T' \in \Omega$ one has $C(T') = i$.

Corollary

$\mathbb{AS}$ *adds Cohen* κ*-reals.*

Proof.

Given T_G generic tree added via $\mathbb{AS}^{\mathcal{U}}$, we can define $z \in 2^\kappa$ as follows: Let $\{t_\alpha : \alpha < \kappa, \alpha$ limit ordinal$\}$ be an incresing subsequence of the leftmost splitnodes in T_G and let T_α be the ω-perfect tree generated by $\textsc{Split}_{\alpha+\omega}(T_{t_{\alpha+1}})$. Then define $z(\alpha) = C(T_\alpha)$, for every $\alpha < \kappa$.
To show that z is Cohen we argue as follows: given $(p, T) \in \mathbb{AS}^{\mathcal{U}}$, and $t \in 2^{<\kappa}$ arbitrary, let $\bar{z}$ be the part of z and $\{t_\alpha : \alpha \leq \lambda\}$ the part of leftmost splitnodes decided by (p, T) (w.l.o.g. we can assume λ limit ordinal). Pick t_λ and let p_0 be the ω-perfect tree generated by $\textsc{Split}_{\lambda+\omega}(T_{t_{\lambda+1}})$.

....

By definition of C, we can always find $q_0 \subseteq p_0$, $q_0 \in \Omega$, such that $C(q_0) = t(0)$. Then replace p_0 by q_0 in T, i.e., put

$$T^1 := \{t : \exists s \in \textsc{Term}(p_0)(t \not\perp s) \Rightarrow \exists s_0 \in \textsc{Term}(q_0)(t \not\perp s)\}.$$

Then proceed by induction on $\xi < |t|$: let p_ξ be the ω-perfect tree generated by $\textsc{Split}_{\lambda+\xi+\omega}(T^{\xi}_{t_{\lambda+\xi+1}})$ and pick an ω-perfect tree $q_\xi \subseteq p_\xi$ such that $C(q_\xi) = t(\xi)$ and put

$$T^{\xi+1} := \{t : \exists s \in \textsc{Term}(p_\xi)(t \not\perp s) \Rightarrow \exists s \in \textsc{Term}(q_\xi)(t \not\perp s)\}.$$

Finally let $T' := \bigcap_{\xi<|t|} T_\xi$ and p' be the tree generated by $p \cup \bigcup_{\xi<|t|} q_\xi$. By construction, T' end-extends p' and $(p', T') \Vdash z \supseteq \bar{z}^\frown t$, and this shows that z is κ-Cohen. □

Mathias forcing in 2^κ and generalized Ramsey property

κ-Mathias forcing $\mathbb{MA}$ for κ uncountable is defined as the poset of pairs (s, A), where $s \subseteq \kappa$ of size $< \kappa$ and $A \subseteq \kappa$ of size κ such that $\sup(s) < \min(A)$, ordered by
$(s, A) \leq (t, B) \Leftrightarrow t \supseteq s \wedge A \subseteq B \wedge t \setminus s \subseteq A$.

Remark

Note that for $\mathbb{MA}^{Club}$ we have the following two straightforward facts:

1. $\mathbb{MA}^{Club}$ *adds Cohen κ-reals. Let z be the canonical $\mathbb{MA}^{Club}$-generic set and then define $c \in 2^\kappa$ by: $c(\alpha) = 0$ iff the $\alpha + 1$-st element of z is in S. One can easily check that c is κ-Cohen.*
2. $\mathbb{MA}^{Club}$ *does not have pure decision. In fact, let $T \in \mathbb{MA}^{Club}$ and $\alpha \in \kappa$ successor ordinal such that T does not decide the α-th element of z. Consider the formula $\varphi =$ "the α-th element of z is in S"; then φ cannot be purely decided by T.*

We now want to show that we can build a somehow more general κ-Cohen sequence.

For every $a, b \in [\kappa]^\omega$, we define the following equivalence relation: $a \approx b \Leftrightarrow |a \triangle b| < \omega$. We also choose a representative for any equivalence class. We then define a coloring $C : [\kappa]^\omega \to \{0,1\}$ as follows:

for $b \in [\kappa]^\omega$, let a be the representative of $[b]_\approx$. Then put:

$$C(b) := \begin{cases} 0 & \text{iff } a \triangle b \text{ is even} \\ 1 & \text{else.} \end{cases}$$

Let $x \subseteq \kappa$ be the Mathias generic and $\{\alpha_j : j < \kappa\}$ enumerate the limit ordinals $< \kappa$. $i^x(\xi)$ denotes the ξth element of x. Define, for $j < \kappa$,

$$z(j) := \begin{cases} 0 & \text{iff } C(\{i^x(\xi) \in x : \alpha_j < \xi < \alpha_{j+1})\}) = 0 \\ 1 & \text{else} \end{cases}$$

Claim

z is κ-Cohen.

Proof.

W.l.o.g. assume $\sup s$ is limit and let λ be such that $\alpha_\lambda = \sup s$. Then let $b_j := \{i^A(\xi) \in A : \alpha_{\lambda+j} < \xi < \alpha_{\lambda+j+1}\}$, a_j be the corresponding representative, and $\xi_j := \min(b_j \cap a_j)$. We then recursively define $b'_j \subseteq b_j$, for $j < |t|$, as follows:

$$b'_j := \begin{cases} b_j & \text{if } C(b_j) = t(j) \\ b_j \setminus \{\xi_j\} & \text{if } C(b_j) \neq t(j) \end{cases}$$

Let $\Gamma := \{\xi_j : b'_j \neq b_j\}$ and $A' := A \setminus \Gamma$. Moreover, let $s' = s^\frown\sigma$, where $\sigma \in 2^{<\kappa}$ is the sequence corresponding to the set $\bigcup_{j<|t|} b'_j$. Hence, for every $j < |t|$, $(s', A') \Vdash z(\lambda + j) = t(j)$, since $(s', A') \Vdash z(\lambda + j) = C(b'_j) = t(j)$. □

Relationship between Ramsey property and Baire property at κ

Beyond underlining a big difference from the ω-case about Mathias forcing, the previous construction suggests a way to prove the following rather surprising connection between Ramsey property and Baire property in 2^κ.

Theorem

Let Γ *be a topologically reasonable family of subsets of* 2^κ. *Then* $\Gamma(\mathbb{MA}) \Rightarrow \Gamma(\mathbb{C})$.

Notation: $\Gamma(\mathbb{P}) :=$ every set in Γ is $\mathbb{P}$-measurable.

Table about tree-measurability at κ

Forcing notion	$\boldsymbol{\Sigma}^1_1$-counterexample	Forceable	Unknown
Sacks	$\mathbb{S}^{\text{Club}}$	$\mathbb{S}$	$\mathbb{S}^{\mathcal{U}}$
Silver	$\mathbb{V}^{\text{Club}}$	$\mathbb{V}$	$\mathbb{V}^{\mathcal{U}}$
Miller	$\mathbb{M}^{\text{Club}}$	$\mathbb{M}$	$\mathbb{M}^{\mathcal{U}}$
Mathias	$\mathbb{MA}^{\text{Club}}$, $\mathbb{MA}$, $\mathbb{MA}^{\mathcal{U}}$		
Cohen	$\mathbb{C}$		

Some interesting (I hope) questions

Question

Let $\mathcal{M}_\kappa$ be the ideal of κ-meager sets, $I_{\mathbb{S}^{\text{club}}}$ is the ideal of $\mathbb{S}^{\text{club}}$-meager sets, and $\leq_T$ denotes Tukey embedding. Is $\mathcal{M}_\kappa \leq_T I_{\mathbb{S}^{\text{club}}}$?

Question

Can we prove that if $N \supseteq V$ is a ZFC-model containing an absolute $\mathbb{S}^{\mathcal{U}}$-generic tree over V, then there is $c \in 2^\kappa \cap N$ Cohen over V?

Question

Can one prove κ-axiom A for the amoebas and tree-forcings, for κ regular successor?

Thank you for your attention!