Infinite utility streams and irregular sets

Giorgio Laguzzi

Albert-Ludwigs-Universität Freiburg

Logic Colloquium - August 2017

Social welfare relations

We consider:

- a set of *utility levels* Y with some given topology (e.g., $Y = \{0, 1\}, [0, 1], \omega$)
- $X := Y^\omega$ the *space of infinite utility streams*, endowed with the product topology

Given $x, y \in X$ we use the following notation:

- $x \leq y$ iff forall $n \in \omega$, $x(n) \leq y(n)$
- $x < y$ iff $x \leq y$ and $\exists n \in \omega$, $x(n) < y(n)$
- $\mathcal{F} := \{\pi : \omega \to \omega : \text{ finite permutation}\}$
- $x \in X$, $f_\pi(x) := (x_{\pi(0)}, x_{\pi(1)}, \dots, x_{\pi(n)}, \dots)$.

Definition

In this context pre-orders $\preceq$ (reflexive and transitive relations) are usually called *social welfare relations* (SWR). Moreover a SWR $\preceq$ is called:

- *strongly Paretian* iff $x < y \Rightarrow x \prec y$
- *intermediate Paretian* iff $\exists^\infty n(x(n) < y(n)) \Rightarrow x \prec y$
- *weakly Paretian* iff $\forall n(x(n) < y(n)) \Rightarrow x \prec y$.

Moreover, if $\preceq$ is Paretian SWR we say that $\preceq$ is an *ethical preference relations* (EPR) iff for every $\pi \in \mathcal{F}$ we have $f_\pi(x) \sim x$. (The latter property is usually called *finite anonimity* or *intergenerational equity*.)
We say $\preceq$ is *continuous* iff for every $x \in X$ the sets $\{y \in X : x \preceq y\}$ and $\{y \in X : y \preceq x\}$ are closed.

Example: The lex-order is a total SWR which is not continuous.

In some cases, total SWR can be well-represented

Definition

Let $\preceq$ be a SWR. Then $\preceq$ is said to be *represented* by the *utility function* $u : X \to \mathbb{R}$ iff

$$x \preceq y \Leftrightarrow u(x) \leq u(y).$$

Lemma (Debreu's Lemma)

Let $\preceq$ be a SWR. $\preceq$ is total and continuous iff there exists a utility function u that represents $\preceq$.

Remark: The lex-order is not representable via utility functions.

What about total EPR? Do they exists?

Proposition (Folklore)

AC implies the existence of total EPR.

Proposition (Lawners, 2011)

If there is a total EPR, then there is a non-Ramsey set.

Proposition (Zame, 2007)

If there is a total EPR, then there is a non-measurable set.

- **Question 1**: Does the existence of a non-Ramsey set imply the existence of a total EPR?
- **Question 2**: Does the existence of a non-measurable set imply the existence of a total EPR?

non-measurable set without total EPR

non-measurable set without total EPR

Idea

- We first prove that the existence of a total EPR gives a set without Baire property.
- We then use Shelah's model where all sets have the Baire property (and so there are no total EPR) but there is a non measurable set.

Question: How *many* incompatiple elements are there?

We start with a basic example. Let $\lhd$ be defined as follows: for every $x, y \in X$, we say $x \lhd y$ iff there exists $\pi \in \mathcal{F}$ such that $f_\pi(x) < y$.

Lemma

$A := \{(x, y) \in X \times X : x \ntrianglelefteq y \wedge y \ntrianglelefteq x\}$ *is comeager.*

Proof.

Let A' be the complement of A. We show that A' is meager. First note that A' can be partitioned into two pieces:
$E := \{(x, y) \in X \times X : x \trianglerighteq y\}$ and
$D := \{(x, y) \in X \times X : y \trianglerighteq x\}$. We prove E is meager, since the proof for D works similarly.
Fix $y \in X$ so that supp(y) is infinite (i.e., y is not eventually 0) and consider $E^y := \{x \in X : (x, y) \in E\}$. Let
$H^y := \{x \in X : x \geq y\}$. Note that

$$E^y := \bigcup_{\pi \in \mathcal{F}} H^{f_\pi(y)}.$$

Since $\mathcal{F}$ is countable it is enough to prove that for each $\pi \in \mathcal{F}$, $H^{f_\pi(y)}$ is meager.

Proof.

Actually we show that H^y is nowhere dense, for every $y \in X$ with $|\text{supp}(y)| = \omega$. Indeed, fix $U \subseteq X$ basic open set, and let $k \in \omega$ be sufficiently large that for all $n \geq k$, $U_n = [0,1]$. Then pick $n^* > k$ such that $n^* \in \text{supp}(y)$ and pick $U' \subseteq U$ so that:

- $\forall n \neq n^*$, $U_n = U'_n$;
- $U'_{n^*} := [0, y(n^*))$

Then it is clear that $U' \cap H^y = \emptyset$. This concludes the proof that each H^y is nowhere dense, when $|\text{supp}(y)| = \omega$. Note that if $\pi \in \mathcal{F}$ we get $|\text{supp}(f_\pi(y))| = \omega$ as well, and so $H^{f_\pi(y)}$ is nowhere dense too.

Proof.

By Ulam-Kuratowski theorem, we conclude the proof if we show that the set $\{y \in X : |\text{supp}(y)| = \omega\}$ is comeager. So let B be the complement of such a set, i.e., B consists of those y that are eventually 0. Define $B_n := \{y \in B : |\text{supp}(y)| \leq n\}$. Clearly $B := \bigcup_{n\in\omega} B_n$. Moreover each B_n in nowhere dense. Indeed, let U be a basic open set and pick $k > n$ so that for all $m \geq k$, $U_m = [0, 1]$. Then define $U' \subseteq U$ by replacing the kth of U with $(0, 1]$. It is clear that $U' \cap B_n = \emptyset$. Hence, we have proved that for comeager many y, E^y is meager, and that implies E is meager by Kuratowski-Ulam theorem. □

It is not hard to show that one can generalize the previous proof in order to obtain the following.

Proposition

Let $\preceq$ be a partial EPR, and $A := \{(x, y) \in X \times X : x \not\preceq y \wedge y \not\preceq x\}$. If A has the Baire property, then A is comeager.

Question: But what about total EPR?

Proposition

Let $\preceq$ be a total EPR, and $A := \{(x, y) \in X \times X : x \not\prec y \wedge y \not\prec x\}$. Then A does not have the Baire property.

Proof.

Note that in this case the EPR is total and so the set $A = \{(x, y) \in X \times X : x \sim y\}$. To reach a contradiction, assume A has the Baire property. By the previous proposition, A has to be comeager.

Proof.

Hence, by Kuratowski-Ulam's there is $y \in X$ such that A_y is comeager. For $0 < r < 1$, define the function $i : X \to X$ such that $i(x(0)) := x(0) + r$ and $\forall n > 0$, $i(x(n)) = x(n)$.
Note also that for every $x \in X$, $i(x) \succ x$ and so in particular $x \sim y \Rightarrow x \not\sim i(y)$. Hence, $A_y \cap i[A_y] = \emptyset$.
Since A_y is comeager, it should be $A_y \cap i[A_y] \neq \emptyset$, yielding to a contradiction. □

non-Ramsey set without total EPR

non-Ramsey set without total EPR

Idea

- Use Shelah's amalgamation to build a model where all sets in $L(\mathbb{R}, \{Y\})$ are measurable (and so there are no total EPRs) but Y is non-Ramsey.
- Consider the $L(\mathbb{R}, \{Y\})$ of such a forcing-extension in order to get a model where all sets are measurable but there is a non-Ramsey set.

The main property

Definition ($(\mathbb{B}, \dot{Y})$-homogeneity)

Let B be a complete Boolean algebra, $\dot{Y}$ be B-names. One says that B is $(\mathbb{B}, \dot{Y})$-homogeneous if and only if for any isomorphism ϕ between two complete subalgebras B_1, B_2 of B, such that $B_1 \approx B_2 \approx \mathbb{B}$, there exists $\phi^* : B \to B$ automorphism extending ϕ such that $\Vdash_B$ " $\phi^*(\dot{Y}) = \dot{Y}$". (Intuitively, we want B-names fixed by any automorphism constructed by the amalgamation).

Shelah's amalgamation

Let $B_0, B_1 \lessdot B$ isomorphic complete subalgebras and $\phi : B_0 \to B_1$. Let $e_0 : B \to B \times B$ such that $e_0(b) = (b, 1)$ (and analogously $e_1(b) = (1, b)$).

Step 1: define $\mathbf{Am}_1(B, \phi_0) \lessdot B \times B$ and $\phi_1 : e_0[B] \to e_1[B]$ so that ϕ_1 is an isomorphism extending ϕ.

Step n: define $\mathbf{Am}_n = \mathbf{Am}(\mathbf{Am}_{n-1}, \phi_{n-1})$ and ϕ_n extends ϕ_{n-1}.

Step ω: define $\mathbf{Am}_\omega(B, \phi)$ as the direct limit of the B_n's and ϕ_ω the limit of the ϕ_n's.

The main construction

Let κ be inaccessible. We recursively build a sequence of complete Boolean algebras $\{B_i : i > \kappa\}$ and a sequence of sets of names for reals $\{Y_i : i < \kappa\}$ such that $\forall i < j < \kappa$, $B_i \lessdot B_j$ and $Y_i \subseteq Y_j$ as follows:

- Using a book-keeping argument we cofinally often amalgamate over random algebras and we fix the set Y_i under the isomorphisms generated by the amalgamation. (To get $(\mathbb{B}, Y)$-homogeneity)
- for cofinally many i we put $B_{i+1} = B_i * \mathbb{A}$ and $Y_{i+1} = Y_i$
- for cofinally many i we put $B_{i+1} = B_i * \mathbb{MA}$ and $Y_{i+1} = Y_i$
- for cofinally many i we put $B_{i+1} = B_i * \mathbb{MA}$ and $Y_{i+1} = Y_i \cup \{x_T : T \in \mathbb{MA}\}$
- at limit steps $j < \kappa$ put $B_j = \lim_{i<j} B_i$ and $Y_j = \bigcup_{i>j} Y_i$.

Two key-steps

- Dominating reals are preserved under iteration with random forcing.
- Dominating reals are in a sense preserved by amalgamation.

non-Ramsey set without total EPR

Let $B = \lim_{i<\kappa} B_i$, $Y = \bigcup_{i<\kappa} Y_i$ and let G be B-generic over V.

$$L(\mathbb{R}, \{Y\})^{V[G]} \models \text{ no total EPR and } Y \text{ is non-Ramsey.}$$

Further questions

Further questions

Egalitarian principles

Compare Paretian principles with the following:

- **Pigou-Dalton's principle**: For every $x, y \in X$, $\varepsilon > 0$, $i, j \in \mathbb{N}$, if $y_i = x_i + \varepsilon < x_j - \varepsilon = y_j$ and for all $k \neq i, j$ one has $x_k = y_k$, then $x \prec y$.
- **Hammond's equity:** For every $x, y \in X$, if there are $i, j \in \mathbb{N}$ such that $x_i < y_i < y_j < x_j$ and for all $k \neq i, j$ one has $x_k = y_k$, then $x \prec y$.

Axiom of Determinacy

Investigate the connections with infinite games and AD.

References

1. Haim Judah, Andrej Roslanowsky, *On Shelah's amalgamation*, Israel Mathematical Conference Proceedings, Vol. 6 (1993), pp. 385-414.
2. Luc Lauwers, *Ordering infinite utility streams comes at the cost of a non-Ramsey set*, Journal of Math. Economics, Volume 46, pp 32-37, (2009).
3. Tadeusz Litak, *Infinite populations, choice and determinacy*, Studia Logica, (2017).
4. William R. Zame, *Can intergenerational equity be operationalized?*, Theoretical Economics 2, pp 187-202, (2007).

THANK YOU FOR YOUR ATTENTION!