Full-splitting Miller trees and infinitely often equal trees

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$\mathbb{P}$-dichotomies

Let $\mathcal{I}$ be a σ-ideal over the reals, and let $\mathbb{P}$ be a forcing with tree conditions.

Definition

We say that a set of reals X satisfies the $(\mathcal{I}, \mathbb{P})$-dichotomy iff either $X \in \mathcal{I}$ or there exists $T \in \mathbb{P}$ such that $[T] \subseteq X$.

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Well-known examples:

- Perfect set property: $\mathcal{I} =$ ideal of countable sets / $\mathbb{P} =$ Sacks forcing
- K_σ-regularity: $\mathcal{I} =$ ideal of bounded sets / $\mathbb{P} =$ Miller forcing

Such dichotomies provide a dense embedding

$$\mathbb{P} \hookrightarrow \text{BOREL} \setminus \mathcal{I}.$$

They are useful because we can use both the combinatorial properties of trees and the properties of the σ-ideal for studying the forcing notion associated. As an example, one can consider the following result.

Theorem (Zapletal)

If $\mathcal{I}$ is a σ-ideal on ω^ω σ-generated by closed sets then the forcing $\text{BOREL} \setminus \mathcal{I}$ is proper and preserves Baire category (non-meager ground-model sets remain non-meager in the extension).

The perfect set property is related to Davis' game on the Cantor space 2^ω. The analogous game played on the Baire space ω^ω gives rise to the following dichotomy.

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- Given $f : \omega^{<\omega} \to \omega$, let
  $D_f := \{x \in \omega^\omega : \forall^\infty n(f(x{\restriction}n) \neq x(n))\}$ and then
  $\mathfrak{D}_\omega := \{D_f : f : \omega^{<\omega} \to \omega\}$.

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  $\mathfrak{D}_\omega := \{D_f : f : \omega^{<\omega} \to \omega\}$.
- We say that a set $X \subseteq \omega^\omega$ satisfies the $(\mathfrak{D}_\omega, \mathbb{FM})$-dichotomy (or *Rosłanowski dichotomy*) iff either $X \in \mathfrak{D}_\omega$ or there exists $T \in \mathbb{FM}$ such that $[T] \subseteq X$.

Theorem (Rosłanowski)

Every $\mathbf{\Sigma}^1_1$ set satisfies the Rosłanowski dichotomy.

A slightly different σ-ideal has been studied by Spinas.

Definition

For every $x \in \omega^\omega$ let $K_x := \{y \in \omega^\omega \mid \forall^\infty n(x(n) \neq y(n))\}$, and let $\mathfrak{I}_{\mathrm{ioe}}$ be the σ-ideal generated by K_x, for $x \in \omega^\omega$.

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The two ideals are very similar, and in fact the following equalities hold.

Proposition

1. $cov(\mathfrak{I}_{\mathrm{ioe}}) = cov(\mathfrak{D}_\omega) = cov(\mathcal{M})$
2. $non(\mathfrak{I}_{\mathrm{ioe}}) = non(\mathfrak{D}_\omega) = non(\mathcal{M})$.
3. $add(\mathfrak{I}_{\mathrm{ioe}}) = add(\mathfrak{D}_\omega) = \omega_1$
4. $cof(\mathfrak{I}_{\mathrm{ioe}}) = cof(\mathfrak{D}_\omega) = \mathfrak{c}$.

The notion of a full-splitting Miller tree is not sufficient to get the right dichotomy for $\mathfrak{I}_{\mathrm{ioe}}$, as the following example shows.

Example

Let T be the tree on $\omega^{<\omega}$ defined as follows:

- If $|s|$ is even then $\textsc{Succ}_T(s) = \{0, 1\}$.
- If $|s|$ is odd then
$$\textsc{Succ}_T(s) = \begin{cases} 2\mathbb{N} & \text{if} \quad s(|s|-1) = 0 \\ 2\mathbb{N}+1 & \text{if} \quad s(|s|-1) = 1 \end{cases}$$

where $\textsc{Succ}_T(s) := \{n \mid s^\frown \langle n \rangle \in T\}$. Clearly T is $\mathfrak{I}_{\mathrm{ioe}}$-positive but cannot contain a full-splitting subtree.

The right dichotomy for $\mathfrak{I}_{\mathrm{ioe}}$ involves a subtle modification of the notion of a full-splitting Miller tree.

Definition

A tree $T \subseteq \omega^\omega$ is called an *infinitely often equal tree*, or simply *ioe-tree*, if for each $t \in T$ there exists $N > |t|$, such that for every $k \in \omega$ there exists $s \in T$ extending t such that $s(N) = k$. Let $\mathbb{IE}$ denote the partial order of ioe-trees ordered by inclusion.

Definition

We say that a set $X \subseteq \omega^\omega$ satisfies the $(\mathfrak{I}_{\mathrm{ioe}}, \mathbb{IE})$-dichotomy (or *Spinas dichotomy*) iff either $X \in \mathfrak{I}_{\mathrm{ioe}}$ or there exists $T \in \mathbb{IE}$ such that $[T] \subseteq X$.

Theorem (Spinas)

Every $\mathbf{\Sigma}^1_1$ set satisfies the Spinas dichotomy.

Simple remarks about $\mathbb{FM}$ and $\mathbb{IE}$

- $\mathbb{FM}$ adds a Cohen real (let $\{s_n : n \in \omega\}$ be a fixed enumeration of $\omega^{<\omega}$ and consider the function φ defined by $\varphi(x) = s_{x(0)}{}^\frown s_{x(1)}{}^\frown s_{x(2)}{}^\frown \dots$).
- $\mathbb{IE} * \mathbb{IE}$ adds a Cohen real.
- $\mathbb{IE}$ below a certain condition is equivalent to $\mathbb{FM}$. Such a condition is constructed in the following way:
  1. If $s \neq t$ are splitting nodes of T^{GS} then $|s| \neq |t|$.
  2. If $t \in T^{\mathrm{GS}}$ is a not the immediate successor of a splitting node of T then $t(|t|-1) = 0$.

Hence, $\mathbb{IE}$ also adds Cohen reals.

Dichotomies for higher projective levels

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Theorem (Khomskii - L.)

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2. $\mathbf{\Sigma}^1_2(\mathbb{IE}\text{-dich})$
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Theorem (Khomskii - L.)

Let κ be inaccessible and let G be $Coll(\omega, < \kappa)$-generic over V. Then in $V[G]$ all sets definable from countable sequences of ordinals satisfy the $\mathbb{FM}$- and the $\mathbb{IE}$-dichotomy, and in $L(\mathbb{R})^{V[G]}$ all sets of reals satisfy the $\mathbb{FM}$- and $\mathbb{IE}$-dichotomy.

Marczewski-type regularity properties

$\mathbb{FM}$ and $\mathbb{IE}$-measurability . . .

In the following definitions $\mathbb{P}$ is any tree-like forcing notion.

Definition

- A set of reals X is said to be $\mathbb{P}$-measurable iff $\forall T \in \mathbb{P} \exists T' \in \mathbb{P}, T' \subseteq T (X \cap [T'] = \emptyset \vee [T'] \subseteq X)$.
- A set of reals X is said to be weakly-$\mathbb{P}$-measurable iff $\exists T \in \mathbb{P} (X \cap [T] = \emptyset \vee [T] \subseteq X)$.

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Such a notion of measurability generalizes the Lebesgue measurability ($\mathbb{P} =$ random forcing), the Bernstein partition property ($\mathbb{P} =$Sacks forcing) and the Ramsey property ($\mathbb{P} =$ Mathias forcing).

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Theorem (Brendle - Löwe)

Let Γ be a topologically reasonable family of sets of reals. Then for many tree forcings $\mathbb{P}$ one has $\Gamma(\mathbb{P}) \Leftrightarrow \Gamma(w\mathbb{P})$.

. . . vs Baire property

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Theorem (Khomskii - L.)

Let Γ be a pointclass closed under continuous pre-images. Then the following are equivalent:

1. Γ(Baire)
2. $\Gamma(\mathbb{FM})$
3. $\Gamma(\mathbb{IE})$

Proof.

- $2 \Rightarrow 3$. We say that an ioe-tree T is *in strict form* if it can be written as follows:
  - for every $\sigma \in \omega^{<\omega}$, there exists $N_\sigma \subseteq \omega^n$, for some $n \geq 1$, such that
  - $\forall k \, \exists! s \in N_\sigma \, (s(n-1) = k)$, and
  - for $m < (n-1)$, there is some k such that $s(m) \neq k$ for all $s \in N_\sigma$. We use $\mathrm{len}(N_\sigma) = n$ to denote the length of N_σ, and we canonically enumerate N_σ as $\{s_k^\sigma \mid k < \omega\}$, in such a way that $s_k^\sigma(n-1) = k$.
  - T is the tree generated by sequences of the form

$$s_{n_0}^{\varnothing} \frown s_{n_1}^{\langle n_0 \rangle} \frown s_{n_2}^{\langle n_0, n_1 \rangle} \frown \ldots \frown s_{n_\ell}^{\langle n_0, n_1, n_2, \ldots, n_{\ell-1} \rangle}$$

for some sequence $\langle n_0, n_1, \ldots, n_\ell \rangle$.

. . . .

If T is in strict form, $t \in T$, and we need to find the first N such that $\forall k\ \exists s \supseteq t$ with $s(N) = k$, we only need to find the first sequence of the form s_n^σ extending t, and then $N := |s_n^\sigma| + \operatorname{len}(N_{\sigma^\frown\langle n\rangle})$. Every ioe-tree T can be pruned to an ioe-subtree $S \leq T$ in strict form.
So, let $A \in \Gamma$ and let T be an ioe-tree in strict form. Define a function $\psi' : \omega^{<\omega} \to T$ by setting:

$$\psi'(\langle x(0), \ldots, x(\ell)\rangle) := s_{x(0)}^{\varnothing} \frown s_{x(1)}^{\langle x(0)\rangle} \frown s_{x(2)}^{\langle x(0),x(1)\rangle} \frown \ldots \frown s_{x(\ell)}^{x\restriction\ell}.$$

This gives rise to a natural homeomorphism $\psi : \omega^\omega \cong [T]$. Since $\psi^{-1}[A]$ is also in Γ we can find a full-splitting tree S such that $[S] \subseteq \psi^{-1}[A]$ or $[S] \cap \psi^{-1}[A] = \varnothing$. But $\psi``[S]$ generates an ioe-subtree of T; in fact, if t is a splitting node of S then for every n we have $\psi'(t \frown \langle n\rangle) = \psi'(t) \frown s_n^t$, and the last digit of s_n^t is n by definition.

. . . .

- $3 \Rightarrow 2$. Let $A \in \Gamma$ and recall the Golstern-Shelah tree T^{GS} introduced before. Since A is $\mathbb{IE}$-measurable there exists $S \leq T^{\mathrm{GS}}$ such that $[S] \subseteq A$ or $[S] \cap A = \varnothing$. But then S is an $\mathbb{FM}$-tree , so A is weakly $\mathbb{FM}$-measurable, which is sufficient.

. . . .

- $1 \Rightarrow 2$. Let $A \subseteq \omega^\omega$ be a set in Γ. By Γ(Baire) we can find a basic open set $[s]$ such that $[s] \subseteq^* A$ or $[s] \cap A =^* \varnothing$, where $\subseteq^*$ and $=^*$ stands for "modulo a meager set". Without loss of generality, assume the former. Then there is a G_δ co-meager set $B \subseteq [s] \cap A$. Since $\mathfrak{D}_\omega \subseteq \mathcal{M}$, B cannot be $\mathfrak{D}_\omega$-small, hence it contains an $\mathbb{FM}$-tree. This is sufficient, as $\mathbb{FM}$- and weak $\mathbb{FM}$-measurability are classwise equivalent.
- $2 \Rightarrow 1$. Use the function φ described before. Let $A \in \Gamma$ and let $A' := \varphi^{-1}[A]$, also in Γ. By $\Gamma(\mathbb{FM})$ we can find a $T \in \mathbb{FM}$ such that $[T] \subseteq A'$ or $[T] \cap A' = \varnothing$, without loss of generality the former. Then $\varphi``[T]$ cannot be meager, but it is analytic, so it is comeager in some $[s]$. Then $[s] \subseteq^* A$. This is sufficient because Γ(Baire) is equivalent to the assertion that for all $A \in \Gamma$ there exists a basic open $[s]$ such that $[s] \subseteq^* A$ or $[s] \cap A =^* \varnothing$.

$\square$

$\mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$ vs $\mathbf{\Sigma}^1_2(\mathbb{IE})$

Theorem (Khomskii - L.)

$\mathbf{\Delta}^1_2(\text{Baire}) \Rightarrow \mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$.

Corollary

It is consistent that $\mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$ is true while $\mathbf{\Sigma}^1_2(\mathbb{IE})$ is false; in particular $\mathbb{IE}$- and weak $\mathbb{IE}$-measurability are not classwise equivalent.

Proof of Theorem.

Assume $\mathbf{\Delta}^1_2$(Baire). Let $A \subseteq \omega^\omega$ be a $\mathbf{\Sigma}^1_2$ set. We have to find an $\mathbb{IE}$-tree T such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.

We may assume that for some r, $\omega_1^{L[r]} = \omega_1$, since otherwise $\mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$ follows easily (for example from $\mathbf{\Sigma}^1_2$(Baire)). We may also assume, without loss of generality, that the parameters in the definition of A are in $L[r]$. Using the Borel decomposition of $\mathbf{\Sigma}^1_2$ sets we can write $A = \bigcup_{\alpha<\omega_1} B_\alpha$, where B_α are Borel sets coded in $L[r]$. If there exists at least one α such that $B_\alpha \notin \mathfrak{I}_{\mathrm{ioe}}$, then there is an $\mathbb{IE}$-tree T with $[T] \subseteq B_\alpha \subseteq A$ and we are done. So suppose that all B_α are $\mathfrak{I}_{\mathrm{ioe}}$-small. For each α, since $L[r] \models$ "$B_\alpha \in \mathfrak{I}_{\mathrm{ioe}}$" we can fix a sequence $\langle x_i^\alpha \mid i < \omega \rangle$ of reals in $L[r]$ such that $B_\alpha \subseteq \bigcup_{i<\omega} K_{x_i^\alpha}$.

. . . .

Let $\rho : \omega^\omega \to \omega^\omega$ be defined by $\rho(x) := \langle x(0), x(2), x(4), \dots \rangle$. By $\mathbf{\Delta}^1_2$(Baire), we know that in V there is a Cohen real c over $L[r]$. Then c is infinitely often equal over $L[r]$, and in particular, infinitely often equal to $\rho(x^\alpha_i)$ for all $\alpha < \omega_1, i < \omega$. Let T_c be the $\mathbb{FM}$-tree such that

$$[T_c] = \{y \mid \rho(y) = c\}.$$

We claim that $[T_c] \cap A = \varnothing$. Let $a \in A$, then there is some $\alpha < \omega_1$ such that $a \in B_\alpha$. By absoluteness of "$B_\alpha \subseteq \bigcup_{i<\omega} K_{x^\alpha_i}$", there is some $i < \omega$ such that a is eventually different from x^α_i. Let $N \in \omega$ be such that $\forall n > N\, (a(n) \neq x^\alpha_i(n))$. But since c is ioe to $\rho(x^\alpha_i)$, we can easily find $n > N$ such that $c(n) = x^\alpha_i(2n) \neq a(2n)$. By definition this implies that $a \notin [T_c]$. $\square$

Open questions

A couple of open questions

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2. Investigate the ideal σ-generated by the sets X satisfying $\forall T \in \mathbb{P} \exists T' \in \mathbb{P}(T' \leq T \wedge [T'] \cap X = \emptyset)$, where $\mathbb{P} \in \{\mathbb{FM}, \mathbb{IE}\}$.
   (Work in progress, joint with Yurii Khomskii and Wolfgang Wohofsky.)

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3. Investigate analogous Rosłanowski and Spinas dichotomies in κ^κ. (Related to Philipp Schlicht's work)

Thank you for your attention!

A game for $\mathbb{IE}$-dichotomy

Definition

Let $G^{\mathbb{IE}}(A)$ be the game in which players I and II play as follows:

I:	N_0		(s_0, N_1)		(s_1, N_2)		$\dots$	
II:		k_0		k_1		k_2		$\dots$

where $s_i \in \omega^{<\omega} \setminus \{\varnothing\}$, $N_i \geq 1$, $k_i \in \omega$, and the following rules must be obeyed for all i:

- $|s_i| = N_i$,
- $s_i(N_i - 1) = k_i$.

Then player I wins iff $z := s_0{}^\frown s_1 \frown s_2{}^\frown \cdots \in A$.

Theorem (Khomskii - L.)

1. *Player I has a winning strategy in* $G^{\mathbb{IE}}(A)$ *iff there is an* $\mathbb{IE}$*-tree* T *such that* $[T] \subseteq A$.
2. *Player II has a winning strategy in* $G^{\mathbb{IE}}(A)$ *iff* $A \in \mathfrak{I}_{\mathrm{ioe}}$.