Generalized Silver measurability

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Definition

A set of reals X is *Sacks-measurable* iff there exists a perfect tree T such that $[T] \subseteq X$ or $[T] \cap X = \emptyset$.

Theorem (Solovay, 1970)

$$\mathbf{\Sigma}^1_2(\textsc{Baire}) \Leftrightarrow \forall x \in \omega^\omega, \mathbb{C}(L[x]) \textit{ is comeager}$$

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Theorem (Brendle-Löwe,1999)

$$\mathbf{\Delta}^1_2(\mathbb{S}) \Leftrightarrow \mathbf{\Sigma}^1_2(\mathbb{S}) \Leftrightarrow \forall x \in \omega^\omega, \omega^\omega \cap L[x] \neq \omega^\omega$$

$$\mathbf{\Delta}^1_2(\mathbb{M}) \Leftrightarrow \mathbf{\Sigma}^1_2(\mathbb{M}) \Leftrightarrow \forall x \in \omega^\omega, \omega^\omega \cap L[x] \text{ does not dominate } \omega^\omega$$

$$\mathbf{\Delta}^1_2(\mathbb{L}) \Leftrightarrow \mathbf{\Sigma}^1_2(\mathbb{L}) \Leftrightarrow \forall x \in \omega^\omega, \omega^\omega \cap L[x] \text{ is not unbounded in } \omega^\omega$$

Theorem (Solovay, 1970)

$$L(\omega^\omega)^{V[G]} \models \textit{all sets are "regular"},$$

where G is $\mathrm{Coll}(\omega, < \kappa)$-generic, with κ inaccessible.

Theorem (Shelah, 1984)

$$L(\omega^\omega)^{V[G]} \models \textit{all sets have the Baire property},$$

where G is B_{ω_1}-generic and B_{ω_1} is a ccc algebra built by using "sweetness" and Shelah's amalgamation.

Theorem (Brendle-Halbeisen-Löwe, 2005)

$$L(\omega^\omega)^{V[G]} \models \textit{all sets are Silver measurable},$$

where G is $\mathbb{C}_{\omega_1}$-generic.

From ω^ω to κ^κ

Question. What happens if we move from Baire space ω^ω to the generalized Baire space κ^κ, with κ uncountable?

Differences and problems

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- In L there is a Σ^1_1-good well-ordering of the "κ-reals";
- In L there exists a $\mathbf{\Delta}^1_1$ set which is not regular;
- there is a $\mathbf{\Sigma}^1_1$ set without Baire property (namely the club filter $\textsc{Cub}$);
- it is not clear how to generalize the Lebesgue measure;

Generalized tree forcings

Generalizing tree-forcings

- $T \subseteq 2^{<\kappa}$ is *Sacks* iff every node has a splitting extension. We call $\mathbb{S}$ the poset of Sacks trees ordered by inclusion. (Note that $\mathbb{S}$ is not $< \kappa$-closed)

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- $T \subseteq 2^{<\kappa}$ is *club Sacks* iff it is Sacks and for every $x \in [S]$ we have $\{\alpha < \kappa : x{\restriction}\alpha \in \textsc{Split}(T)\}$ is closed unbounded (we write $T \in \mathbb{S}^{\text{club}}$)

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- $T \subseteq 2^{<\kappa}$ is *Silver* iff it is Sacks and moreover for every $s, t \in T$ such that $|s| = |t|$ one has $s^\frown i \in T \Leftrightarrow t^\frown i \in T$, for $i \in \{0, 1\}$ (we write $T \in \mathbb{V}$);
- $T \subseteq 2^{<\kappa}$ is *club Silver* iff it is Silver and $\mathrm{Lv}(T) := \{\alpha < \kappa : \exists t \in T(t \in \textsc{Split}(T) \wedge |t| = \alpha)\}$ is closed unbounded (we write $T \in \mathbb{V}^{\text{club}}$); analogously for $\mathbb{V}^{\text{stat}}$;

...

- $T \subseteq \kappa^{<\kappa}$ is called *Miller* iff $\forall t \in T \exists t' \in T(t \subseteq t' \wedge t' \in \textsc{Split}(T) \wedge |\textsc{Succ}(t,T)| = \kappa)$ (we write $T \in \mathbb{M}$);
- $T \subseteq \kappa^{<\kappa}$ is called *club Miller* ($T \in \mathbb{M}^{\text{club}}$) iff it is Miller and the following hold:
  - for every $x \in [T]$ one has $\{\alpha < \kappa : x{\restriction}\alpha \in \textsc{Split}(T)\}$ is closed unbounded,
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  - for every $t \in \textsc{Split}(T)$ one has $\{\alpha < \kappa : t^\frown\alpha \in T\}$ is closed unbounded.
- $T \subseteq \kappa^{<\kappa}$ is called *full Miller* ($T \in \mathbb{M}_{\text{full}}$) iff it is Miller and for every $t \in \textsc{Split}(T)$ for every $\alpha < \kappa$, one has $t^\frown\alpha \in T$.

$\mathbb{P}$-measurability

Definition

Let $\mathbb{P} \in \{\mathbb{S}^{\text{club}}, \mathbb{V}^{\text{club}}, \mathbb{M}^{\text{club}}, \mathbb{S}, \mathbb{V}, \mathbb{M}\}$. We say that a set of κ-reals X is $\mathbb{P}$-measurable iff there exists $T \in \mathbb{P}$ such that $[T] \subseteq X$ or $[T] \cap X = \emptyset$.

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For Γ *topologically reasonable* class of subsets of κ-reals, the following implications easily generalize from the standard framework.

- $\Gamma(\text{BAIRE}) \Rightarrow \Gamma(\mathbb{M}^{\text{club}})$
- $\Gamma(\mathbb{M}^{\text{club}}) \Rightarrow \Gamma(\mathbb{S}^{\text{club}})$
- $\Gamma(\mathbb{V}^{\text{club}}) \Rightarrow \Gamma(\mathbb{S}^{\text{club}})$
- $\Gamma(\text{BAIRE}) \Rightarrow \Gamma(\mathbb{V}^{\text{club}})$, for κ inaccessible.

Measurability for “club” tree forcing

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Problem. If $\mathbb{P} \in \{\mathbb{S}^{\text{club}}, \mathbb{V}^{\text{club}}, \mathbb{M}^{\text{club}}\}$, then $\mathbf{\Sigma}^1_1(\mathbb{P})$ fails in ZFC, because of the club filter. This is shown in a general framework by Friedman, Khomskii and Kulikov. (We will see later a direct and specific proof for $\mathbb{V}^{\text{club}}$).
This suggests that we should look at $\mathbf{\Delta}^1_1$ sets.

$\mathbf{\Delta}^1_1$-measurability

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Proposition

$\mathbf{\Delta}^1_1(\mathbb{S}^{\text{club}}) \Rightarrow \forall x \in \kappa^\kappa(\kappa^\kappa \cap L[x] \neq \kappa^\kappa)$.

Proposition (Friedman-Khomskii-Kulikov, 2013)

$\mathbf{\Delta}^1_1(\mathbb{M}^{\text{club}}) \Rightarrow \forall x \in \kappa^\kappa(\kappa^\kappa \cap L[x]$ *is not dominating*).

Proposition (L., 2012)

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So in general we are able to generalize only one direction.

Proposition (Friedman-Wu-Zdomskyy, 2013)

Suppose κ is successor. There exists an extension $M \supseteq L$ such that

$$M \models \neg \mathbf{\Delta}^1_1(\textsc{Baire}) \land \forall x \in \kappa^\kappa (\mathbb{C}(L[x]) \neq \emptyset).$$

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$$M \models \neg \mathbf{\Delta}^1_1(\textsc{Baire}) \wedge \forall x \in \kappa^\kappa (\mathbb{C}(L[x]) \neq \emptyset).$$

Corollary

$$M \models \neg \mathbf{\Delta}^1_1(\mathbb{S}^{\text{club}}) \wedge \forall x \in \kappa^\kappa (\kappa^\kappa \cap L[x] \neq \kappa^\kappa).$$

$$M \models \neg \mathbf{\Delta}^1_1(\mathbb{M}^{\text{club}}) \wedge \forall x \in \kappa^\kappa (\kappa^\kappa \cap L[x] \text{ is not dominating}).$$

$\mathbf{\Delta}^1_1$-separations

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Even if one cannot prove exact characterizations for statements $\mathbf{\Delta}^1_1(\mathbb{P})$, one can anyway prove some *separation theorems* for different notions of measurability.

Proposition (Friedman-Khomskii-Kulikov, 2013)

Let κ be inaccessible. Then

- $\mathbb{S}_{\kappa^+} \Vdash \mathbf{\Delta}^1_1(\mathbb{S}^{\text{club}}) \wedge \neg\mathbf{\Delta}^1_1(\mathbb{M}^{\text{club}})$
- $\mathbb{V}_{\kappa^+} \Vdash \mathbf{\Delta}^1_1(\mathbb{V}^{\text{club}}) \wedge \neg\mathbf{\Delta}^1_1(\mathbb{M}^{\text{club}})$.

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We have already mentioned that it fails for every "club" tree forcing.
Question. But what if we consider the notions of measurability with non-club tree forcing?
The club filter becomes measurable in this case. As a consequence, in such a case one might be able to build a model where all projective sets are measurable.
The first step is to look at the method used in the standard setting.

In the standard setting, we have:

if $\mathbb{P} \in \{\mathbb{S}, \mathbb{V}\}$, then $\mathbb{C}_{\omega_1} \Vdash \textsc{Pr}(\mathbb{P})$;

if $\mathbb{P} \in \{\mathbb{C}, \mathbb{M}, \mathbb{B}\}$, λ inaccessible, then $\mathrm{Coll}(\omega, {<}\lambda) \Vdash \textsc{Pr}(\mathbb{P})$.

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The key point which fails in the generalized setting is the so called factor lemma.

Lemma (Factor Lemma)

- *Let G be $\mathrm{Coll}(\omega, < \lambda)$ and $x \in \omega^\omega \cap V[G]$. Then there exists H $\mathrm{Coll}(\omega, < \lambda)$-generic over $V[x]$ such that $V[x][H] = V[G]$.*
- *Let G be $\mathbb{C}_{\omega_1}$ and $x \in \omega^\omega \cap V[G]$. Then there exists H $\mathbb{C}_{\omega_1}$-generic over $V[x]$ such that $V[x][H] = V[G]$.*

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- *Let G be $\mathbb{C}_{\omega_1}$ and $x \in \omega^\omega \cap V[G]$. Then there exists H $\mathbb{C}_{\omega_1}$-generic over $V[x]$ such that $V[x][H] = V[G]$.*

Even if we do not have an analogue for $\mathbb{C}_{\kappa^+}$ and $\mathrm{Coll}(\kappa, < \lambda)$, the main scope will be to find "sufficiently" many κ-reals with good quotient. This clever idea is due to Philipp Schlicht.

Stationary Silver vs club Silver

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Proposition (L., 2013 / Friedman-Khomskii-Kulikov, 2013)

$\textsc{Cub}$ *is not* $\mathbb{V}^{\text{club}}$*-measurable.*

Proposition (L., 2013)

Let κ *be inaccessible and* G *be* $\mathbb{C}_{\kappa^+}$*-generic over* N*. Then*

$$\mathrm{N}[G] \models \textit{all } On^{\kappa}\textit{-definable subsets of } 2^{\kappa} \textit{ is } \mathbb{V}^{\text{stat}}\textit{-measurable.}$$

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CUB *is not* $\mathbb{V}^{\mathsf{club}}$*-measurable.*

Proof.

We show that for every $T \in \mathbb{V}^{\mathsf{club}}$,

$$\exists x \in 2^\kappa (x \in \mathrm{CUB} \cap [T]) \wedge \exists y \in 2^\kappa (y \in \mathrm{NS} \cap [T]),$$

Define $x \in 2^\kappa$ as follows:

$$x(\alpha) := \begin{cases} f_T(\alpha) & \text{if } \alpha \in \mathrm{dom}(f_T), \\ 1 & \text{otherwise.} \end{cases}$$

Then obviously $x \supseteq \mathrm{Lv}(T)$ and so $x \in \mathrm{CUB} \cap [T]$. Analogously, we can define $y \in 2^\kappa$ so that $y(\alpha) = 0$ iff $\alpha \notin \mathrm{dom}(f_T)$, in oder to obtain $y \in \mathrm{NS} \cap [T]$. □

Proving that $\mathbb{C}_{\kappa^+}$ forces that all On^κ-definable sets are stationary Silver measurable requires more work.

Main idea. We want to find a stationary Silver tree whose branches have good quotient, i.e., they satisfy the factor lemma.

Perfect trees of generic reals

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Lemma

Let κ be inaccessible. Let $\mathbb{VT} := \{p : \exists T \in \mathbb{V} \exists \alpha \in \kappa (p = T \restriction \alpha)\}$, ordered by end-extension, i.e., $p' \leq p$ iff $p \subseteq p' \wedge \forall t \in p' \setminus p \exists s \in \mathrm{TERM}(p)(s \subseteq t)$. Let $T_G := \bigcup \{p : p \in G\}$, where G is $\mathbb{VT}$-generic filter over the ground model N. Then

$$\mathrm{N}[G] \models \quad T_G \in \mathbb{V} \wedge \forall x \in [T_G](x \text{ is Cohen over } \mathrm{N}) \\ \wedge\, Lv(T_G) \text{ is stationary and co-stationary,}$$

Moreover, $\mathbb{VT}$ is a forcing of size κ and is $< \kappa$-closed. So it is actually equivalent to κ-Cohen forcing.

Proof.

Fix $p \in \mathbb{VT}$ and $D \subseteq \mathbb{C}$ open dense and let $\{t_\alpha : \alpha < \delta < \kappa\}$, enumerate all terminal nodes of p (w.l.o.g. assume δ is a limit ordinal). Then consider the following recursive construction:

- pick $s_0 \supseteq t_0$ such that $s_0 \in D$;
- for $\alpha + 1$, pick $s_{\alpha+1} \supseteq t_{\alpha+1} \oplus s_\alpha$ such that $s_{\alpha+1} \in D$.
- for α limit, put $s'_\alpha = \bigcup_{\xi<\alpha} s_\xi$ and pick $s_\alpha \supseteq t_\alpha \oplus s'_\alpha$ such that $s_\alpha \in D$.
- once the procedure has been done for every $\alpha < \delta$, we put $s_\delta := \bigcup_{\alpha<\delta} t_0 \oplus s_\alpha$ and then $t'_\alpha := t_\alpha \oplus s_\delta$.

Finally, let p' be the downward closure of $\bigcup_{\alpha<\delta} t'_\alpha$. By construction, $p' \in \mathbb{VT}$, $p' \leq p$ and for every terminal node $t \in p'$, we get $t \in D$. Hence $p' \Vdash \forall x \in [T_G](H_x \cap D \neq \emptyset)$, where $H_x := \{s \in \mathbb{C} : s \subset x\}$.

. . . .

We now want to further extend p' in order to catch the second property as well, i.e., $\mathrm{Lv}(T_G)$ is both stationary and co-stationary. So fix $\dot{C}$ name for a club of κ. Build sequences $\{q_n : n \in \omega\}$ and $\{\xi_n : n \in \omega\}$ such that: $q_0 = p'$, and $q_{n+1} \leq q_n$ such that $q_{n+1} \Vdash \xi_n \in \dot{C}$ and $\xi_n > \mathrm{ht}(q_n)$ and $\mathrm{ht}(q_{n+1}) > \xi_n$. Finally put $\xi_\omega = \lim_{n<\omega} \xi_n$, $q_\omega := \bigcup_{n\in\omega} q_n$, and then

$$p^* := q_\omega \cup \bigcup \{t^\frown i : t \in \textsc{Term}(q_\omega) \wedge i \in \{0,1\}\}.$$

Hence $p^* \Vdash \forall n(\xi_n \in \dot{C})$, and then $p^* \Vdash \xi_\omega \in \dot{C}$. But $\xi_\omega = \mathrm{ht}(q_\omega)$, since the ξ_n's and the $|\mathrm{ht}(q_n)|$'s are mutually cofinal, and hence $p^* \Vdash \xi_\omega \in \mathrm{Lv}(T_G) \cap \dot{C}$. This shows that $\mathrm{Lv}(T_G)$ is stationary.

. . . .

For proving that it is co-stationary as well, we can further extend p^*, by using the same procedure, in order to find $\{q'_n : n \in \omega\}$ and $\{\xi'_n : n \in \omega\}$ as above and then $p^{**} \leq q'_\omega$ such that $p^{**} := q'_\omega \cup \bigcup\{t^\frown 0 : t \in \textsc{Term}(q'_\omega)\}$. Hence

$$p^{**} \Vdash \xi_\omega \in \mathrm{Lv}(T_G) \cap \dot{C} \wedge \xi'_\omega \notin \mathrm{Lv}(T_G) \cap \dot{C},$$

which completes the proof. □

Brief digression: Miller trees of generic branches

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Remark

Coll$(\kappa, 2^\kappa)$ *adds a full Miller tree of Cohen branches. Indeed, define the forcing* $\mathbb{MT} := \{p : \exists T \in \mathbb{M}_{\mathsf{full}} \exists \alpha < \kappa (p \sqsupseteq T[\alpha])\}$, *ordered by end-extension. Then* $\mathbb{MT}$ *adds a full Miller tree of Cohen branches and* $\mathbb{MT} \cong$ Coll$(\kappa, 2^\kappa)$.

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Lemma (L., 2014)

Let M *be a ZFC-model extending the ground model* N. *If for all* $x \in \kappa^\kappa \cap \mathrm{M}$ *there exists* $y \in \kappa^\kappa \cap \mathrm{N}$ *such that* $\forall \alpha < \kappa \exists \beta \geq \alpha (x(\beta) < y(\beta))$, *then in* M *there is no club Miller tree of Cohen branches.*

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Lemma

Let $\alpha < \kappa^+$. Let $\dot{T}$ be the canonical $\mathbb{VT}_0$-name for the generic Silver tree added by $\mathbb{VT}_0$, and $\dot{x}$ be a $\mathbb{VT}_\alpha$-name for a Cohen branch through $\dot{T}$. Let G be a $\mathbb{VT}_\alpha$-generic filter over N and $z = \dot{x}^G$. Then $\mathbb{VT}_\alpha / \dot{x}{=}z$ is equivalent to $\mathbb{VT}_\alpha$.

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We need some preliminary results.

Claim

$\mathbb{VT}^*_\alpha := \{p \in \mathbb{VT}_\alpha : |x_p| \geq \mathrm{ht}(p(0))\}$ *is dense in* $\mathbb{VT}_\alpha$.

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Proof.

Given $p \in \mathbb{VT}_\alpha$ we have to find $p' \leq p$ in $\mathbb{VT}^*_\alpha$. Start with $p_0 := p$ and then, for every $n \in \omega$, pick $p_{n+1} \leq p_n$ such that $|x_{p_{n+1}}| > \mathrm{ht}(p_n(0))$. Let $p_\omega := \bigcup_{n\in\omega} p_n$ and $w := \bigcup_{n\in\omega} x_{p_n}$. Then $w \subseteq x_{p_\omega}$ and $|w| = \mathrm{ht}(p_\omega(0))$. Hence $p' := p_\omega$ has the required property. □

Claim

For every $p \in \mathbb{VT}^*_\alpha$ *we have* $|x_p| = \mathrm{ht}(p(0))$.

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Proof.

By contradiction, assume $x_p = t^\frown s$, for some $t \in \mathrm{TERM}(p(0))$ and non-empty $s \in 2^{<\kappa}$. Let S be the downward closure of $\bigcup\{t \oplus t' : t \in \mathrm{TERM}(p(0))\}$, for some $t' \perp t^\frown s$ with $t' \supset t$. Let $p' \in \mathbb{VT}_\alpha$ be defined as

$$p'(\iota) := \begin{cases} S & \text{if } \iota = 0, \\ \dot{p}(\iota) & \text{if } \iota > 0. \end{cases}$$

Then pick $p^* \leq p'$ such that $p^* \in \mathbb{VT}^*_\alpha$. Since $p^* \Vdash S \sqsubset \dot{T}$ and $|x_{p^*}| \geq \mathrm{ht}(S)$, it follows that $t' \subseteq x_{p^*}$ and so $x_{p^*} \perp x_p$, contradicting $p^* \leq p$. □

Claim

$$\forall p \in \mathbb{VT}^*_\alpha \forall s \in 2^{<\kappa}(x_p \subseteq s \Rightarrow \exists p^* \in \mathbb{VT}^*_\alpha(s \subseteq x_{p^*})).$$

Proof.

Completely analogous to the previous one. □

Claim

$\forall p \in \mathbb{VT}^*_\alpha \forall s \in 2^{<\kappa}(x_p \subseteq s \Rightarrow \exists p^* \in \mathbb{VT}^*_\alpha(s \subseteq x_{p^*}))$.

Proof.

Completely analogous to the previous one. □

Corollary

Let $D \subseteq \mathbb{VT}^*_\alpha$ be open dense. Then $W_q := \{x_p \in 2^{<\kappa} : p \in D \wedge p \leq q\}$ is dense in $\mathbb{C}$ below x_q.

Proof of the main Lemma.

We know that $\mathbb{VT}^*_\alpha/\dot{x}{=}z = \mathbb{VT}^*_\alpha \setminus \bigcup_{\beta<\gamma} A_\beta$, where the elements of this union are recursively defined in $\mathrm{N}[z]$ as follows:

$$A_0 := \{p \in \mathbb{VT}^*_\alpha : \exists \xi < \kappa (p \Vdash \dot{x}(\xi) \neq z(\xi))\}.$$

$$A_{\beta+1} := \{p \in \mathbb{VT}^*_\alpha : \exists D \subseteq A_\beta \text{ open dense below } p\ , D \in \mathrm{N}\}.$$

$$A_\lambda := \bigcup_{\beta<\lambda} A_\beta, \text{ for } \lambda \text{ limit ordinal},$$

and finally γ is chosen so that $A_\gamma = A_{\gamma+1}$. Note that $\gamma = 0$

. . . .

We argue by contradiction, pick $p \in A_1 \setminus A_0$. Since $p \in A_1$, it follows that there exists $D \subseteq A_0$ such that $D \in \mathrm{N}$ and D is dense below p. Then the set $W_p := \{x_{p'} \in 2^{<\kappa} : p' \in D \wedge p' \leq p\}$ is dense in $\mathbb{C}$ below x_p, by the corollary previously mentioned, and so there exists $p' \in D$ such that $x_{p'} \subset z$, as z is Cohen over N (and $x_p \subset z$, by $p \notin A_0$). Also since $D \subseteq A_0$, it follows $p' \in A_0$. But, by definition,

$$p' \in A_0 \Leftrightarrow p' \Vdash \dot{x}(\xi) \neq z(\xi), \text{ for some } \xi < \kappa$$

providing us with a contradiction. Hence we get

$$\mathbb{VT}^*_\alpha / \dot{x}{=}z = \{p \in \mathbb{VT}^*_\alpha : \forall \xi < \kappa (p \nVdash \dot{x}(\xi) \neq z(\xi))\} = \\ = \{p \in \mathbb{VT}^*_\alpha : x_p \subset z\}.$$

□

Projective stationary Silver measurability

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Proposition

Let κ be inaccessible and G be $\mathbb{C}_{\kappa^+}$-generic over N. Then

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Proof sketch.

Since we have obtained a method to add a stationary Silver tree of Cohen branches with good quotient, the method is completely analogous to the standard one.

. . . .

If $\dot{x}$ is a Cohen branch through the generic Silver tree added by $\mathbb{VT}_0$, $\mathbb{C}_{\kappa^+}$ can be viewed as $Q_{\dot{x}} * \dot{R}_0 * \dot{R}_1$, where $Q_{\dot{x}}$ is the forcing generated by $\dot{x}$, while $\Vdash_{Q_{\dot{x}}} \dot{R}_0 \cong \mathbb{C}_\alpha$ and finally $\dot{R}_1$ is just a "tail" of $\mathbb{C}_{\kappa^+}$, and so it is equivalent to $\mathbb{C}_{\kappa^+}$ itself. So let us put $\dot{R} = \dot{R}_0 * \dot{R}_1$, so to have $\mathrm{N}[x] \models \dot{R}^x \cong \mathbb{C}_{\kappa^+}$.
Let x be Cohen over N with good quotient. Then

$$\mathrm{N}[x] \models \text{“} \Vdash_{\dot{R}^x} \varphi(x)\text{”} \quad \text{or} \quad \mathrm{N}[x] \models \text{“} \nVdash_{\dot{R}^x} \varphi(x)\text{”}.$$

Assume the former, and put $\theta(x) := \text{“} \Vdash_{\dot{R}^x} \varphi(x)\text{”}$ Then there exists $s \in \mathbb{C}$ such that $s \Vdash \theta(\dot{x})$. Pick T stationary-Silver tree of good Cohen branches over N such that $\textsc{Stem}(T) = s$. Hence, for every $z \in [T]$, we have $\mathrm{N}[z] \models \theta(z)$, and so $\mathrm{N}[z] \models \text{“} \Vdash_{\dot{R}^z} \varphi(z)\text{ ”}$.

. . . .

Since any z has good quotient, it follows that $\dot{R}^z$ is $\mathbb{C}_{\kappa^+}$. That means that there exists H filter $\dot{R}^z$-generic (i.e., $\mathbb{C}_{\kappa^+}$-generic) over $\mathrm{N}[z]$ such that $\mathrm{N}[z][H] = \mathrm{N}[G]$. Hence $\mathrm{N}[G] \models \varphi(z)$, that gives us $\mathrm{N}[G] \models [T] \subseteq X$.
For the case $\mathrm{N}[x] \models$ "$\not\Vdash_{\dot{R}^x} \varphi(x)$", simply note that "$\not\Vdash_{\dot{R}^x} \varphi(x)$" is equivalent to "$\Vdash_{\dot{R}^x} \neg\varphi(x)$", by weak homogeneity. Hence, a specular argument provides us with $T \in \mathbb{V}^{\mathsf{stat}}$ such that $\mathrm{N}[G] \models [T] \cap X = \emptyset$. □

Full Miller measurability

Full Miller measurability

Concerning Miller measurability, the situation looks different. It is not clear whether one can use $\mathbb{C}$ for adding a Miller tree of Cohen branches. We have seen before that this cannot be done if we require the tree to have club splitting nodes, but we conjecture that a similar method could be used even to obtain the same "negative" result for *simple* Miller tree.

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However, we have mentioned before that $\text{Coll}(\kappa, 2^\kappa)$ adds a full Miller tree of Cohen branches. So the idea to get projective full Miller measurability is to work with the Levy collapse $\text{Coll}(\kappa, <\lambda)$, with $\lambda > \kappa$ inaccessible.

Lemma

Let G be $\mathrm{Coll}(\kappa, <\lambda)$-generic over N. Let $\dot{T}$ be the canonical name for the generic Miller tree added by $\mathrm{Coll}(\kappa, 2^\kappa)$, $\dot{x}$ a $\mathrm{Coll}(\kappa, <\lambda)$-name for a branch in $\dot{T}$, and $z = \dot{x}^G$. Then $\mathrm{Coll}(\kappa, <\lambda)/\dot{x} = z$ is forcing-equivalent to $\mathrm{Coll}(\kappa, <\lambda)$.

Proposition (L., 2013)

Let λ be inaccessible greater than κ, and let G be $\mathrm{Coll}(\kappa, <\lambda)$-generic over N. Then

$$\mathrm{N}[G] \models \text{" all } On^\kappa\text{-definable subsets of } \kappa^\kappa \text{ are } \mathbb{M}_{\mathsf{full}}\text{-measurable ".}$$

Full Miller measurability vs Hurewicz dichotomy

We say that $X \subseteq \kappa^\kappa$ has the *Hurewicz dichotomy* (HD) iff either it is K_κ-compact or it contains a homeomorphic copy of κ^κ.

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Their result and mine are anyway independent, since there seems not to be a direct implication with the full Miller measurability. Moreover, for κ not weakly compact it is not even clear whether HD implies the *simple* Miller measurability.

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About generalized Lebesgue measurability and random-like forcing

- Can we find a random-like forcing for κ inaccessible, i.e., a poset that is κ^+-cc, κ^κ-bounding and $< \kappa$-closed?
- Investigate the corresponding regularity properties.

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General question about tree forcing and notions of measurabilty

- Find the "right" generalization of a tree forcing $\mathbb{P}$ so that $\mathbb{P}$ is $< \kappa$-closed (and possibly has fusion) and simultaneously we can force $\mathrm{PR}(\mathbb{P})$. E.g., Can we find $\mathcal{F}$ ultrafilter on κ such that $\mathbb{V}^{\mathcal{F}}$ is $< \kappa$-closed and we can also force $\mathrm{PR}(\mathbb{V}^{\mathcal{F}})$?

Open questions

About Miller measurability vs HD and PSP (joint project with Luca Motto Ros):

- In which cases is the inaccessible λ strictly necessary?
- Separate projective (full) Miller measubility from projective HD.
- Investigate other separations involving (full) Miller measurability, HD and PSP.
- Look at the generalization of other dichotomies coming from the standard setting, such as u-regularity, l-property and Spinas dichotomy.

Thanks for your attention!