Δ_2 DEGREES WITHOUT Σ_1 INDUCTION

WEI LI

Abstract. In this paper, we study the structure of Turing degrees below $\mathbf{0}'$ in the theory that is a fragment of Peano arithmetic without Σ_1 induction, with special focus on proper d-r.e. degrees and non-r.e. degrees. We prove

(1) $P^- + B\Sigma_1 + \text{Exp} \vdash$ There is a proper d-r.e. degree.
(2) $P^- + B\Sigma_1 + \text{Exp} \vdash I\Sigma_1 \leftrightarrow$ There is a proper d-r.e. degree below $\mathbf{0}'$.
(3) $P^- + B\Sigma_1 + \text{Exp} \nvdash$ There is a non-r.e. degree below $\mathbf{0}'$.

1. Introduction

Let P^- denote the axioms of Peano Arithmetic (PA) concerning rules governing the standard arithmetic operations such as the associative law of "+", the distributive law with respect to "+" and "·", etc, excluding the induction scheme. Paris and Kirby [11] defined fragments of PA by restricting the induction scheme to instances of bounded logical complexity and showed the relative logical strengths of the resulting theories. For $n \geq 1$, let $I\Sigma_n$ (Σ_n induction) denote the restriction of the induction scheme to Σ_n formulas, and let $B\Sigma_n$ (Σ_n bounding) be the statement saying that every Σ_n function maps a finite set in the sense of the model onto a finite set. It is known that $I\Sigma_n$ is strictly stronger than $B\Sigma_n$, and $B\Sigma_{n+1}$ is strictly stronger than $I\Sigma_n$, over the base theory $P^- + I\Sigma_0 + \text{Exp}$ ("Exp" says that $x \mapsto 2^x$ is a total function, and is a theorem of $P^- + I\Sigma_1$). It is possible to develop a theory of computation within a weak system of arithmetic. In fact, all the notions of classical recursion theory concerning primitive recursive functions, partial and total recursive functions, recursively enumerable (r.e.) sets etc. studied by Kleene and Post have their analogs in the system $P^- + B\Sigma_1 + \text{Exp}$. The research area in which we analyze the strength of induction required to establish theorems in recursion theory is called reverse recursion theory.

A Turing degree is r.e. if it contains an r.e. set. The degree of a complete r.e. set is denoted $\mathbf{0}'$. In the 1980's, S. Simpson first proved (unpublished) the Friedberg-Muchnik Theorem (the existence of a pair of incomparable r.e. degrees, originally proved in the standard model of PA using the $\mathbf{0}'$-priority method) within the system $P^- + I\Sigma_1$. Slaman and Woodin [14] then studied Post's problem in models of the weaker theory $P^- + B\Sigma_1 + \text{Exp}$. They provided examples of models of $P^- + B\Sigma_1 + \text{Exp}$ where the Sacks Splitting Theorem failed. Thus, $P^- + B\Sigma_1 + \text{Exp}$ is not strong enough for the implementation of the $\mathbf{0}'$-priority method involving the Sacks preservation strategy. Mytilinaios [9] continued the study and proved that $I\Sigma_1$ suffices to prove the Sacks Splitting Theorem. Later, Chong and Mourad [2] showed (without using the priority method) that the Friedberg-Muchnik Theorem is provable in $P^- + B\Sigma_1 + \text{Exp}$. In general, any construction which is priority-free or involves not more than the use of a $\mathbf{0}'$-priority argument may be successfully implemented in a model of $P^- + I\Sigma_1$. Similarly, the $\mathbf{0}''$-priority method is applicable in

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models of $P^- + I\Sigma_2$ (see [3, 9, 10, 14]). It is reasonable to conjecture, in view of the success story concerning the Friedberg-Muchnik Theorem, that all theorems proved using the $\mathbf{0}'$-priority method with effective bounds on the number of injuries for each requirement (a hallmark of the construction of a pair of r.e. sets with incomparable Turing degrees for the Friedberg-Muchnik Theorem) remain valid in models of $P^- + B\Sigma_1 + \text{Exp}$, even if the original methods of proof do not carry over in the new setting. This conjecture is, however, false. The existence of a nonrecursive low set, originally proved using a $\mathbf{0}'$-priority construction with effective bounds, is known to be equivalent to $I\Sigma_1$ over $P^- + B\Sigma_1 + \text{Exp}$ (see Chong and Yang [4]).

In this paper we consider problems about non-r.e. sets in the system $P^- + B\Sigma_1 + \text{Exp}$. In particular, we study the structure of the degrees below $\mathbf{0}'$. In classical recursion theory, i.e. in the standard model of PA, these degrees are precisely those which contain as members only sets that are Δ_2 definable, but in models of $P^- + B\Sigma_1 + \text{Exp}$, the situation may be different.

For any two r.e. sets A and B, $A \setminus B$ is said to be a d-r.e. set (*difference* of two r.e. sets). A degree is d-r.e. if it contains a d-r.e. set. The degree is called *proper* d-r.e., if it is d-r.e. but not r.e. Clearly every r.e. degree is d-r.e., and every d-r.e. set in a model of $P^- + B\Sigma_1 + \text{Exp}$ is Δ_2 definable. Furthermore, in classical recursion theory, we have the following result.

Theorem 1.1 (Cooper [5]). *There is a proper d-r.e. degree.*

We first investigate the existence of a proper d-r.e. degree from the point of view of reverse recursion theory. By the general observation on the $\mathbf{0}'$-priority method described above, Cooper's proof of the existence of a proper d-r.e. degree may be carried out in models of of $P^- + I\Sigma_1$. The situation becomes particularly interesting when working with a model that precludes the use of a priority construction, such as in a model where Σ_1 induction fails, and so the $\mathbf{0}'$-priority method fails in general. We show that in a model of $P^- + B\Sigma_1 + \text{Exp}$ where $I\Sigma_1$ fails (called a $B\Sigma_1$ model), by adopting a new approach, we can still construct a proper d-r.e. degree. The key to the new approach is to exploit the definition of Turing reducibility in the setting of $B\Sigma_1$ models. In a model of weak induction, finite sets in the sense of the model are used in place of singletons in the definition of Turing reducibility to ensure the transitivity of $\leq_T$. This fine difference in the definition of reducibility enables one to construct a d-r.e. degree $\mathbf{d}$ that does not lie below $\mathbf{0}'$. For such a $\mathbf{d}$, $\mathbf{d}$ is not r.e., since every r.e. degree is Turing reducible to $\mathbf{0}'$. In fact, the existence of a proper d-r.e. degree not below $\mathbf{0}'$ is not accidental. In any $B\Sigma_1$ model, we show that every d-r.e. degree below $\mathbf{0}'$ is r.e. Beyond this, we also exhibit a $B\Sigma_1$ model in which every degree below $\mathbf{0}'$ is r.e. The conclusion one draws from these results is that in the absence of Σ_1 induction, the structure of Turing degrees below $\mathbf{0}'$ presents a relatively neater picture. The fact that it is possible for $\mathbf{0}'$ to bound only r.e. degrees also looks intriguing and calls for further investigation.

The organization of this paper is as follows: In the next section, we present some notions and theorems which will be used subsequently in this paper. In Section 3, we study the existence of proper d-r.e. degrees. We first construct a proper d-r.e. degree for any $B\Sigma_1$ model, and then show that over the base theory $P^- + B\Sigma_1 + \text{Exp}$, there is a proper d-r.e. degree below $\mathbf{0}'$ if and only if Σ_1 induction holds. In Section 4, we consider degrees below $\mathbf{0}'$ in a saturated $B\Sigma_1$ model and show that in such a model degrees below $\mathbf{0}'$ are all r.e. We end this paper with several questions related to the topics discussed.

2. Preliminaries

We begin by recalling some useful facts about first order arithmetic. More details can be found in [3, 7, 9, 11].

2.1. **Fragments of Peano Arithmetic.** The language of first order arithmetic is the language $\mathcal{L}(0, 1, +, \cdot, <)$. Σ_n formulas are defined as usual. As mentioned in the introduction, P^- consists of the usual axioms on arithmetical operations without induction.

An induction principle may have different forms of expression. One of them, called *induction scheme*, is the following:

$$\forall x\, [(\forall y < x\, \varphi(y)) \to \varphi(x)] \to \forall x\, \varphi(x),$$

for every φ, possibly with parameters.

Another form is the *bounding scheme*:

$$\forall x\, (\forall y < x\, \exists w\, \varphi(y, w) \to \exists b\, \forall y < x\, \exists w < b\ \varphi(y, w)),$$

for any φ, possibly with parameters.

The Σ_n *induction* (Σ_n *bounding*, resp.) scheme, denoted $I\Sigma_n$ ($B\Sigma_n$, resp.), is the induction scheme (bounding scheme, resp.) restricted to Σ_n formulas.

Theorem 2.1. *Let $n \geq 1$. Assume $P^- + I\Sigma_0 + \mathrm{Exp}$, where* Exp *says that $\forall x\, \exists y\, (y = 2^x)$.*

(1) (Pairs and Kirby) $B\Sigma_{n+1} \Rightarrow I\Sigma_n \Rightarrow B\Sigma_n$. Furthermore, the arrows do not reverse.
(2) (Slaman) $B\Sigma_n \Leftrightarrow I\Delta_n$.

By Clause (1) of Theorem 2.1, the hierarchy of restricted induction

$$\cdots \Rightarrow I\Sigma_2 \Rightarrow B\Sigma_2 \Rightarrow I\Sigma_1 \Rightarrow B\Sigma_1 \tag{2.1}$$

does not collapse.

2.2. **Models of Fragments of** PA**.**

Sets. Let $\mathcal{M}$ be a model of $P^- + I\Sigma_0 + \mathrm{Exp}$. A subset of $\mathcal{M}$ is *r.e.*, if it is Σ_1 definable. If the complement of an r.e. set is also Σ_1 definable, then the set is *recursive*. The set difference of two r.e. sets is called *d-r.e.* (difference of r.e. sets).

A set $D \subset \mathcal{M}$ is *bounded*, if there is a $b \in \mathcal{M}$ such that $D \subseteq [0, b)$. A bounded set is *$\mathcal{M}$-finite*, if it is represented by the binary expansion of some element in $\mathcal{M}$. A set is *regular* if its intersection with any $\mathcal{M}$-finite set is $\mathcal{M}$-finite. To distinguish between sets and numbers, in this paper, we use lower case letters to denote numbers and capital letters to denote sets.

Lemma 2.2 (H. Friedman)**.** *Let $n \geq 1$ and $\mathcal{M} \models P^- + I\Sigma_n$. Then any Σ_n subset of $\mathcal{M}$ is regular, and any partial Σ_n function maps a bounded set to a bounded set. In particular, if $\mathcal{M} \models P^- + I\Sigma_1$, then all r.e. sets and d-r.e. sets of $\mathcal{M}$ are regular.*

Given an r.e. set A, let $A_s \subseteq A$ be the collection of numbers enumerated into A by stage s. Then A_s is $\mathcal{M}$-finite for any s. For any d-r.e. set D, D_s is defined similarly.

Suppose $f : \mathcal{M}^2 \to \mathcal{M}$ is a partial function. We define its *limit* at x as follows:

$$\lim_s f(s, x) = y \ \leftrightarrow\ \exists t\, \forall s > t\, (f(s, x) \downarrow = y).$$

Clearly, for every $x \in \mathcal{M}$ and r.e. (d-r.e.) set F, $\lim_s F_s(x) = F(x)$.

Computation and Degrees. Fix a Δ_0 bijection $\langle \cdot, \cdot \rangle : \mathcal{M}^2 \to \mathcal{M}$ such that

(1) $\langle a, b\rangle \geq \max\{a, b\}$ for all $a, b \in \mathcal{M}$, and
(2) $\langle \cdot, \cdot \rangle$ is strictly increasing with respect to each component.

By Σ_1 induction, we define

$$\langle z_0, z_1, \ldots, z_{n+1}\rangle = \langle\langle z_0, z_1, \ldots, z_n\rangle, z_{n+1}\rangle,$$

for every $n \in \mathcal{M} \setminus \{0\}$ and $z_0, z_1, \ldots, z_{n+1} \in \mathcal{M}$. (Without Σ_1 induction, functions $\langle z_0, z_1, \ldots, z_{n+1}\rangle$ are defined for every $n \in \mathbb{N}$.)

An r.e. set Φ is a *Turing functional* if it satisfies the universal closure of the following conditions:

(1) $\langle X, z, P, N\rangle \in \Phi \to ((z = 0 \vee z = 1) \wedge (P \cap N = \emptyset))$,
(2) $(\langle X, z, P, N\rangle \in \Phi \wedge P' \cap N' = \emptyset \wedge P' \supseteq P \wedge N' \supseteq N \wedge X' \subseteq X) \to \langle X', z, P', N'\rangle \in \Phi$,
(3) $(\langle X, z, P, N\rangle, \langle X, z', P, N\rangle \in \Phi) \to z = z'$.

Here, $\mathcal{M}$-finite sets X, P, N etc. are identified with their representations of binary expansion. Intuitively, for a Turing functional Φ, $\langle X, z, P, N\rangle \in \Phi$ means the program Φ with input X produces output z, whenever P is some positive part of the oracle and N is some negative part of the oracle.

Let $\{W_e\}_{e\in\mathcal{M}}$ be an effective enumeration of all r.e. sets. Any W_e and its enumeration could be modified uniformly and recursively to produce an r.e. Turing functional Φ_e such that:

(1) If W_e is a Turing functional, then $\Phi_e = W_e$.
(2) For every stage s and computation $\langle X, z, P, N\rangle \in \Phi_{e,s}$, the $\mathcal{M}$-finite sets X, P, N are subsets of $[0, s)$.
(3) Φ_e satisfies the *local downward closer property with respect to* $\Phi_{e,s}$:
For any stage s and computation $\langle X, z, P, N\rangle \in \Phi_{e,s}$, if Y is an $\mathcal{M}$-finite subset of X, then $\langle Y, z, P, N\rangle \in \Phi_{e,s}$.

Note that the modification could be uniformly recursive, so the enumeration of all r.e. Turing functionals $\{\Phi_e\}_{e\in\mathcal{M}}$ is recursive.

Given $A, B \subseteq \mathcal{M}$, A is said to be *Turing reducible* (or *reducible*, for short) to B, denoted by $A \leq_T B$, if there is an r.e. Turing functional Φ such that for every $\mathcal{M}$-finite set X,

$$X \subseteq A \;\Leftrightarrow\; \exists P\, \exists N\, (P \subseteq B \wedge N \subseteq \overline{B} \wedge \langle X, 1, P, N\rangle \in \Phi),$$
$$X \subseteq \overline{A} \;\Leftrightarrow\; \exists P\, \exists N\, (P \subseteq B \wedge N \subseteq \overline{B} \wedge \langle X, 0, P, N\rangle \in \Phi).$$

In the above definition, if $\Phi = \Phi_e$, then we say $A \leq_T B$ *via* Φ_e (in symbols $A = \Phi_e^B$). Turing degrees, r.e. degrees, etc, are defined as usual.

Turing reducibility is also called *strong reducibility* or *setwise reducibility.* They are defined so against the notion of *weak reducibility* or *pointwise reducibility* (denoted by $\leq_p$), which is obtained by substituting an element "x" for an $\mathcal{M}$-finite set "X" in the definition of Turing functional and Turing reducibility. Turing reducibility is transitive and stronger than weak reducibility, but weak reducibility need not be transitive.

Now we fix the following notations. Suppose Φ_i is a Turing functional, W_e is an r.e. set. Then for any stage s,

$$\Phi_i^{W_e}[s] = \{\langle X, z, P, N\rangle \in \Phi_{i,s} : P \subseteq W_{e,s},\ N \subseteq \overline{W}_{e,s}\}.$$

That is, $\Phi_i^{W_e}[s]$ is a collection of computations consistent with W_e from the view of stage s. Since Φ_i is a Turing functional, $\Phi_i^{W_e}[s]$ is also *self consistent*, i.e. the universal closure of the following formula holds:

$$\langle X, z, P, N\rangle \in \Phi_i^{W_e}[s] \to \langle X, 1-z, P', N'\rangle \notin \Phi_i^{W_e}[s].$$

Note that $\Phi_i^{W_e}$ also satisfies the *local downward closure property* with respect to $\Phi_i^{W_e}[s]$.

Now suppose $\mathcal{M}$ is a model of $B\Sigma_1$. If $\langle X, z, P, N\rangle \in \Phi_i$ such that $P \subseteq W_e$ and $N \subseteq \overline{W}_e$, then $\langle X, z, P, N\rangle \in \Phi_i^{W_e}[s]$ for all large enough stages s. If $\mathcal{M}$ also satisfies $I\Sigma_1$, then we can define $\Phi_i^D[s]$ similarly. Here, $I\Sigma_1$ is required to ensure that whenever $\langle X, z, P, N\rangle \in \Phi_i$, $P \subseteq D$, $N \subseteq \overline{D}$ and s is large enough, we have $P \subseteq D_s$ and $P \subseteq \overline{D_s}$ so that $\langle X, z, P, N\rangle \in \Phi_i^D[s]$.

$B\Sigma_n$ Model. Let $n \geq 1$. A model $\mathcal{M} \models P^- + I\Sigma_0 + \mathrm{Exp}$ is said to be a $B\Sigma_n$ *model*, if $\mathcal{M} \models B\Sigma_n$ and $\mathcal{M} \not\models I\Sigma_n$. Clause (1) of Theorem 2.1 asserts that there exists a $B\Sigma_n$ model.

An analysis of $B\Sigma_n$ models is needed to study the relationship between fragments of PA and theorems in recursion theory proved under $I\Sigma_n$. A theorem is equivalent to $I\Sigma_n$ over $B\Sigma_n$, if it is provable by $I\Sigma_n$ but fails in every $B\Sigma_n$ model.

A subset I of $\mathcal{M}$ is a *cut*, if I is a nonempty proper initial segment of $\mathcal{M}$ and closed under successor. A partial function on $\mathcal{M}$ is *cofinal* if its range is unbounded in $\mathcal{M}$.

Lemma 2.3. *Let $\mathcal{M} \models P^- + B\Sigma_n + \mathrm{Exp}$. Then $\mathcal{M}$ is a $B\Sigma_n$ model if and only if there exists a Σ_n cut I with a Δ_n function $f : I \to \mathcal{M}$ such that f is strictly increasing and cofinal.*

Assume $A \subseteq \mathcal{M}$. A set $G \subseteq A$ is said to be *coded on A* if there is an $\mathcal{M}$-finite set X such that $X \cap A = G$. Let $n \geq 1$. A set $G \subseteq A$ is Δ_n *on A* if G and $A \setminus G$ are both Σ_n.

Lemma 2.4 (Chong and Mourad [1])**.** *Suppose $\mathcal{M} \models P^- + B\Sigma_n + \mathrm{Exp}$ and $A \subseteq \mathcal{M}$. Then every set bounded and Δ_n on A is coded on A. In particular, any Δ_n set of $\mathcal{M}$ is regular and any bounded Δ_n set is $\mathcal{M}$-finite.*

The above lemma makes more sense for $B\Sigma_n$ models. In a $B\Sigma_n$ model, as its induction principle is weak, classical proofs of a theorem usually do not work. Nevertheless, by Lemma 2.4, some information, which is Δ_n on a Σ_n cut I, is coded on I. Such a code is employed as a parameter in a proof of either the theorem or its negation. This idea is seen more clearly in Sections 3 and 4.

Computation and Cut. Suppose $\mathcal{M}$ is a $B\Sigma_1$ model, I is a Σ_1 cut in $\mathcal{M}$, $a \in \mathcal{M}$ is greater than all numbers in I, and $\{\Phi_e\}_{e\in\mathcal{M}}$ is a recursive enumeration of all r.e. Turing functionals of $\mathcal{M}$. The following two lemmas are straightforward.

Lemma 2.5. *For every nonempty $\mathcal{M}$-finite set X,*

$$X \subseteq I \leftrightarrow \max X \in I,$$
$$X \subseteq \overline{I} \leftrightarrow \min X \in \overline{I},$$

where $\max X$ *($\min X$, respectively) is the maximum (minimum) element in X.*

Lemma 2.6. *For any set $G \subseteq \mathcal{M}$, $I \leq_T G$ if and only if $I \leq_p G$.*

For every $e, s \in \mathcal{M}$, we define

(2.2) $\Psi_e^I[s] = \{\langle X, z, n\rangle : \exists P \subseteq I_s\, \exists N \subseteq \overline{I}_s\, (\langle X, z, P, N\rangle \in \Phi_e^I[s] \wedge n = \min(N \cup \{a\}))\}.$

That is, we only consider the minimum element of the negative condition of the computation. Ψ_e^I also satisfies *local downward closure property* with respect to $\Psi_e^I[s]$. If $G = \Phi_e^I$, then for any $\mathcal{M}$-finite set X,

$$X \subseteq G \leftrightarrow \exists n \in \overline{I}\, (\langle X, 1, n\rangle \in \Psi_e^I),$$
$$X \subseteq \overline{G} \leftrightarrow \exists n \in \overline{I}\, (\langle X, 0, n\rangle \in \Psi_e^I).$$

Therefore, we also say that $G \leq_T I$ *via* Ψ_e or $G = \Psi_e^I$. $\{\Psi_e^I\}_{e\in\mathcal{M}}$ can be seen as a *recursive enumeration of all r.e. Turing functionals with oracle* I.

3. Proper d-r.e. degree and Σ_1 induction

Cooper [5] proved the existence of a proper d-r.e. degree in the standard model $\mathbb{N}$, using a $\mathbf{0'}$-priority construction. As we see in Section 3.1, his proof remains valid under the weaker assumption of Σ_1 induction. The remain problem is therefore the converse: is Σ_1 induction necessary for the existence of a proper d-r.e. degree? In Section 3.2, we give a negative answer to this question.

3.1. $\boldsymbol{I\Sigma_1}$ Implies the Existence of a Proper d-r.e. Degree.

Theorem 3.1. *Let* $\mathcal{M} \models P^- + I\Sigma_1$. *Then there exists a d-r.e. set* D *such that* $D \not\equiv_T W$ *for any r.e. set* W.

Proof. Suppose $\mathcal{M} \models P^- + I\Sigma_1$. Let $\{W_e\}_{e\in\mathcal{M}}$, $\{\Phi_e\}_{e\in\mathcal{M}}$, and functions $\langle\cdot,\dots,\cdot\rangle$ be as above. The objective in the construction is to meet, for all $e, i, j \in \mathcal{M}$, the requirement

$$R_{\langle e,i,j\rangle} : D \neq \Phi_i^{W_e} \text{ or } W_e \neq \Phi_j^D.$$

In $\mathcal{M}$, we can perform Cooper's construction by Σ_1 induction. Moreover, by Σ_1 induction again, each requirement $R_{\langle e,i,j\rangle}$ is injured at most $3^{\langle e,i,j\rangle} - 1$ times. To show $R_{\langle e,i,j\rangle}$ is satisfied, we consider

$$A_{\langle e,i,j\rangle} = \{\langle e',i',j',k\rangle : \langle e',i',j'\rangle < \langle e,i,j\rangle \wedge R_{\langle e',i',j'\rangle} \text{ receives attention at least } k \text{ times}\}$$

is bounded and Σ_1, therefore is $\mathcal{M}$-finite by Lemma 2.2. Thus the range of the recursive function $f : A_{\langle e,i,j\rangle} \to \mathcal{M}$ defined by

$$f(\langle e',i',j',k\rangle) = \mu s\,(R_{\langle e',i',j'\rangle} \text{ receives attention at least } k \text{ times by stage } s)$$

is bounded. Suppose $\operatorname{ran}(f) \subseteq [0, s)$. Then after stage s, $R_{\langle e,i,j\rangle}$ is never be injured and receives attention at most twice and $R_{\langle e,i,j\rangle}$ is satisfied eventually. □

Remark. Let $D = A \setminus B$, where A and B are r.e. such that $B \subseteq A$. $I\Sigma_1$ implies that A and B are regular. Then by $I\Sigma_1$,

$$X \subseteq D \leftrightarrow X \subseteq A \wedge X \subseteq \overline{B}, \tag{3.1}$$

$$X \subseteq \overline{D} \leftrightarrow \exists X_1 \subseteq \overline{A}\, \exists X_2 \subseteq B\, (X = X_1 \cup X_2), \tag{3.2}$$

for every $\mathcal{M}$-finite set X. Hence, $P^- + I\Sigma_1$ is sufficient to show that every d-r.e. set is Turing reducible to $\emptyset'$ and that there is a proper d-r.e. degree below $\mathbf{0'}$ by Theorem 3.1.

Suppose $\mathcal{M}$ is a $B\Sigma_1$ model. Then (3.1) remains valid but (3.2) fails: if $X \subseteq \overline{D}$ and A is not regular, then $X_2 = X \cap A$ (a subset of B) may not be $\mathcal{M}$-finite. For this reason, D may not be reducible to $\emptyset'$. The observation here will be important for our construction of a proper d-r.e. degree (not below $\mathbf{0'}$) in a $B\Sigma_1$ model.

3.2. $\boldsymbol{B\Sigma_1}$ Implies the Existence of a Proper d-r.e. Degree.

Theorem 3.2. *If* $\mathcal{M} \models P^- + B\Sigma_1 + \mathrm{Exp}$, *then there is a proper d-r.e. degree in* $\mathcal{M}$.

By Theorem 3.1, we only need to show the existence of a proper d-r.e. degree in any $B\Sigma_1$ model $\mathcal{M}$. Suppose $I \subseteq \mathcal{M}$ is a Σ_1 cut, a is an upper bound of all numbers in I, and $f : I \to \mathcal{M}$ is a Δ_1 strictly increasing cofinal function.

The difficulty of applying the $\mathbf{0'}$-priority method in a $B\Sigma_1$ model is as follows. Fix a requirement R_e and suppose each requirement $R_{e'}$, $e' < e$ is injured only $\mathcal{M}$-finitely many times. Then the set $A_e = \{\langle e', n\rangle : e' < e \wedge R_{e'} \text{ requires attention at least } n \text{ times}\}$ is r.e.

Without $I\Sigma_1$, the enumeration of A_e may not terminate at any stage s and there may not be any opportunity to satisfy R_e.

Proof of Theorem 3.2. We will construct a d-r.e. set D such that $D \not\leq_T \emptyset'$ in stages along the cut I, without the use of a priority argument. (At any stage $i \in I$, we compute $f(i)$ many steps.) For every $e \in \mathcal{M}$, the *requirement* is

$$Q_e : D \neq \Phi_e^{\emptyset'}.$$

The strategy of meeting a requirement Q_e is to attach a *witness* $X_e = [ea, (e+1)a)$ to Q_e and to look for a stage $i > 0$ such that

$$\Phi_e^{\emptyset'}[f(i-1)] \restriction X_e = \emptyset.$$

If no such stage exists, Q_e is automatically satisfied with witness X_e. If i exists, then we enumerate $ea+i$ into D at stage i. Now consider whether there is a stage $j > i$ such that

$$\Phi_e^{\emptyset'}[f(j-1)] \restriction X_e = \{ea + i\}.$$

If there is no such stage j, then Q_e is satisfied, as $\Phi_e^{\emptyset'} \restriction X_e \neq \{ea + i\} = D \restriction X_e$. If j exists, then we extract $ea + i$ from D at stage j, look for a stage $k > j$ such that

$$\Phi_e^{\emptyset'}[f(k-1)] \restriction X_e = \emptyset,$$

and repeat the strategy over again.

Notice that different requirements here do *not conflict* with one another and this strategy allows us to accommodate all requirements simultaneously.

According to the strategy, the function $\langle i, e\rangle \mapsto \langle f(i), D_{f(i)} \restriction X_e\rangle$ is recursive, where $e \in \mathcal{M}$ and $i \in I$. Thus by $B\Sigma_1$ which is equivalent with $I\Delta_1$ according to Theorem 2.1, we can easily prove the following results:

(1) For every $x \in \mathcal{M}$, x is enumerated into D at most once and is extracted from D at most once,
(2) For every $e \in \mathcal{M}$ and $i \in I$, there is at most one element in $D_{f(i)} \restriction X_e$.

Therefore, D is d-r.e. and $D \restriction X_e$ contains at most one element.

For the sake of a contradiction, suppose Q_e is not satisfied.

Case 1. $D \restriction X_e = \Phi_e^{\emptyset'} \restriction X_e = \emptyset$. Suppose $i \in I$ is a stage such that there is a computation $\langle X_e, 0, P, N\rangle \in \Phi_e^{\emptyset'}[f(i-1)]$, where P has been enumerated as a subset of $\emptyset'$ by stage $f(i-1)$ and $N \subseteq \overline{\emptyset'}$. Let $j \geq i$ be the first stage by which the element in $D_{f(i)} \restriction X_e$, if any, is extracted from D (such a stage exists for $D \restriction X_e = \emptyset$). Then at stage $j+1$, the element $ea + j + 1$ is enumerated into D and will never be extracted, contradicting the assumption that $D \restriction X_e = \emptyset$.

Case 2. $D \restriction X_e = \Phi_e^{\emptyset'} \restriction X_e = \{ea + i\}$. Then $D \restriction X_e = D_{f(i)} \restriction X_e = \Phi_e^{\emptyset'}[f(j)] \restriction X_e$ for some $j > i$. Thus $ea+i$ is extracted from D at stage j. Again, that is a contradiction. ☐

The cut I plays a significant role in the proof of Theorem 3.2. It exploits the recursive cofinal function f and *compresses* time and space to achieve the diagonalization against $\Phi_e^{\emptyset'}$ for every e. Notice that the set D constructed in the proof of Theorem 3.2 is unbounded. With the aid of I, we can actually further compress the space and construct a bounded d-r.e. set $D \not\leq_T \emptyset'$.

3.3. **Bounded Sets.** Let $\mathcal{M}, I, a, f$ be as in Section 3.2. Suppose $D = A \setminus B$, where A and B are bounded r.e. sets and $B \subseteq A$. Let b be an upper bound of all elements in A. We may further assume that $A_{f(0)} = B_{f(0)} = \emptyset$ and for all $i \in I$, $B_{f(i+1)} \subseteq A_{f(i)}$ (this is to ensure that, along the time axis I, none appears in B before it is enumerated in A). Since the set

$$H = \{(x, i) : x < b,\, i \in I,\, x \in (A_{f(i)} \setminus A_{f(i-1)}) \cup (B_{f(i)} \setminus B_{f(i-1)})\}, \tag{3.3}$$

which records the stages of enumeration, is Δ_1 on $[0, b) \times I$, H is coded by Lemma 2.4. Suppose $\hat{H} \subseteq [0, b) \times [0, a)$ is a code of H satisfying for every $x < b$, there are exactly two i's such that $(x, i) \in \hat{H}$. Define $i_x = \min\{i < a : (x, i) \in \hat{H}\}$ and $j_x = \max\{i < a : (x, i) \in \hat{H}\}$. Then for every $x < b$,

$$j_x \in I \to x \in \overline{D}, \quad (i_x \in I \wedge j_x \in \overline{I}) \to x \in D, \quad i_x \in \overline{I} \to x \in \overline{D}.$$

Fix $e \in \mathcal{M}$. To ensure $D \neq \Phi_e^{\emptyset'}$, we need to implement a diagonalization strategy as in Theorem 3.2. Given any r.e. set R, we say x that *escapes from computation* Φ_e^R *at stage* s, if $x \in B$ and for every computation of the form $\langle \{x\}, 0, P, N\rangle$ in $\Phi_e^R[s]$, $N \cap R \neq \emptyset$. Note that if $X \subseteq \overline{D}$ is $\mathcal{M}$-finite such that for every stage s, there is an $x \in X$ such that x escapes from computation Φ_e^R at stage s, then $\Phi_e^R \upharpoonright X \neq D \upharpoonright X = \emptyset$ by the local downward closure property of Φ_e^R. This idea leads to the following lemma.

Lemma 3.3. *If R is r.e. and $\Phi_e^R = D$, then for some stage s, there is no $x < b$ with $f(i_x) \geq s$ such that x escapes from computation Φ_e^R at stage s.*

Proof. We prove this by contradiction. Suppose $\Phi_e^R = D$ and for every $i \in I$, there is an $x < b$ with $i_x \geq i$ such that x escapes from computation Φ_e^R at stage $f(i)$. Then the function $\alpha : I \to I^2$ defined by $i \mapsto (i_x, j_x)$ where $x < b$ is the first enumerated number satisfying $i_x \geq i$, $j_x \in I$ and x escapes from computation Φ_e^R at stage $f(i)$. The function α is total on I by assumption.

Since α is recursive, α is coded on I^3 by a function $\hat{\alpha} : [0, a) \to [0, a)^2$. We denote the first coordinate of $\hat{\alpha}(i)$ by $\hat{\alpha}_1(i)$ and the second by $\hat{\alpha}_2(i)$. We may further assume that for every $i < a$, $\hat{\alpha}_2(i) > \hat{\alpha}_1(i) \geq i$.

Now let $X = \{x < b : (i_x, j_x) \in \mathrm{ran}(\hat{\alpha})\}$. Then for every $x \in X$,

Case 1. There is an $i \in I$ such that $\hat{\alpha}(i) = (i_x, j_x)$. Then $\alpha(i) = (i_x, j_x)$. Thus, $i_x, j_x \in I$.

Case 2. There is an $i \in \overline{I}$ such that $\hat{\alpha}(i) = (i_x, j_x)$. Then $i_x, j_x \in \overline{I}$ since $i_x \geq i$.

Therefore, $X \subseteq \overline{D}$. Moreover, by the definition of α, for every stage $f(i)$, there is an $x \in X$ such that x escapes from computation Φ_e^R at stage $f(i)$, and so $\Phi_e^R \upharpoonright X \neq \emptyset$, contradicting the assumption that $\Phi_e^R = D$. □

Corollary 3.4. *If $\Phi_e^{\emptyset'} = D$, then for some stage s, there is no $x < b$ such that x escapes from computation $\Phi_e^{\emptyset'}$ at stage s and $f(i_x) \geq s$.*

The above definition can be generalized to computations $\{\Psi_e\}_{e \in \mathcal{M}}$. We say an element x *escapes from computation* Ψ_e^I *at stage* s, if $x \in B$ and $n \in I$ for all $\langle \{x\}, 0, n\rangle$ in $\Psi_e^I[s]$.

Corollary 3.5. *If $\Psi_e^I = D$, then for some $i \in I$, there is no $x < b$ such that x escapes from computation Ψ_e^I at stage $f(i)$ and $i_x \geq i$.*

Lemma 3.6. *$D \leq_T \emptyset'$ if and only if $D \leq_T I$.*

Proof. We only need to show the "only if" part. Since every $\mathcal{M}$-finite set X,

$$X \subseteq D \leftrightarrow (X \subseteq [0, b) \wedge X \subseteq D),$$
$$X \subseteq \overline{D} \leftrightarrow (X \cap [0, b) \subseteq \overline{D}),$$

we only need to consider subsets of $[0, b)$ in the computation of D.

Suppose $D = \Phi_e^{\emptyset'}$. Define G to be a set that codes the approximation of $\Phi_e^{\emptyset'}$ as follows:

$$G = \{(X, i, j, z) : X \subseteq [0, b) \wedge i < j \in I \wedge (z = 0 \vee z = 1) \wedge \\ \exists P \, \exists N \, (\langle X, z, P, N\rangle \in \Phi_e^{\emptyset'}[f(i)] \wedge N \subseteq \overline{\emptyset'_{f(j)}})\}.$$

That is, $(X, i, j, z) \in G$ if and only if the computation $\Phi_e^{\emptyset'}(X)[f(i)] = z$ is still valid at stage $f(j)$. Since G is Δ_1 on $[0, 2^b) \times I \times I \times [0, 2)$, by Lemma 2.4, G is coded by $\hat{G} \subseteq [0, 2^b) \times [0, a) \times [0, a) \times [0, 2)$.

Suppose $X \subseteq [0, b)$. If $X \subseteq D$, then there is a quadruple $\langle X, 1, P, N\rangle \in \Phi_e^{\emptyset'}[f(i)]$ such that $N \subseteq \overline{\emptyset'}$. Thus, for every $j > i$, $j \in I$, we have $N \subseteq \overline{\emptyset'_{f(j)}}$ and $(X, i, j, 1) \in \hat{G}$. Since I is not $\mathcal{M}$-finite,

$$u = \sup\{j < a : \forall j' \, (i < j' < j \to (X, i, j', 1) \in \hat{G})\} \in \overline{I}. \tag{3.4}$$

Therefore,

$$\exists i \in I \, \exists u \in \overline{I} \, \forall j \, (i < j < u \to (X, i, j, 1) \in \hat{G}). \tag{3.5}$$

Conversely, if (3.5) holds, then there is some $\langle X, 1, P, N\rangle \in \Phi_e^{\emptyset'}[f(i)]$ such that $N \subseteq \overline{\emptyset'}$. Thus, for every $\mathcal{M}$-finite set $X \subseteq [0, b)$,

$$X \subseteq D \leftrightarrow \exists i \in I \, \exists u \in \overline{I} \, \forall j \, (i < j < u \to (X, i, j, 1) \in \hat{G}),$$

and similarly,

$$X \subseteq \overline{D} \leftrightarrow \exists i \in I \, \exists u \in \overline{I} \, \forall j \, (i < j < u \to (X, i, j, 0) \in \hat{G}).$$ □

To construct a bounded d-r.e. set $D \not\leq_T \emptyset'$, by Lemma 3.6, it is enough to ensure that $D \not\leq_T I$: For each i and e, if there is an $x \in D$ with $i_x \geq i$ such that x escapes from Ψ_e^I at stage $f(i)$, then $D \neq \Psi_e^I$ by Corollary 3.5.

Lemma 3.7. *For every $e, s \in \mathcal{M}$, if $\Psi_e^I = D$, then the set*

$$J_{e,s} = \{j \in I : \exists x < b \, \exists i \in I \, \exists n \in \overline{I} \, (\langle \{x\}, 0, n\rangle \in \Psi_e^I[s] \wedge f(i) \leq s \wedge i = i_x \wedge j = j_x)\}$$

is bounded in I. Moreover, there is a recursive function $\beta : \mathcal{M}^2 \to \mathcal{M}$ such that whenever $\Psi_e^I = D$, $\beta(e, s) \in I$ is an upper bound of all elements in $J_{e,s}$.

Proof. Fix s and e. Let $i^* \in I$ be the largest i such that $f(i^*) \leq s$. Then

$$\begin{aligned} J_{e,s} &= \{j \in I : \exists x < b \, \exists i \leq i^* \, \exists n \in \overline{I} \, (\langle \{x\}, 0, n\rangle \in \Psi_e^I[s] \wedge i = i_x \wedge j = j_x)\} \\ &\subseteq \{j < a : \exists x < b \, \exists i \leq i^* \, \exists n > j \, (\langle \{x\}, 0, n\rangle \in \Psi_e^I[s] \wedge i = i_x \wedge j = j_x)\}. \end{aligned} \tag{3.6}$$

We denote the set in the second line of (3.6) by $\tilde{J}_{e,s}$, which is $\mathcal{M}$-finite. Let $\beta(e, s) = \sup \tilde{J}_{e,s}$. We only need to show that if $\Psi_e^I = D$, then $\tilde{J}_{e,s} \subset I$.

Suppose $\Psi_e^I = D$ and $\langle \{x\}, 0, n\rangle \in \Psi_e^I$, where $x < b$, $i_x \leq i^*$ and $j_x < n$. Since $i_x \in I$, $x \in A$. If $n \in \overline{I}$, then $x \in \overline{D}$, and so $x \in B$ with $j_x \in I$. If $n \in I$, then $j_x \in I$ since $j_x < n$. In any case $j_x \in I$, so $\tilde{J}_{e,s} \subset I$. □

Proof of Theorem 3.2 (bounded set D). I, a, f are defined as above. Let

$$\hat{H} = \{(x, i) \in [0, 2^a) \times [0, a) : \text{ If the } \mathcal{M}\text{-finite set } X \text{ represented by the binary expansion of } x \text{ has at least two elements, then } i = \min X \text{ or } i = \max X, \text{and if } X \text{ has less than two elements, then } i = 0 \text{ or } i = 1\}.$$

That is, for every pair $(i, j) \in [0, a) \times [0, a)$ with $i < j$, we have some $x < 2^a$ such that $(i, j) = (i_x, j_x)$, where

$$i_x = \min\{i : (x, i) \in \hat{H}\}, \quad j_x = \max\{i : (x, i) \in \hat{H}\}.$$

Then $\hat{H}$ codes the enumeration of a d-r.e. set D:

$$D = \{x < 2^a : i_x \in I, j_x \in \overline{I}\}.$$

Let β be defined as in Lemma 3.7.

Now we claim that $D \not\leq_T I$, so $D \not\leq_T \emptyset'$ by Lemma 3.6. For the sake of contradiction, suppose $D = \Psi_e^I$. For each $i \in I$, let $x(i)$ be the least such that $(i_{x(i)}, j_{x(i)}) = (i, \beta(e, f(i))+1)$. By Lemma 3.7, $x(i)$ escapes from Ψ_e^I at stage $f(i)$. Then by Lemma 3.5, $D \neq \Psi_e^I$. That is a contradiction. □

3.4. **$B\Sigma_1 + \neg I\Sigma_1$ Implies d-r.e. degrees below $\mathbf{0'}$ are r.e.** In Sections 3.2 and 3.3, it was shown that in a $B\Sigma_1$ model, there is a proper d-r.e. degree not below $\mathbf{0'}$. In this section, we prove that in any $B\Sigma_1$ model, it is impossible to find a proper d-r.e. degree below $\mathbf{0'}$.

Let $\mathcal{M}, f, I, a$ be as in Section 3.2 and 3.3. Let $D = A \setminus B$, where A and B are r.e. (may not be bounded) and $B \subseteq A$. As before, we may assume that $A_{f(0)} = B_{f(0)} = \emptyset$ and for all $i \in I$, $B_{f(i+1)} \subseteq A_{f(i)}$. If D is recursive, then clearly $\deg(D)$ is r.e. For the rest of this section, we always assume that $D = \Phi_e^{\emptyset'}$ is not recursive. The object is to construct an r.e. set $W \equiv_T D$.

Regular Sets. We first consider the case when D is regular. The idea of considering regular and non-regular sets can be traced back to Chong and Mourad [1].

Lemma 3.8. *If $D = \Phi_e^{\emptyset'}$ is regular, then $D \leq_T I$.*

Proof. Suppose D is regular. The method here is similar to that in Lemma 3.6. For each $k \in I$ and stage s, we say (E_0, E_1) is a *partition of* $[0, f(k))$ *at stage* s, if

(1) $E_0 \cup E_1 = [0, f(k))$, $E_0 \cap E_1 = \emptyset$, and
(2) There are $\mathcal{M}$-finite sets P_0, P_1, N_0, N_1 such that $\langle E_0, 0, P_0, N_0\rangle, \langle E_1, 1, P_1, N_1\rangle \in \Phi_e^{\emptyset'}[f(i)]$.

Note that at each stage $f(i)$, there is at most one partition of $[0, f(k))$. Let

$$G = \{(i, j, k) \in I^3 : i \leq j \wedge \text{There are partitions of } [0, f(k)) \text{ at stage } f(i) \text{ and } f(j) \\ \wedge \text{ The two partitions are equal}\}.$$

By Lemma 2.4, G is coded on I^3. Suppose $E_{0,k} = \overline{D} \cap [0, f(k))$ and $E_{1,k} = D \cap [0, f(k))$, then there is some $i \in I$ such that

$$\exists P_0, P_1 \subseteq \emptyset' \, \exists N_0, N_1 \subseteq \overline{\emptyset'} \, (\langle E_{0,k}, 0, P_0, N_0\rangle \in \Phi_e^{\emptyset'}[f(i)] \wedge \langle E_{1,k}, 1, P_1, N_1\rangle \in \Phi_e^{\emptyset'}[f(i)]).$$

That is, at stage $f(i)$, Φ_e *correctly computes a partition of* $[0, f(k))$. Then for any stage $s \geq f(i)$, the partition of $[0, f(k))$ at stage s must be $(E_{0,k}, E_{1,k})$, so $(i, j, k) \in \hat{G}$ for all $j \geq i$. Thus for any $k \in I$,

$$E_{0,k} = \overline{D} \cap [0, f(k)) \wedge E_{1,k} = D \cap [0, f(k)) \to \exists i \in I \, \exists i' \in \overline{I} \, \forall j \, ((E_{0,k}, E_{1,k}) \text{ is a} \\ \text{partition of } [0, f(k)) \text{ at stage } f(i) \wedge (i \leq j \leq i' \to (i, j, k) \in \hat{G})). \tag{3.7}$$

Now suppose the conclusion in (3.7) holds for (E_0, E_1), and we prove the hypothesis in (3.7). Thus, the hypothesis and conclusion in (3.7) are actually equivalent and $D \leq_T I$.

For the sake of a contradiction, we assume that $\overline{D} \cap [0, f(k)) \neq E_0$, $D \cap [0, f(k)) \neq E_1$, $i \in I$, (E_0, E_1) is a partition of $[0, f(k))$ at stage $f(i)$, $i' \in \overline{I}$, and $\forall j \, (i \leq j \leq i' \to (i, j, k) \in \hat{G})$. Let $\tilde{j} > i$ such that $\tilde{j} \in I$ and Φ_e correctly computes a partition of $[0, f(k))$ at stage $f(\tilde{j})$. Then $(i, \tilde{j}, k) \notin \hat{G}$. That is a contradiction. □

Lemma 3.9. *If $D = \Phi_e^{\emptyset'}$ is regular and non-recursive, then $D \geq_T I$.*

Proof. Consider the following set:

$$G_0 = \{(i, j, k) \in I^3 : i < j < k \wedge \exists x \in A_{f(i)} (x \in B_{f(k)} \setminus B_{f(j)})\},$$

i.e. $f(k)$ is a stage that we find $A_{f(i)} \setminus B_{f(j)}$ is not a subset of D. By Lemma 2.4, G_0 can by coded on I^3 by a set $\hat{G}_0 \subseteq [0, a)^3$ such that for every $(i, j, k) \in \hat{G}_0$, $i < j < k$ and for each $i < j \in I$, there is some $k < a$ such that $(i, j, k) \in \hat{G}_0$. (If $(i, j, k) \in I^2 \times [0, a)$ and $A_{f(i)} \setminus B_{f(j)} \subseteq D$, then $k \in \overline{I}$.)

Suppose the set $C_0 = \{k < a : \exists i, j \in I (A_{f(i)} \setminus B_{f(j)} \subseteq D \wedge (i, j, k) \in \hat{G}_0)\} \subseteq \overline{I}$ is *unbounded in* $\overline{I}$, i.e. no $i' \in \overline{I}$ is a lower bound of numbers in C_0, then

$$i' \in \overline{I} \leftrightarrow \exists k < i' (k \in C_0)$$

It follows that $\overline{I}$ is r.e. in D. Hence $I \leq_p D$. By Lemma 2.6, $I \leq_T D$.

Now suppose C_0 is bounded in $\overline{I}$. Then there is an $i' \in \overline{I}$ such that

$$\forall i, j \in I (A_{f(i)} \setminus B_{f(j)} \subseteq D \leftrightarrow \exists k > i' ((i, j, k) \in \hat{G}_0)).$$

Furthermore, since D is regular, for every $i \in I$ there is $j \in I$ such that $A_{f(i)} \setminus B_{f(j)} \subseteq D$. Thus, $D = \{x : \exists i, j \in I \exists k > i' ((i, j, k) \in \hat{G}_0 \wedge x \in A_{f(i)} \setminus B_{f(j)})\}$ is r.e. In that case, by modifying the enumeration of D, we may assume that $B = \emptyset$. Let

$$G_1 = \{(i, j, k) \in I^3 : i < j \wedge \exists x < f(k) (x \in A_{f(j)} \setminus A_{f(i)})\},$$

i.e. $f(j)$ is a stage that we find $A_{f(i)} \upharpoonright [0, f(k))$ is not $D \upharpoonright [0, f(k))$. Again G_1 is coded by $\hat{G}_1 \subseteq [0, a)$ in I^3. We also assume that for all $i < k \in I$, there is some j such that $(i, j, k) \in \hat{G}_1$. (If $A_{f(i)} \upharpoonright [0, f(k)) = D \upharpoonright [0, f(k))$ and $(i, j, k) \in \hat{G}_1$, then $j \in \overline{I}$.)

Suppose $C_1 = \{j : \exists i, k \in I (A_{f(i)} \upharpoonright [0, f(k)) = D \upharpoonright [0, f(k)) \wedge (i, j, k) \in \hat{G}_1)\}$ is unbounded in $\overline{I}$. Then for all i'',

$$i'' \in \overline{I} \leftrightarrow \exists i \in I \exists k \in I \exists j < i'' (A_{f(i)} \upharpoonright [0, f(k) = D \upharpoonright [0, f(k)) \wedge (i, j, k) \in \hat{G}_1).$$

Hence, $I \leq_T D$.

If C_1 is bounded in $\overline{I}$, then for some $i'' \in \overline{I}$,

$$\forall i, k \in I (A_{f(i)} \upharpoonright [0, f(k)) = D \upharpoonright [0, f(k)) \leftrightarrow \exists k > i'' (i, j, k) \in \hat{G}_1). \quad (3.8)$$

Again, since D is regular, for all $k \in I$, there is some $i \in I$ such that $A_{f(i)} \upharpoonright [0, f(k)) = D \upharpoonright [0, f(k))$. Then (3.8) implies that D is recursive. That is a contradiction. □

Corollary 3.10. *If $D = \Phi_e^{\emptyset'}$ is regular and non-recursive, then $D \equiv_T I$.*

Non-regular Sets. Similar to Lemma 3.9, we have

Lemma 3.11. *If D is non-regular, then $D \geq_T I$.*

Proof. Suppose $d \in \mathcal{M}$ and $D \upharpoonright [0, d)$ is not $\mathcal{M}$-finite. As in Section 3.3, let

$$H = \{(x, i) : x < d \wedge i \in I \wedge (x \in A_{f(i)} \setminus A_{f(i-1)} \vee x \in B_{f(i)} \setminus B_{f(i-1)})\},$$

and let $\hat{H} \subseteq [0, d) \times [0, a)$ be a code of H in $[0, d) \times I$ such that for every $x < d$, there are exactly two i's with $(x, i) \in \hat{H}$. Define $i_x = \min\{i < a : (x, i) \in \hat{H}\}$ and $j_x = \max\{i < a : (x, i) \in \hat{H}\}$.

For every $x < d$, if $x \in D$, then $i_x \in I$ and $j_x \in \overline{I}$. Suppose such j_x's are *unbounded in* $\overline{I}$, i.e.

$$\forall i' \in \overline{I} \exists x < d (x \in D \wedge j_x < i'). \quad (3.9)$$

Then for any i',

$$i' \in \overline{I} \leftrightarrow \exists x < d (x \in D \wedge j_x < i').$$

Thus, $I \leq_T D$.

If (3.9) fails, then let $i' \in \overline{I}$ be such that $\forall x < d\,(x \in D \rightarrow j_x > i')$. Then we consider the x's in $\overline{D}$ such that $j_x > i'$. For such an x, $i_x \in \overline{I}$. If

$$\forall i'' \in \overline{I}\,\exists x < d\,(x \in \overline{D} \wedge j_x > i' \wedge i_x < i''). \tag{3.10}$$

Then for every i'',

$$i'' \in \overline{I} \leftrightarrow \exists x < d\,(x \in \overline{D} \wedge j_x > i' \wedge i_x < i'').$$

Thus, $I \leq_T D$.

Suppose (3.10) fails again, and let $i'' \in \overline{I}$ be such that $\forall x < d\,(x \in \overline{D} \wedge j_x > i' \rightarrow i_x > i'')$. Then for all $x < d$,

(1) If $j_x > i'$, then $x \in D$ if and only if $i_x < i''$.
(2) If $j_x \leq i'$, then $x \in \overline{D}$.

Thus, $D \restriction [0, d)$ is Δ_1. According to Lemma 2.4, $D \restriction [0, d)$ is $\mathcal{M}$-finite, contradicting our assumption. □

Given $k \in I$, we say $r_k \in I$ is a *separating point of* $[0, f(k))$ if

$$\forall x \in ([0, f(k)) \setminus A_{f(r_k)})\,(x \in B \rightarrow \exists P \subseteq \emptyset'\,\exists N \subseteq \overline{\emptyset'}\,(\langle\{x\}, 0, P, N\rangle \in \Phi_e^{\emptyset'}[f(r_k)])).$$

By Corollary 3.4, a separating point of $[0, f(k))$ exists for every $k \in I$. Moreover, the predicate "r is not a separating point of $[0, f(k))$", whose variables are r and k, is Σ_1, so it is reducible to I by Lemma 3.12.

Lemma 3.12 (Chong and Yang [3])**.** *Every bounded r.e. set is reducible to I.*

If $B = \emptyset$, then clearly D is of r.e. degree. For the general case, intuitively, we modify the enumeration of D to be more "effective": we enumerate x into D only if we see that x is enumerated into A at some stage s and all the computations of the form $\langle\{x\}, 0, P, N\rangle \in \Phi_e^{\emptyset'}[s]$ are fake by $N \cap \emptyset' \neq \emptyset$. That is, we define

$$A^* = \{x : \exists s\,\exists t > s\,[x \in A_s \setminus B_t \wedge \forall P, N\,(\langle\{x\}, 0, P, N\rangle \in \Phi_e^{\emptyset'}[s] \rightarrow N \cap \emptyset'_t \neq \emptyset)]\}$$
$$B^* = A^* \cap B.$$

Then $D = A^* \setminus B^*$. We show that $D \equiv_T A^* \oplus B^* \oplus I$. By Lemma 3.11, $D \geq_T I$. Thus, we only need to show:

Lemma 3.13. *If D is non-regular, then $D \oplus I \equiv_T A^* \oplus B^* \oplus I$.*

Proof. Fix any $k \in I$. By Lemma 3.12, we can get a separating point r (may not be unique) of $[0, f(k))$ recursively in I. Then the interval $[0, f(k))$ is separated into two parts: $[0, f(k)) \setminus A_{f(r)}$ and $[0, f(k)) \cap A_{f(r)}$. By definition of separating point,

$$\forall x \in [0, f(k)) \setminus A_{f(r)}\,(x \in B \rightarrow \exists P \subseteq \emptyset'\,\exists N \subseteq \overline{\emptyset'}\,(\langle\{x\}, 0, P, N\rangle \in \Phi_e^{\emptyset'}[f(r)])).$$

Therefore,

$$\forall x \in [0, f(k)) \setminus A_{f(r)}\,(x \in B \rightarrow x \notin A^*),$$

i.e.

$$D \restriction ([0, f(k)) \setminus A_{f(r)}) = A^* \restriction ([0, f(k)) \setminus A_{f(r)}), \quad B^* \restriction ([0, f(k)) \setminus A_{f(r)}) = \emptyset. \tag{3.11}$$

In addition, we claim that

$$A^* \restriction ([0, f(k)) \cap A_{f(r)}) \text{ is recursive.}$$

For every $x \in [0, f(k)) \cap A_{f(r)}$, since x is enumerated into A at some stage s, there exists a stage $t > s$ such that either

Case 1. x is enumerated into B at stage t, or

Case 2. All computations of the form $\langle\{x\}, 0, P, N\rangle$ in $\Phi_e^{\emptyset'}[s]$ are fake, i.e. $N \cap \emptyset'_t \neq \emptyset$.

Thus, $x \in A^*$ if and only if Case 2 occurs first, which can be determined recursively. Therefore, to determine whether X is a subset of $D \upharpoonright ([0, f(k)) \cap A_{f(r)})$ or a subset of $\overline{D} \upharpoonright ([0, f(k)) \cap A_{f(r)})$, we only need to take $B^* \upharpoonright ([0, f(k)) \cap A_{f(r)})$ as an oracle, and vice versa. This property and (3.11) combine to produce $D \oplus I \equiv_T A^* \oplus B^* \oplus I$. □

Theorem 3.14. *In a $B\Sigma_1$ model, every d-r.e. degree below $\mathbf{0}'$ is r.e.*

We therefore have:

Corollary 3.15. *Assume $P^- + B\Sigma_1 + \mathrm{Exp}$. Then*

(1) There is a proper d-r.e. degree;
(2) $I\Sigma_1$ is equivalent with the existence of a proper d-r.e. degree below $\mathbf{0}'$.

4. Degrees below $\mathbf{0}'$ in a Saturated model

As shown in Section 3, any proper d-r.e. degree in a $B\Sigma_1$ model is not below $\mathbf{0}'$. In this section, we expand our investigation with an analysis of the degrees below $\mathbf{0}'$ for $B\Sigma_1$ models. The main result is:

Theorem 4.1. $P^- + B\Sigma_1 + \mathrm{Exp} \nvdash$ *There is a non-r.e. degree below $\mathbf{0}'$.*

To show this theorem, we consider a $B\Sigma_1$ model $\mathcal{M}$ with the following properties:

(1) $\mathbb{N} \subset \mathcal{M}$ is a Σ_1 cut of $\mathcal{M}$.
(2) Every subset of $\mathbb{N}$ is coded (on $\mathbb{N}$) in the model $\mathcal{M}$.

Such a model $\mathcal{M}$ is called a *saturated* $B\Sigma_1$ (or *saturated*, for short) model. In [14], Slaman and Woodin showed that a saturated $B\Sigma_1$ model exists. Let I denote $\mathbb{N}$, $a \in \mathcal{M}$ be such that $I \subset [0, a)$, and $f : I \to \mathcal{M}$ be a strictly increasing cofinal Δ_1 function with $f(0) = 0$. We may further assume that $\langle\cdot,\cdot\rangle \upharpoonright I^2$ maps onto I. The proof that every d-r.e. set $D = A \setminus B$ reducible to $\emptyset'$ is of r.e. degree in Section 3.4 could be simplified if $\mathcal{M}$ is saturated: Suppose $D = \Phi_e^{\emptyset'}$ and

$$G = \{\langle k, r\rangle : r \text{ is the least separating point of } [0, f(k+1))\}.$$

Then G is coded by $\hat{G} \subseteq [0, a)$. For each $k < a$, let r_k be the least r such that $\langle k, r\rangle \in \hat{G}$ and $[0, f(k))$ can be recursively separated into two parts:

- $P_0 = \{x : \exists k \in I\, (x \in [f(k), f(k+1)) \wedge x \notin A_{f(r_k)})\}$ and
- $P_1 = \{x : \exists k \in I\, (x \in [f(k), f(k+1)) \wedge x \in A_{f(r_k)})\}$.

For any $x \in [f(k), f(k+1))$,

- If $x \in A_{f(r_k)}$, then $x \in D$ if and only if $x \notin B$.
- If $x \notin A_{f(r_k)}$, then by the definition of separating point, $x \in D$ if and only if all computations of the form $\langle\{x\}, 0, P, N\rangle$ in $\Phi_e^{\emptyset'}[f(r_k)]$ are fake, (i.e. $N \cap \emptyset' \neq \emptyset$) and $x \in A$.

Thus, $D \upharpoonright P_0$ is Σ_1, $D \upharpoonright P_1$ is Π_1 and D is of r.e. degree. Clearly, the key to this proof is the separating points.

Now suppose $V = \Phi_e^{\emptyset'}$, which may not be d-r.e. We generalize the notion of separating points as follows: Let $k \in I$ and

$$H_k = \{(x, i) : x \in [f(k), f(k+1)) \wedge i \in I \wedge \Phi_e^{\emptyset'}(x)[f(i)] = 1\}.$$

That is, H_k records the approximation of $\Phi_e^{\emptyset'} \upharpoonright [f(k), f(k+1))$. Since H_k is recursive on $[f(k), f(k+1)) \times I$, it is coded by some $\hat{H}_k \subseteq [f(k), f(k+1)) \times [0, a)$. For each k, we fix a code $\hat{H}_k$. For any $i < a$ and $x \in [f(k), f(k+1))$, we define

$$V_i(x) = \begin{cases} 1 & \text{if } (x,i) \in \hat{H}_k \\ 0 & \text{otherwise} \end{cases}$$

and so $V(x) = \lim_{i \in I} V_i(x)$.

Suppose $i \in I$. Then

(1) x is said to be *i-honest*, if for any $j \in I$ greater than i, $V_j(x) = V_i(x)$; otherwise, x is an *i-liar*.

(2) x is *found to be an i-liar by stage j*, if x is an i-liar, $j \in I$, $j > i$ and

$$\exists k \leq j\, (k > i \wedge V_k(x) \neq V_i(x)).$$

(3) x is called an *i-white liar*, if x is an i-liar and $V(x) = V_i(x)$.

(4) x is an *i-malicious liar*, if x is an i-liar and $V(x) \neq V_i(x)$.

We observe that *white liars* correspond to *escaping elements* in Section 3.3. Similar to Lemma 3.3, we have

Lemma 4.2. *For any $i, k \in I$, there is a $j > i$ such that all i-white liars in $[f(k), f(k+1))$ are found by stage j.*

Proof. For the sake of a contradiction, we suppose $i, k \in I$ and for each $j > i$ in I, there is an i-white liar not founded by stage j, and without loss of generality, we assume all such i-white liars are not in V. Then consider the function $\delta : I \setminus [0, i] \to I$, $j \mapsto \langle n_j, z^j_0, z^j_1, \ldots, z^j_{n_j-1}\rangle$, where

(1) z^j is the first i-white liar not in V that is found at a least stage $j' > j$ but is not found by stage j, and

(2) $z^j_0 < z^j_1 < \ldots < z^j_{n_j-1}$ is a list of all stages $l \in I$ such that $V_l(z^j) \neq V_{l-1}(z^j)$.

According to the saturation of $\mathcal{M}$, δ is coded on I^2 by an $\mathcal{M}$-finite partial function $\hat{\delta} : [i+1, a) \to [0, a)$ with the following properties:

(1) $\mathrm{dom}(\hat{\delta}) \supset \mathrm{dom}(\delta)$, and

(2) For each $j \in \mathrm{dom}(\hat{\delta})$, $\hat{\delta}(j) = \langle n_j, z^j_0, z^j_1, \ldots, z^j_{n_j-1}\rangle$ for some $n_j, z^j_0, \ldots, z^j_{n_j-1}$ such that
    (a) $z^j_0 < z^j_1 < \ldots, < z^j_{n_j-1} < a - 1$, and
    (b) $\forall m\, (z^j_m > i \leftrightarrow z^j_m > j)$.

Then for any $j \in \mathrm{dom}(\hat{\delta})$, we may recursively find an $x \in [f(k), f(k+1))$ with $V_i(x) = 0$ such that $z^j_0, \ldots, z^j_{n_j-1}$ are the first n_j many l's satisfying $V_l(x) \neq V_{l-1}(x)$ and the $(n_j+1)^{\text{th}}$ l is the largest possible according to $\hat{H}_k$. This x is said to be *corresponding to j*. Notice that if x corresponds to a $j \in I$, then x is also an i-white liar not found by j, and if x corresponds to a $j \in \overline{I}$, then x is i-honest.

Now let

$$X = \{x \in [f(k), f(k+1)) : \exists j \in \mathrm{dom}(\hat{\delta})\, (x \text{ is corresponding to } j)\}.$$

Since each $x \in X$ is either i-honest or an i-white liar, $X \subseteq \overline{V}$. But then there is some $j > i$ such that $\Phi^{\emptyset'}_e[f(j)] \restriction X = \emptyset$. According to the local downward closure property of $\Phi^{\emptyset'}_e$, all i-liars in X are found by stage j. That is a contradiction. ☐

Suppose all i-white liars in $[f(k), f(k+1))$ are found by stage j, $x \in [f(k), f(k+1))$ and the approximation $V_l(x)$ does not "change its mind" between i and j, i.e. $\forall l \in [i, j]\, (V_i(x) = V_l(x))$. Then x cannot be an i-white liar. Thus, for all such x's,

$$V(x) = V_i(x) \leftrightarrow \neg\exists j \in I\, (j > i \wedge V_j(x) \neq V_i(x)). \tag{4.1}$$

Conversely, for any $x \in [f(k), f(k+1))$, there are $i < j \in I$ such that x is i-honest and all i-white liars in $[f(k), f(k+1))$ are found by stage j. Then the approximation $V_l(x)$ does not "change its mind" between i and j for all $j > i$.

Lemma 4.3. *There exists $u \in I$ with property $\rho(k, u)$:*
For any $x \in [f(k), f(k+1))$, there are $i < j < u$ such that all i-white liars in $[f(k), f(k+1))$ are found by stage j and $V_i(x) = V_l(x)$ for all $l \in [i, j]$.

Proof. Let $T_k = \{\langle i, j\rangle \in I^2 : i < j \wedge \text{All } i\text{-white liars in } [f(k), f(k+1)) \text{ are found by stage } j\}$. Since $\mathcal{M}$ is saturated, T_k is coded by $\hat{T}_k \subseteq [0, a)$ so that for all $\langle i, j\rangle \in \hat{T}_k$, $i < j$. Now consider the function $\epsilon : [f(k), f(k+1)) \to \hat{T}_k$, $x \mapsto \mu\langle i, j\rangle(\langle i, j\rangle \in \hat{T}_k \wedge \forall l \in [i, j]\, V_i(x) = V_l(x))$. For every x in $[f(k), f(k+1))$, since $\langle \cdot, \cdot\rangle$ maps I^2 onto I and there is a pair $\langle i, j\rangle \in T_k$ such that $\forall l \in [i, j]\, (V_i(x) = V_l(x))$, we have $\epsilon(x) \in I$. By $B\Sigma_1$, $\text{ran}(\epsilon)$ is bounded in I. Let $u \in I$ be an upper bound of all elements in $\text{ran}(\epsilon)$ and it is straightforward to verify that $\rho(k, u)$ holds. □

For each $k \in I$, let u_k be the least u satisfying $\rho(k, u)$ and

$$F = \{\langle k, i, j\rangle \in I^3 : i < j < u_k \wedge j \text{ is the least such that all } i\text{-white liars in } [f(k), f(k+1)) \text{ are found by stage } j\}.$$

Suppose $\hat{F} \subseteq [0, a)$ is a code of F such that for all $\langle k, i, j\rangle \in \hat{F}$ with $k \in I$, $i < j < u_k$ and $\langle k, i, j\rangle \in F$.

We recursively separate $\mathcal{M}$ into countably many parts $\{E_{k,i}\}_{k \in I, i < u_k}$:

$$E_{k,i} = \{x \in [f(k), f(k+1)) \setminus \textstyle\bigcup_{i' < i} E_{k,i'} : \exists j\, (\langle k, i, j\rangle \in \hat{F} \wedge \forall l \in [i, j]\, (V_i(x) = V_l(x)))\}.$$

For every $k \in I$ with $\langle k, i, j\rangle \in \hat{F}$ and every $x \in E_{k,i}$, x cannot be an i-white liar since all i-white liars in $[f(k), f(k+1))$ are found by stage j. Thus (4.1) holds for all $x \in E_{k,i}$. Define the r.e. set A on each $E_{k,i}$ by

$$x \in A \restriction E_{k,i} \leftrightarrow \exists j \in I\, (j > i \wedge V_j(x) \neq V_i(x)).$$

By (4.1) again, $A \equiv_T V$ and $\deg(V)$ is r.e.

5. Open problems

We conclude this paper with two open problems.

(1) In any $B\Sigma_1$ model, a proper d-r.e. degree exists. Does a proper 3-r.e. degree exist? In general, is there a proper n-r.e. degree in $B\Sigma_1$ models, for $n \geq 3$? If R is r.e. and Q is a subset of $\overline{R}$, then for all stage s, $Q \subseteq \overline{R_s}$. However, if R is 2-r.e. for $n \geq 2$ and $Q \subseteq \overline{R}$, then Q may not be a subset of $\overline{R_s}$ for any s and the computation Φ_e^R may not be correctly approximated. This raises a main difficulty to diagonalize Φ_e^R.

(2) Is there a $B\Sigma_1$ model with a non-r.e. degree below $\mathbf{0'}$? We have seen a $B\Sigma_1$ model in which every degree below $\mathbf{0'}$ is r.e. A careful analysis of the proof shows if $B\Sigma_1$ model has a Σ_1 cut on which every Π_2 subset is coded, then in the model, all degrees below $\mathbf{0'}$ are r.e. It is tempting to conjecture that there is a characterization of the existence of non-r.e. degrees below $\mathbf{0'}$ in terms of the existence of codes in the model.

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Department of Mathematics, National University of Singapore, Singapore 119076
E-mail address: wei.li@nus.edu.sg