FRIEDBERG NUMBERING IN FRAGMENTS OF PEANO ARITHMETIC AND α-RECURSION THEORY

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ABSTRACT. In this paper, we investigate the existence of a Friedberg numbering in fragments of Peano Arithmetic and initial segments of Gödel's constructible hierarchy L_α, where α is Σ_1 admissible. We prove that
(1) Over $P^- + B\Sigma_2$, the existence of a Friedberg numbering is equivalent to $I\Sigma_2$, and
(2) For L_α, there is a Friedberg numbering if and only if the tame Σ_2 projectum of α equals the Σ_2 cofinality of α.

1. INTRODUCTION

The idea of coding information using numbers was introduced by Kurt Gödel. In the proof of his famous Incompleteness Theorem [9], Gödel effectively assigned to each formula a unique natural number. Generally, any map from $\mathbb{N}$ onto a set of objects, such as formulas, is called a *numbering* of the objects. For example, one can follow Gödel to effectively list all Σ_1 formulas, hence all recursively enumerable (r.e.) sets, which we shall refer to as the *Gödel numbering* of r.e. sets. In this paper, we focus on numberings f of r.e. sets such that the relation $\{(x, e) : x \in f(e)\}$ is r.e.

A universal numbering is a recursive list of all r.e. sets. Gödel numbering is universal. Yet, Gödel numbering is not one-one, as two Σ_1 formulas may define the same r.e. set. A natural question was raised by S. Tennenbaum: "Is there a recursive list of all r.e. sets without repetition?" Essentially, the question asks for an effective choice function of r.e. sets. Friedberg [8] gave an affirmative answer to Tennenbaum's question for the standard model $\mathbb{N}$ of natural numbers. Thus, a one-one universal numbering is said to be a Friedberg numbering. In [14], Kummer simplified Friedberg's proof by a priority-free argument. Kummer's proof and Friedberg's proof both use the method of effective approximation to search for the least index of an r.e. set and obtain as a result a one-one enumeration of r.e. sets.

Our purpose in this paper is to investigate the existence of Friedberg numbering in different models of computation: models of fragments of Peano Arithmetic (PA), and initial segments L_α of Gödel's constructible universe, where α is Σ_1 admissible.

To see the stage, let P^- denote the axioms of PA on rules governing the standard arithmetic operations such as the associative law of "+", the distributive law with respect to "+" and "$\cdot$", etc, excluding the induction scheme. Paris and Kirby [20] defined fragments of PA by restricting the induction scheme to instances of bounded logical complexity and then showed the relative logical strengths of the resulting theories over $P^- + I\Sigma_0 + \exp$ (here, "exp" says that $\forall x\, \exists y\, (y = 2^x)$). In the study of r.e. sets, two important examples of fragments of PA are: $P^- + I\Sigma_n$, where $I\Sigma_n$ is the restriction of the induction scheme to Σ_n formulas, and $P^- + B\Sigma_n$, where $B\Sigma_n$ states that every Σ_n function maps a finite set in the sense of the model onto a finite set. It is known that $I\Sigma_n$ is strictly stronger

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than $B\Sigma_n$ over the base theory P^- (See Section 2). In the 1980's, S. Simpson first proved (unpublished) the Friedberg-Muchnik Theorem assuming $P^- + I\Sigma_1$. Slaman and Woodin [23] then studied Post's problem in the absence of a priority method in models satisfying the weaker theory $P^- + B\Sigma_1$. In general, any construction which involves the use of the $0'$-priority method is applicable in models of $I\Sigma_1$. Similarly, the $0''$-priority method is applicable in models of $I\Sigma_2$. Nevertheless, these are general principles for priority methods. To identify the necessary and sufficient fragment of PA for theorems in recursion theory to hold, a closer analysis of models in fragments of PA is required. (See [5, 6, 7, 18, 19, 23]).

Historically, the investigation of recursion theoretic aspects of fragments of PA was partially motivated by α-recursion theory and a number of ideas and methods of α-recursion theory have been successfully adapted. An ordinal α is Σ_1 admissible if L_α satisfies Σ_1 replacement. An exploitation of the model theoretic properties of Gödel's constructible sets led Sacks and Simpson [22] to prove the Friedberg-Muchnik Theorem for every Σ_1 admissible ordinal α. One of the strategies used in the proof was the use of an indexing of α-r.e. sets different from the standard one. This indexing is not α-recursive for many α's. An alternative system of indexing for the generalization of Friedberg-Muchnik Theorem was given by Lerman [15]. Different indexing systems for α-r.e. sets is a powerful tool in α-recursion theory (see [1, 17, 21]), and will also be used in this paper.

Proofs of theorems about α-r.e. sets in the case when $\alpha = \omega$ quite often make strong use of Σ_n replacement for $n \geq 2$, which L_α may not satisfy. Thus generalizing theorems about α-r.e. sets to arbitrary admissible ordinals may be regarded as the austere art of making Σ_1 admissibility do the work of Σ_n. This is quite often a challenging task and introduces additional complexity to the constructions. And there are instances where Σ_1 admissibility simply cannot successfully perform the task assigned.

An intuitive approach to analyzing the existence of a Friedberg numbering in models of fragments of PA or L_α is illustrated in the following paragraphs. Let $\{W_e\}$ be a Gödel numbering in such a model. Then e is the least index of W_e if

$$\forall i < e\,(W_i \neq W_e). \tag{1.1}$$

(1.1) is a Σ_2 sentence preceded by a bounded quantifier. A careful examination of known proofs shows that $P^- + I\Sigma_2$ and α satisfying Σ_2 replacement suffice to prove the existence of a Friedberg numbering in the model. The most interesting situation is then when $I\Sigma_2$ or Σ_2 replacement fails.

Though no priority method is required to construct a Friedberg numbering, interestingly, we will show that $I\Sigma_2$ is in fact necessary for the existence of a Friedberg numbering in models that satisfy $P^- + B\Sigma_2$. Observe that $B\Sigma_2$ reduces (1.1) to a Σ_2 formula as in the standard model $\mathbb{N}$. However, in a model satisfying $B\Sigma_2$ but not $I\Sigma_2$, for an r.e. set W, there may not be an e satisfying (1.1) such that $W_e = W$. Therefore, the straightforward extension of known proofs does not work. In the other direction, if e is the least index, $B\Sigma_2$ suffices to establish an upper bound of the least differences between W_e and all W_i, $i < e$. That property provides a possible way to do a diagonalization argument to show that no one-one numbering is universal, so that there is no Friedberg numbering.

For an L_α not satisfying Σ_2 replacement, the lifting of the construction from ω to α has another complication. Because of the failure of Σ_2 replacement, (1.1) is in fact Π_3 and not Σ_2. Hence the least index of an α-r.e. set, while it exists, may not be effectively approximated. An analysis of this situation leads to different outcomes. We give two examples to illustrate this point by way of the ordinals: ω_1^{CK} and $\aleph_\omega^L$. Though $L_{\omega_1^{CK}}$ does not satisfy Σ_2 replacement, the collection of α-r.e. sets can be arranged in order type ω through a Σ_1 projection from ω_1^{CK} into ω. Then the construction may be carried out in the new ordering and yields the existence of a Friedberg numbering. The second example

$\aleph_\omega^L$, however, does not have the advantage of a Σ_1 projection into a smaller ordinal, as $\aleph_\omega^L$ is a cardinal of L. Here the lack of Σ_2 admissibility and a Σ_1 projection to a suitably smaller regular ordinal results in the nonexistence of a Friedberg numbering for $L_{\aleph_\omega^L}$. Our plan is to extend the diagonalization argument in $B\Sigma_2$ models to $L_{\aleph_\omega^L}$. Since $L_{\aleph_\omega^L}$ does not satisfy Σ_2 replacement, in general, for W_e from (1.1), the least upper bound of the least differences of W_e and all W_i, $i < e$, may be $\aleph_\omega^L$. Nevertheless the situation is different when W_e is α-finite. Suppose W_e is an α-finite set satisfying (1.1), and $\zeta = \sup W_e < \aleph_\omega^L$. Then for every $i < e$, if $W_i \not\supseteq W_e$, then the least difference between W_i and W_e is less than ζ. If $W_i \supsetneq W_e$, then there exists a large enough $\aleph_n^L > \zeta$ such that $W_{i,\aleph_n^L} \supsetneq W_e$, since for every $m < \omega$, $\langle L_{\aleph_m^L}, \in\rangle$ is a Σ_1 elementary substructure of $\langle L_{\aleph_\omega^L}, \in\rangle$. Also, note that the Π_1 function: $n \mapsto \aleph_n^L$, allows an arrangement of α-r.e. sets in blocks of length $\aleph_0^L, \aleph_1^L, \ldots$. By considering α-finite sets, the diagonalization strategy for $B\Sigma_2$ models may be extended to $L_{\aleph_\omega^L}$ block by block. The argument for $L_{\aleph_\omega^L}$ can be generalized to an arbitrary Σ_2 inadmissible cardinal α. A further analysis leads to the characterization in this paper of the existence of Friedberg numberings in terms of the notions of tame Σ_2 projectum (a Σ_1 projection is also tame Σ_2) and Σ_2 confinality of α (denoted by $t\sigma 2p\,(\alpha)$ and $\sigma 2cf(\alpha)$ respectively). The notion of $t\sigma 2p\,(\alpha)$ was introduced by Lerman [15] and $\sigma 2cf(\alpha)$ was introduced by Jensen [11] in his study of the fine structure theory of Gödel's L. The precise definitions of $t\sigma 2p\,(\alpha)$ and $\sigma 2cf(\alpha)$ are given in Section 3. In the two examples shown here, $t\sigma 2p\,(\omega_1^{CK}) = \sigma 2cf(\omega_1^{CK}) = \omega$, and $t\sigma 2p\,(\aleph_\omega^L) = \aleph_\omega^L > \sigma 2cf(\aleph_\omega^L) = \omega$. They give some hints about the characterization of the existence of a Friedberg numbering in L_α.

The rest of the paper is organized in two sections. Section 2 is devoted to the study of Friedberg numbering in models of fragments of PA. It will be shown that the existence of a Friedberg numbering is equivalent to $I\Sigma_2$ over the base theory $P^- + B\Sigma_2$. Friedberg numbering in L_α is discussed in Section 3: for a Σ_1 admissible ordinal α, a Friedberg numbering exists if and only if $t\sigma 2p(\alpha) = \sigma 2cf(\alpha)$. Each section is further divided into subsections, roughly following the order: review of basic definitions and notions in the subject area, discussion of preliminary lemmas related to Friedberg numbering, and the proof of the characterization theorem.

2. Weak Fragments of PA

The known constructions of a Friedberg numbering ([8, 14]) make strong use of existence of the least index for each r.e. set, in order to construct a Friedberg numbering for $\mathbb{N}$. This is equivalent to proving the theorem in the theory $P^- + I\Sigma_2$, as we discuss below. We will prove in this section that over the base theory $P^- + B\Sigma_2$, $I\Sigma_2$ is both sufficient and necessary for the existence of such a numbering.

2.1. **Background in Fragments of Peano Arithmetic.**

We begin with recalling some useful facts about first order arithmetic. The reader may consult [4, 6, 12, 13] for details.

Axioms of Peano Arithmetic and its fragments. The language of Peano Arithmetic (PA) is the language of first order arithmetic $\mathcal{L}(0, 1, +, \cdot, <)$. The Levy hierarchy of Σ_n formulas for first order arithmetic is defined as usual. P^- consists of the usual axioms on arithmetical operations without induction. The *induction scheme* is

$$\forall x\,[(\forall y < x\,\varphi(y)) \to \varphi(x)] \to \forall x\,\varphi(x),$$

for every formula φ in the language $\mathcal{L}(0, 1, +, \cdot, <)$ (φ may contain parameters).

The fragments $P^- + I\Sigma_n$ ($P^- + I\Pi_n$, resp.), are defined to be P^- together with the induction scheme restricted to Σ_n (Π_n, resp.) formulas. Two variants of induction scheme are the *bounding scheme*

$$\forall x\,(\forall y < x\,\exists w\,\varphi(y,w) \to \exists b\,\forall y < x\,\exists w < b\ \varphi(y,w)),$$

and the *least number scheme*

$$\exists w\,\psi(w) \to \exists w\,(\psi(w) \wedge \forall v < w\,(\neg\psi(v))),$$

where φ and ψ are formulas of $\mathcal{L}(0,1,+,\cdot,<)$ (φ and ψ may contain parameters).

$B\Sigma_n$ ($B\Pi_n$, resp.) is obtained by restricting the bounding scheme to Σ_n (Π_n, resp.) formulas, and $L\Sigma_n$ ($L\Pi_n$, resp.) is obtained by restricting the formulas being considered to Σ_n (Π_n, resp.) formulas.

Suppose $\mathcal{M} \models P^- + I\Sigma_0 + \exp$, where exp asserts that exponentiation is a total function. A subset of $\mathcal{M}$ is *r.e.*, if it is Σ_1 definable; if the complement of an r.e. set is also Σ_1 definable, then the set is *recursive*. A set is *$\mathcal{M}$-finite* if it is represented by the binary expansion of some element in $\mathcal{M}$. A set is *regular* if its intersection with any $\mathcal{M}$-finite set is $\mathcal{M}$-finite.

Given an r.e. set A, let $A_s \subseteq A$ be the collection of all elements that are enumerated by stage s. Then A_s is $\mathcal{M}$-finite for any s.

The following is a list of basic facts.

Theorem 2.1. *Let $n \geq 1$.*

- *(i) (Pairs and Kirby) Assume $P^- + I\Sigma_0 + \exp$. Then the following implications hold:*
  - *(a) $B\Sigma_n \Leftrightarrow B\Pi_{n-1}$;*
  - *(b) $I\Sigma_n \Leftrightarrow I\Pi_n \Leftrightarrow L\Sigma_n \Leftrightarrow L\Pi_n$;*
  - *(c) $B\Sigma_{n+1} \Rightarrow I\Sigma_n \Rightarrow B\Sigma_n$. However, the arrows do not reverse. Thus, the hierarchy of fragments of PA does not collapse.*
- *(ii) (H. Friedman) Suppose $\mathcal{M} \models P^- + I\Sigma_n$. Then any Σ_n subset of $\mathcal{M}$ is regular; any partial Σ_n function maps a bounded set to a bounded set.*

Let $\{W_e\}$ be a Gödel numbering in a model of $P^- + I\Sigma_0 + \exp$. Note that the statement "$W_i = W_e$" is Π_2. Therefore, $L\Pi_2$ suffices to show that every r.e. set has a least index in $\{W_e\}$. By Theorem 2.1, $L\Pi_2 \Leftrightarrow I\Sigma_2$. In fact, the induction needed to carry out the construction of a Friedberg numbering for $\mathbb{N}$ is just $I\Sigma_2$. Thus,

Lemma 2.2 ($P^- + I\Sigma_2$)**.** *There exists a Friedberg numbering.*

$\mathbf{B\Sigma_n}$ *models.* Let $n \geq 1$. A model $\mathcal{M} \models P^- + I\Sigma_0 + \exp$ is said to be a *$B\Sigma_n$ model*, if $\mathcal{M} \models B\Sigma_n$ and $\mathcal{M} \not\models I\Sigma_n$. Clause (i)(c) of Theorem 2.1 asserts that there exists a $B\Sigma_n$ model.

An analysis of $B\Sigma_n$ models is needed to clarify the relationship between fragments of PA and theorems in recursion theory proved in $I\Sigma_n$. A theorem is equivalent to $I\Sigma_n$ over $B\Sigma_n$, if it is provable by $I\Sigma_n$ but fails in every $B\Sigma_n$ model.

In a $B\Sigma_n$ model, notice that Clause (ii) of Lemma 2.1 does not hold, as stated in Lemma 2.3.

A subset I of $\mathcal{M}$ is a *cut*, if I is a nonempty proper initial segment of $\mathcal{M}$ and closed under successor. A partial function on $\mathcal{M}$ is *cofinal* if its range is unbounded in $\mathcal{M}$.

Lemma 2.3. *Let $\mathcal{M} \models B\Sigma_n$. Then $\mathcal{M}$ is a $B\Sigma_n$ model if and only if there exists a Σ_n cut I with a Δ_n function $f : I \to \mathcal{M}$ such that f is nondecreasing and cofinal.*

It is worth noting that the Σ_n cut in Lemma 2.3 is not $\mathcal{M}$-finite, and so is not a regular set.

Assume $A \subseteq \mathcal{M}$. A set $X \subseteq A$ is said to be *coded on* A if there is an $\mathcal{M}$-finite set K such that $K \cap A = X$. Let $n \geq 0$. A set $X \subseteq A$ is Δ_n *on* A if X and $A \setminus X$ are both Σ_n.

Lemma 2.4 (Coding Lemma (Chong and Mourad [4]))**.** *Let $\mathcal{M}$ be a $B\Sigma_n$ model and $A \subseteq \mathcal{M}$. Then every set bounded and Δ_n on A is coded on A. In particular, any Δ_n set of $\mathcal{M}$ is regular and any bounded Δ_n set is $\mathcal{M}$-finite.*

An application of Coding Lemma is Lemma 2.5, which states an induction principle on a Σ_n cut.

To fix notations, we use $[a, b]$ ($[a, b)$ resp.), where $a < b \in \mathcal{M}$, to denote the set $\{x \in \mathcal{M} : a \leq x \leq b\}$ ($\{x \in \mathcal{M} : a \leq x < b\}$ resp.). We use 2^I to represent the set $\{x \in \mathcal{M} : x < 2^i \text{ for some } i \in I\}$. If f is a function, we will use $\text{dom}(f)$ to denote the domain of f and use $\text{ran}(f)$ to denote the range of f. (The notations of $\text{dom}(f)$ and $\text{ran}(f)$ will have the same meaning for functions f in Section 3).

A number z is said to *code a partial function* if it codes an $\mathcal{M}$-finite set D and D is the graph of a partial function.

Lemma 2.5. *Suppose $\mathcal{M}$ is a $B\Sigma_n$ model, $I \subset \mathcal{M}$ is a Σ_n cut, $a_0 \in \{0, 1\}$, and $h : I \times 2^I \to \{0, 1\}$ is total on $I \times 2^I$ and Σ_n definable. Let $G \subseteq I$ be defined by iterating h:*

$$\begin{aligned} G(0) &= a_0 \\ G(i+1) &= h(i, G \upharpoonright [0, i]), \text{ if } i \in I \text{ and } G \upharpoonright [0, i] \text{ is } \mathcal{M}\text{-finite.} \end{aligned}$$

Then for every $i \in I$, $G(i)$ is defined. Thus, G is Δ_n on I and coded on I.

Proof. It follows immediately from the definition that G is Σ_n definable and $\text{dom}(G) \subseteq I$ is a Σ_n cut of $\mathcal{M}$.

To see that $\text{dom}(G) = I$, choose an arbitrary $i \in I$, and we only need to show $\text{dom}(G) \supseteq [0, i]$. For any $j \leq i$,

$$\begin{aligned} (2.1) \quad G(j) = y \leftrightarrow \exists z < 2^{i+1} \, (&z \text{ codes a partial function} \land \\ &z(0) = a_0 \land \forall k < j \, (z(k+1) = h_i(j, z \upharpoonright [0, k])) \land z(j) = y), \end{aligned}$$

where $h_i = h \upharpoonright [0, i] \times [0, 2^{i+1}]$. The function h_i is total on $[0, i] \times [0, 2^{i+1}]$, so h_i is Δ_n definable. In addition, h_i is bounded. Lemma 2.4 implies that h_i is $\mathcal{M}$-finite. Then the right hand side of (2.1) is Σ_0. Thus, $\text{dom}(G) \supseteq [0, i]$. □

2.2. **Towards Friedberg Numbering.** From now on $\mathcal{M}$ is a $B\Sigma_2$ model and $I \subset \mathcal{M}$ is a Σ_2 cut. Let $\{A_e\}_{e \in \mathcal{M}}$ be a one-one numbering of r.e. sets in $\mathcal{M}$. Our purpose is to construct an r.e. set X such that $X \neq A_e$, for all $e \in \mathcal{M}$. Hence, $\{A_e\}_{e \in \mathcal{M}}$ is not a Friedberg numbering.

By Lemma 2.3, let $f : I \to \mathcal{M}$ be a nondecreasing Δ_2 cofinal function with $f(0) = 0$. That makes it possible to establish a partition of $\mathcal{M}$, $\{[f(i), f(i+1)) : i \in I\}$. The interval $[f(i), f(i+1))$ is said to be the i^{th} *block* (or *block* i) of $\mathcal{M}$. Then X is constructed by diagonalizing against A_e's in each block.

For any $a \in \mathcal{M}$,

$$\forall d, e < a \, \exists x \, (d \neq e \to A_d(x) \neq A_e(x)).$$

Since $\{A_e\}_{e \in \mathcal{M}}$ is a one-one numbering. It follows from $B\Sigma_2$ that there is a $b \in \mathcal{M}$ such that

$$\forall d, e < a \, \exists x < b \, (d \neq e \to A_d(x) \neq A_e(x)). \tag{2.2}$$

Here, b is said to be a *bound of differences relative to* $[0, a)$. (2.2) implies that there is at most one $e < a$ such that

$$A_e \upharpoonright [0, b) = X \upharpoonright [0, b). \tag{2.3}$$

Therefore, diagonalizing against $\{A_e\}_{e<a}$ amounts to diagonalizing against the sole A_e satisfying (2.3), if any, by one witness greater than or equal to b. In short, to diagonalize against one block it suffices to diagonalize against one special r.e. set.

Let us recall the definition of limit as follows. Suppose $h : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a total function. Then

$$\lim_s h(s,a) = n,$$

if either $n \in \mathcal{M}$ and there exists t such that

$$\forall s > t\,(h(s,a) = n),$$

or else $n = \infty$ and

$$\forall m\, \exists t\, \forall s > t\,(h(s,a) > m).$$

Since $f : I \to \mathcal{M}$ is Δ_2, (2.2) yields a Δ_2 function $g : I \to \mathcal{M}$ such that $g(i)$ is a bound of differences relative to $[0, f(i))$ for each $i \in I$. A careful examination of the proof of the Limit Lemma [24] in standard model $\mathbb{N}$ shows that the proof of the Limit Lemma only requires $P^- + B\Sigma_1$ and the regularity of Δ_2 sets. Then in the $B\Sigma_2$ model $\mathcal{M}$, the Limit Lemma implies that f and g have recursive approximations. A more precise statement of this situation is that f and g may chosen to have nondecreasing recursive approximations, as proved in Lemma 2.6. Based on those approximations, it will be shown later that X can be constructed in an effective manner.

Lemma 2.6. *Let $\mathcal{M}$ be a $B\Sigma_2$ model and $I \subset \mathcal{M}$ be a Σ_2 cut. Then there exist (total) recursive functions $f', g' : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ such that*

(i) $\lambda s(f'(s,i))$, $\lambda i(f'(s,i))$, $\lambda s(g'(s,i))$ and $\lambda i(g'(s,i))$ are nondecreasing;
(ii) functions f and g given by $f(i) = \lim_s f'(s,i)$, $g(i) = \lim_s g'(s,i)$ are well defined and less than ∞ on I and equal ∞ on $\mathcal{M} \setminus I$;
(iii) $f : I \to \mathcal{M}$ is cofinal;
(iv) $\forall i, j, s, t\,(i \neq j \to g'(s,i) \neq g'(t,j))$. That is, $\mathrm{ran}(\lambda s\, g'(s,i)) \cap \mathrm{ran}(\lambda s\, g'(s,j)) = \emptyset$ for any $i \neq j$;
(v) $\forall i \in I\, \forall d, e < f(i)\, \exists x < g(i)\,(d \neq e \to A_d(x) \neq A_e(x))$, i.e. $g(i)$ is a bound of differences relative to $[0, f(i))$.

Proof. Functions f' and f satisfying (i)-(iii) may be defined from the Σ_2 definition of I (See [2, 3]). We omit the details and directly define g' satisfying (i), (ii) and (v). Then (i), (ii), (iv) and (v) will be satisfied for g'', defined by $g''(s,i) = \langle i, g'(s,i)\rangle$ for any $s, i \in \mathcal{M}$ ($\langle \cdot, \cdot \rangle$ is a recursive code of pairs).

Now define g' by induction on s as follows.

$$\begin{aligned} g'(0,i) &= i \\ g'(s+1,i) &= \begin{cases} g'(s,i) & \text{if } g'(s,i) > f'(s,i) \text{ and} \\ & \forall d, e < f'(s,i)\, \exists x < g'(s,i)\,(d \neq e \to A_{d,s}(x) \neq A_{e,s}(x)), \\ g'(s,i)+1 & \text{otherwise.} \end{cases} \end{aligned}$$

By $I\Sigma_1$, g' is total recursive and $\lambda s(g'(s,i))$ is nondecreasing.

To see that $\lambda i(g'(s,i))$ is nondecreasing, it suffices to show

$$\forall i\,(g'(s,i+1) \geq g'(s,i)). \tag{2.4}$$

by induction on s and $I\Pi_1$. The induction is straightforward and we omit the details here.

Observe that a recursive set either has a maximum element or is unbounded in $\mathcal{M}$ by $L\Pi_1$. Then it follows immediately from the nondecreasing property of $\lambda s(g'(s,i))$ that

$$\lim_s g'(s,i) < \infty \leftrightarrow \{g'(s,i) : s \in \mathcal{M}\} \text{ is bounded.}$$

Then it is easy to check that (ii) and (v) hold by $B\Sigma_2$ and the properties of f and the defintion of g. $\square$

2.3. **Nonexistence of Friedberg Numbering.** Let $\mathcal{M}$, I and $\{A_e\}_{e\in\mathcal{M}}$ be as in Section 2.2. In this section it will be shown that there exists an r.e. set $X \notin \{A_e\}_{e\in\mathcal{M}}$. The method here converts the diagonalization strategy in Section 2.2 to an effective one so as to obtain an r.e. counterexample X.

Theorem 2.7. *There is no Friedberg numbering in a $B\Sigma_2$ model.*

Proof. Again, $\mathcal{M}$, I and $\{A_e\}_{e\in\mathcal{M}}$ are as in Section 2.2. Let f, f', g, g' be as in Lemma 2.6. The construction below defines X such that

(i) $X \subseteq \mathrm{ran}(g')$;
(ii) $\forall i \notin I\, \forall s\, (g'(s,i) \in X)$
(iii) $\forall i \in I\, \forall s\, (g'(s,i) < g(i) \to g'(s,i) \in X)$;
(iv) $\forall i \in I\, [(g(i) \notin X) \leftrightarrow \exists c < f(i)\, (A_c \restriction [0, g(i)) = X \restriction [0, g(i)) \wedge g(i) \in A_c)]$.

According to Lemma 2.6, $g(i)$ is a bound of differences relative to $[0, f(i))$. Then at most one $c < f(i)$ satisfies

$$A_c \restriction [0, g(i)) = X \restriction [0, g(i)).$$

Thus, (iv) implies $X \neq A_c$, for any $c < f(i), i \in I$.

Since $\lambda s(g'(s,i))$ is nondecreasing, (ii) and (iii) are satisfied easily via the approximation g'. However, that strategy fails for Clause (iv). At stage s, it is tempting to (maybe mistakenly) enumerate $g'(s,i)$ if

$$\neg\exists c < f'(s,i)\, (A_{c,s} \restriction [0, g'(s,i)) = X_s \restriction [0, g'(s,i)) \wedge g'(s,i) \in A_{c,s}). \tag{2.5}$$

By (2.5), guessing whether $g'(s,i)$ should be enumerated into X could be wrong even if $g'(s,i) = g(i)$ and $f'(s,i) = f(i)$. We may find a $c < f(i)$ at a later stage satisfying $A_c \restriction [0, g(i)) = X \restriction [0, g(i))$ and $g(i) \in A_c$ in the sense of that stage. But once $g(i) = g'(s,i)$ is mistakenly enumerated into X, $g(i)$ cannot be removed from X.

The problem can be solved with the aid of Lemma 2.5.

A non-effective construction of X is carried out inductively on I, with the intention of finding a set G such that

$$G(i) = 0 \leftrightarrow \exists c < f(i)\, (A_c \restriction [0, g(i)) = X \restriction [0, g(i)) \wedge g(i) \in A_c)]. \tag{2.6}$$

Define

$$X_0 = \bigcup\nolimits_{n\in\mathcal{M}} \{g'(s,n) : g'(s,n) < g(n)\}.$$

$$G(0) = \begin{cases} 0 & \text{if } \exists c < f(0)\, (A_c \restriction [0, g(0)) = X_0 \restriction [0, g(0)) \wedge g(0) \in A_c), \\ 1 & \text{otherwise.} \end{cases}$$

$$X_{i+1} = \begin{cases} X_i & \text{if } G(i) = 0, \\ X_i \cup \{g(i)\} & \text{if } G(i) = 1. \end{cases}$$

$$G(i+1) = \begin{cases} 0 & \text{if } \exists c < f(i+1)\, (A_c \restriction [0, g(i+1)) = X_{i+1} \restriction [0, g(i+1)) \wedge g(i+1) \in A_c), \\ 1 & \text{otherwise.} \end{cases}$$

for all $i \in I$. Here, $g(n) = \infty$, if $n \notin I$, by Lemma 2.6.

Let

$$X = \bigcup\nolimits_{i\in I} X_i.$$

It is immediate from Lemma 2.5 that X_i and $G(i)$ are well defined on I, G is Δ_2 on I and coded on I. Suppose $\hat{G}$ is a code of G on I. Then

$$\begin{aligned} X &= X_0 \cup (\bigcup_{i\in I}\{g(i) : G(i) = 1\}) \\ &= \bigcup_{n\in\mathcal{M}}\{g'(s,n) : \exists t > s\,(g'(s,n) < g'(t,n)) \vee \hat{G}(n) = 1\}. \end{aligned}$$

and X is r.e.

By Lemma 2.6, g is strictly increasing on I. Thus,

$$X \upharpoonright [0, g(i+1)) = X_{i+1} \upharpoonright [0, g(i+1)).$$

(2.6) and Clause (iv) are satisfied according to the construction. □

Theorem 2.7 and Lemma 2.2 combine to yield

Corollary 2.8 ($P^- + B\Sigma_2$)**.** *$I\Sigma_2$ is equivalent to the existence of a Friedberg numbering.*

Remark. A numbering $\{B_e\}_{e\in\mathcal{M}}$ is *acceptable* (*K-acceptable*, resp.) if for any other numbering $\{D_e\}_{e\in\mathcal{M}}$ there is a recursive ($\emptyset'$-recursive, resp.) function f such that $D_e = B_{f(e)}$ for all e. In classical recursion theory, a Friedberg numbering is an example of non-acceptable universal numbering and non-K-acceptable universal numbering.

In a $B\Sigma_2$ model $\mathcal{M}$, no Friedberg numbering exists, but a non-K-acceptable universal numbering, thus a non-acceptable universal numbering still exists.

For instance, suppose $\langle\cdot,\cdot\rangle : \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ is a recursive injection, and let

$$B_e = \begin{cases} \mathcal{M} & \text{if } e = 0, \\ W_i \setminus \{j\} & \text{if } e > 0 \text{ and } \exists i,j < e\,(\langle i,j\rangle = e), \\ W_e \setminus \{0\} & \text{if } e > 0 \text{ and } \neg\,\exists i,j < e\,(\langle i,j\rangle = e). \end{cases} \tag{2.7}$$

where $\{W_e\}_{e\in\mathcal{M}}$ is a Gödel numbering. Then $\{B_e\}_{e\in\mathcal{M}}$ is a universal numbering and

$$B_e = \mathcal{M} \Leftrightarrow e = 0.$$

Thus, there is no K-recursive function $g : \mathcal{M}\to\mathcal{M}$ satisfying

$$W_e = B_{g(e)}.$$

A *K_e-numbering* is a universal numbering for which the grammar equivalence problem is $\emptyset'$-recursive (See [10]). Friedberg numberings are K_e-numberings. In a $B\Sigma_2$ model, no Friedberg numbering exists, and also no K_e-numbering exists. The reason is as follows. Suppose $\mathcal{M}$ is a $B\Sigma_2$ model and $\{C_e\}_{e\in\mathcal{M}}$ is a K_e-numbering. Then for each e in $\mathcal{M}$,

$$\{d < e : C_d = C_e\}$$

is a Δ_2 set, and has a least element. It follows that the least index exists for every r.e. set in the numbering $\{C_e\}_{e\in\mathcal{M}}$. By [14], a Friedberg numbering can be constructed via the numbering $\{C_e\}_{e\in\mathcal{M}}$, a contradiction. Hence,

Corollary 2.9 ($P^- + B\Sigma_2$)**.** *$I\Sigma_2$ is equivalent to the existence of a K_e-numbering.*

3. Σ_1 admissible ordinals

In this section, we investigate the problem of the existence of a Friedberg numbering in the context of admissible ordinals.

3.1. **Background in α-recursion.** We recall some basic definitions and results in α-recursion theory. A detailed introduction to the subject can be found in [1, 16, 17, 21].

Admissible ordinals. The language of α-recursion theory is the language of Zermelo-Fraenkel set theory (ZF). formulas and Levy hierarchy of formulas are defined as usual. Given a formula φ, we write $\mu x\, \varphi(x)$ to denote the least ordinal x such that $\varphi(x)$ holds, and $[x, y]$ ($[x, y)$ resp.) to denote $\{z : x \leq z \leq y\}$ ($\{z : x \leq z < y\}$ resp.). An ordinal α is said to be Σ_1 *admissible* if L_α satisfies Σ_1 replacement.

From now on, α will always denote a Σ_1 admissible ordinal. A set is *α-r.e.*, if it is Σ_1 definable over L_α. If the set is Δ_1 definable over L_α, then it is *α-recursive.* A set is α-finite if it is in L_α. A set is *regular* if its intersection with any α-finite set is still α-finite. For each nonempty α-finite set $C \subset \alpha$, define $\sup C = \mu y\, \forall x \in C(x < y)$, $\max C = \mu y\, \forall x \in C(x \leq y)$, $\min C = \mu x\, (x \in C)$.

Suppose $\beta < \delta \leq \alpha$. β is said to be *δ-stable*, if $L_\beta \prec_1 L_\delta$. β is said to be an *α-cardinal* if there is no α-finite one-to-one correspondence between β and any $\gamma < \beta$. Every α-cardinal greater than ω is α-stable.

Each α-finite set has an α-cardinality. The α-cardinality of an α-finite set C is denoted by $|C|_\alpha$.

Recall that there exists a one-one, α-recursive (total) function f that maps α onto L_α. That is, α-finite sets can be effectively coded as ordinals. Thus, there is no harm in identifying α-finite sets with ordinals below α, and identifying subsets of L_α with subsets of α. From now on, by an α-r.e. set without specification, we always mean an α-r.e. subset of α. Also, f yields a recursive bijection from α^2 to α. Fix such a bijection, and denote it by $\langle \cdot, \cdot \rangle$.

It is straightforward to verify that there is a Gödel numbering of α-r.e. sets, which we denote as $\{W_e\}_{e<\alpha}$. For an arbitrary numbering $\{A_e\}_{e<\alpha}$ and any stage $\eta < \alpha$, the set $A_{e,\eta}$ is defined to be the collection of elements which are less than η and are enumerated into A_e by stage η.

Σ_n projectum and cofinality. Let $n \geq 1$. The *Σ_n projectum of* α, denoted by $\sigma np\,(\alpha)$, is defined to be the least ordinal β such that there is a Σ_n (partial) function from β onto α.

Theorem 3.1 (Jensen, [11])**.** *$\sigma np\,(\alpha)$ is the least β such that some Σ_n (over L_α) subset of β is not α-finite. Thus, if $I \subset \alpha$ is an α-finite set such that $|I|_\alpha < \sigma np\,(\alpha)$, then each Σ_n subset of I is α-finite.*

The *Σ_n cofinality of* $\delta \leq \alpha$, denoted by $\sigma ncf\,(\delta)$, is defined to be

$$\mu\gamma\, \exists f \left[f : \gamma \xrightarrow{\text{one-one}} \delta, (\text{total on } \gamma),\ \text{is } \Sigma_n \text{ over } L_\alpha \text{ and } f \text{ is cofinal (in } \delta) \right].$$

It is obvious that $\sigma np\,(\alpha)$ and $\sigma ncf(\alpha)$ are α-cardinals.

Lemma 3.2 (Local Σ_n Replacement)**.** *Let $n \geq 1$, $a < \sigma ncf\,(\alpha)$ and $R \subseteq \alpha \times \alpha$ be a Σ_n relation. Then*

$$\forall x < a\, \exists y\, R(x, y) \to \exists z\, \forall x < a\, \exists y < z\, R(x, y). \tag{3.1}$$

Tameness. The notion of tameness was introduced by Lerman [15]. It has many applications, especially in constructions involving Σ_2 functions.

Let $f : \beta \to \alpha$ for some $\beta \leq \alpha$. Then f is said to be *tame* Σ_2 if it is total and there exists an α-recursive f' such that

$$\forall \gamma < \beta\, \exists \tau\, \forall x < \gamma\, \forall \eta > \tau\, (f'(\eta, x) = f(x)).$$

Such an f' is said to *tamely generate* f. The tameness of f refers to the way f' approximates f on proper initial segments of dom(f). A Σ_2 function need not be tame Σ_2.

The *tame* Σ_2 *projectum of* α, denoted by $t\sigma 2p\,(\alpha)$, is defined to be

$$\mu\beta\, \exists f \left[f : \beta \xrightarrow[\text{onto}]{\text{one-one}} \alpha, \text{(total on } \beta\text{), is tame } \Sigma_2 \right].$$

A set is *tame* Σ_2 if its characteristic function is tame Σ_2. Analogous to $\sigma 2p\,(\alpha)$, we have

Lemma 3.3 (Simpson, [1, 16])**.** *$t\sigma 2p\,(\alpha)$ is the least β such that not every tame Σ_2 subset of β is α-finite.*

Lemma 3.4 ([16])**.** *For all $\delta \leq \alpha$, there exists a strictly increasing tame Σ_2 cofinal function $f : \sigma 2cf(\delta) \to \delta$, and every Σ_2 function from $\delta \leq \sigma 2cf(\alpha)$ to α is tame.*

Corollary 3.5 ([1, 16, 21])**.** (1) $\omega \leq \sigma 2cf(\alpha) \leq t\sigma 2p\,(\alpha) \leq \sigma 1p\,(\alpha) \leq \alpha$,
(2) $\sigma 2cf(\sigma 1p\,(\alpha)) = \sigma 2cf(t\sigma 2p\,(\alpha)) = \sigma 2cf(\alpha)$.

Corollary 3.6 (Local Σ_2 Replacement)**.** *Let $a < \sigma 2cf\,(\alpha)$ and $R \subseteq \alpha \times \alpha$ be a Σ_2 relation. Then*

$$\forall x < a\, \exists y\, R(x, y) \to \exists z\, \forall x < a\, \exists y < z\, R(x, y). \tag{3.2}$$

Moreover,

$$\forall x < a\, \exists y < \sigma 2cf(\alpha)\, R(x, y) \to \exists z < \sigma 2cf(\alpha)\, \forall x < a\, \exists y < z\, R(x, y). \tag{3.3}$$

Proof. (3.2) is immediate from Lemma 3.2. By Lemma 3.4, it is straightforward to get (3.3) from (3.2). We omit the details here. □

3.2. **Towards Friedberg Numbering.** Assume $\{W_e\}_{e<\alpha}$ is a Gödel numbering. We attempt to lift the construction of a Friedberg numbering from ω to α. No difficulty arises when L_α satisfies Σ_2 replacement. The proof remains valid because Σ_2 replacement suffices to show

$$(e \text{ is the least index for } W_e) \Leftrightarrow \exists b\, \exists \eta\, \forall d < e\, (W_d \restriction b = W_{d,\eta} \restriction b \neq W_{e,\eta} \restriction b = W_e \restriction b). \tag{3.4}$$

Therefore the least index can be approximated effectively.

If L_α does not satisfy Σ_2 replacement, however, the approach fails by noticing that Σ_2 replacement is also necessary for (3.4) to hold. In this situation, if $\{A_e\}_{e<\alpha}$ is a one-one numbering, it is not always true that for an arbitrary $\beta < \alpha$,

$$\exists b\, \forall d, e < \beta\, (d \neq e \to \exists x < b\, (A_d(x) \neq A_e(x))). \tag{3.5}$$

Thus, the straightforward adaptation of the argument in $B\Sigma_2$ models is not applicable if α is not Σ_2 admissible.

In this paper, we introduce three strategies which will either yield a successful construction of a Friedberg numbering for suitable α's or allow a diagonalization argument to be implemented showing the nonexistence of such a numbering.

The intuition is that the shorter the list of α-r.e. sets is, the more likely (3.4) and (3.5) can be made to hold. The first strategy attempts to rearrange the order of α-r.e. sets so as to produce a short, necessarily non-recursive, list of these sets. A further idea is to force every proper initial segment of the list to be correctly approximated from some stage onwards, for the sake of computing the least indices and upper bounds of differences correctly in the limit. To achieve this, we arrange for the list of the α-r.e. sets to have length $t\sigma 2p(\alpha)$. More precisely, α-r.e. sets are listed by a tame Σ_2 projection $g : t\sigma 2p(\alpha) \xrightarrow[\text{onto}]{\text{one-one}} \alpha$. Thus, for an arbitrary numbering $\{A_e\}_{e<\alpha}$, the set A_d respectively is listed before A_e, if $g^{-1}(d) < g^{-1}(e)$.

The second strategy is to exploit the key property of $\sigma 2cf(\alpha)$, i.e. Corollary 3.6. According to Corollary 3.6, it is possible to apply Σ_2 replacement on lengths less than $\sigma 2cf(\alpha)$. The first two strategies combine to suggest the possibility that a Friedberg numbering exists when $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$.

If $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, then Lemma 3.3 implies that coding a tame Σ_2 subset of $\sigma 2cf(\alpha)$ is possible. The problem left to adapt the proof in $B\Sigma_2$ models to L_α is to give an effective method of searching for an upper bound b in (3.5). However, such an upper bound may not exist.

The third strategy is aimed at devising diagonalization method to show the nonexistence of a Friedberg numbering in the situation that $\sigma 2cf(\alpha) < t\sigma 2p\,(\alpha)$. This is done by analyzing α-finite sets together with a property we call *pseudostability*. Pseudostable ordinals will be used to get suitable upper bounds for witnesses that differentiae two α-r.e. sets in a given α-finite collection for the purpose of a diagonalization. (See Section 3.4 and 3.5).

Let $C, I \subset \alpha$ be α-finite. If $|I|_\alpha < \sigma 1p\,(\alpha)$, then for any simultaneous enumeration of α-r.e. sets $\{A_e\}_{e\in I}$, the set
$$I_C = \{e \in I : A_e \supsetneq C\}$$
is α-finite, by Theorem 3.1. Thus $\exists\eta\,\forall e \in I\,(e \in I_C \leftrightarrow A_{e,\eta} \supsetneq C\,)$ by Σ_1 replacement. Therefore, any set $X \supsetneq C$ such that $X \restriction \eta = C$ would not be in $\{A_e\}_{e\in I}$ (recall that $A_{e,\eta} \subseteq [0, \eta)$ for every $e, \eta < \alpha$).

Note that the recursive search for η strongly relies on the parameter I_C. That would be a problem if C varies, as the parameters of I_C may not be recovered effectively. Nevertheless, there are special cases when the parameter I_C can be omitted (i.e. η can be derived directly from C): for example, when C is never in $\{A_e\}_{e\in I}$ as C changes, and a final segment of C is an interval of ordinals with $\sup C$ being an α-stable ordinal, and, roughly speaking, $\sup C$ is large enough, we have
$$\forall e \in I\,(A_e \supsetneq C \leftrightarrow A_e \supseteq C \leftrightarrow A_{e,\sup C} \supseteq C\,).$$

The only problem with the use of α-stable ordinals is that α-stable ordinals need not be cofinal in α. Therefore, the notion of *pseudostablility*, a weak form of α-stability, is introduced. As will be seen in Section 3.4 and 3.5, pseudostable ordinals are cofinal in α and enjoy the properties required for our construction.

3.3. **When $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$.** The main result of this section is

Theorem 3.7. *If $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$, then there exists a Friedberg numbering.*

The strategy here is to adapt Kummer's construction [14] by introducing a shorter list of all α-r.e. sets on $t\sigma 2p\,(\alpha)$ and applying local Σ_2 replacement (Corollary 3.6) on $\sigma 2cf(\alpha)$.

Let $\hat\alpha = t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$, $f : \hat\alpha \xrightarrow[\text{cofinal}]{\text{strictly increasing}} \alpha$ and $g : \hat\alpha \xrightarrow[\text{onto}]{\text{one-one}} \alpha$ be tame Σ_2, and $f', g' : \alpha \times \hat\alpha \to \alpha$ be recursive functions that tamely generate f, g such that for all $\eta < \hat\alpha$,

(i) $\lambda x\,(f'(\eta, x))$ and $\lambda x\,(g'(\eta, x))$ are one-one,
(ii) $\{f'(\eta, x)\}_{x<\alpha}$ and $\{g'(\eta, x)\}_{\eta<x<\alpha}$ are strictly increasing (with respect to x).

For simplicity, f_η, g_η will be used to denote functions $\lambda x\,(f'(\eta, x))$, $\lambda x\,(g'(\eta, x))$ respectively.

Lemma 3.8. *Suppose $\{W_e\}_{e<\alpha}$ is a Gödel numbering. Then there are numberings $\{P_e\}_{e<\alpha}$ and $\{Q_e\}_{e<\alpha}$ such that*

(i) $\{P_e\}_{e<\alpha} \cap \{Q_e\}_{e<\alpha} = \emptyset$;
(ii) $\{P_e\}_{e<\alpha} \cup \{Q_e\}_{e<\alpha} = \{W_e\}_{e<\alpha}$;

(iii) $P_e \neq P_d$ *whenever* $e \neq d$;
(iv) $\{e < \alpha : P_e \supseteq C\}$ *is cofinal in* α, *for every* α*-finite set* C.

Proof. Let

$$P_e = [0, e),$$
$$Q_e = \begin{cases} \alpha & \text{if } e = 0, \\ W_{e'} \cup \{e''\} \setminus \{e'''\} & \text{if } e = \langle\langle e', e''\rangle, e'''\rangle \text{ and } e'' > e''', \\ W_e \cup \{e\} \setminus \{0\} & \text{otherwise.} \end{cases}$$

Then (i)-(iv) are immediate from the definitions of P_e and Q_e. □

Requirements and Strategy. Fix numberings $\{P_e\}_{e<\alpha}$ and $\{Q_e\}_{e<\alpha}$ as in Lemma 3.8. For any $e < \alpha$, e is said to be *the least index for* $\{e' : Q_{e'} = Q_e\}$ *via* g, if

$$\exists i < \hat{\alpha}\, \forall j < i\, (g(i) = e \wedge Q_{g(i)} \neq Q_{g(j)}).$$

We denote the characteristic function of this predicate by $l_{Q,g}(e)$ (or $l(e)$ for short).

A Friedberg numbering $\{A_e\}_{e<\alpha}$ will be constructed and the requirements are as follows.

$$\text{Requirement } e\text{:} \quad R_{P,e} : \exists! \rho\, (P_e = A_\rho),$$
$$R_{Q,e} : \exists! \rho\, (Q_e = A_\rho).$$

The strategy for satisfying requirement e consists of the following:

(1) assign a unique *follower* $\rho = F^*_P(e)$ to P_e with the objective of making $A_{F^*_P(e)}$ equal to P_e;
(2) assign a unique follower $\rho = F^*_Q(e)$ to Q_e, whenever $l(e) = 1$, with the objective of making $A_{F^*_Q(e)}$ equal to Q_e; and
(3) for every $\rho < \alpha$, assign ρ to a unique set from $\{P_e\}_{e<\alpha} \cup \{Q_e\}_{e<\alpha, l(e)=1}$, such that ρ is the follower of the corresponding set.

More precise definitions of F^*_P and F^*_Q will be given in the part of construction and the part of verification.

The strategy works effectively, except for the fact that "$l(e) = 1$" is not a recursive predicate. Nevertheless, it will soon be seen that "$l(e) = 1$" has an effective approximation $l'(\eta, e)$ (see Lemma 3.10). For the moment assume that Lemma 3.10 holds, i.e.

$$l(e) = 1 \leftrightarrow \lim_{\eta \to \alpha} l'(\eta, e) = 1,$$

where l' is α-recursive. Then at each stage η, the construction will proceed as follows.

Step One: assign a follower to Q_e, if $e < \eta$, Q_e has no follower and $l'(\eta, e) = 1$; *release* the follower of Q_e, if any, whenever $e \geq \eta$ or $l'(\eta, e) = 0$;
Step Two: assign a follower to P_e, if $e < \eta$ and P_e has no follower;
Step Three: for all $\rho' \in [0, \eta) \cup \{\rho : \rho \text{ is relased at step one}\}$, if ρ' has not been assigned to any set by the end of step two, then assign ρ' to some P_d such that P_d has not been assigned to any follower and $P_d \supseteq \bigcup_{\delta<\eta} A_{\rho',\delta}$.
Step Four: if $F_P(\eta, e)$ is a follower of P_e and $F_Q(\eta, e')$ is a follower of $Q_{e'}$ by the end of step three, then let $A_{F_P(\eta,e),\eta} = P_{e,\eta}$ and $A_{F_Q(\eta,e'),\eta} = Q_{e',\eta}$.

This strategy succeeds, because

(1) each P_e has a follower and never releases its follower,
(2) eventually Q_e has a permanent follower after some stage if and only if $l(e) = 1$, and
(3) each ρ, as a follower, is released at most once, after which it will be a permanent follower of a P set or a Q set.

More details will be given in Lemma 3.11.

To approximate $l(e)$, the notion of the *greatest common length* of $Q_{g(i)}$ and $Q_{g(j)}$, $\forall j < i$ will be introduced. To define this notion, we first prove Lemma 3.9. Lemma 3.9 claims that for an $i < \hat{\alpha}$, the statement that $W_{g(i)}$ is not equal to $W_{g(j)}$ for any $j < i$ is equivalent to the existence of an upper bound $b < \hat{\alpha}$, such that the least difference of $W_{g(i)}$ with any $W_{g(j)}$ for $j < i$, after the mapping g^{-1}, lies below b and is seen by stage $f(b)$.

Lemma 3.9. *If $i < \hat{\alpha}$, then*

$$\begin{aligned}(3.6)\quad \forall j < i\,(Q_{g(i)} \neq Q_{g(j)}) \leftrightarrow\ & \exists b < \hat{\alpha} \forall j < i\, \exists x < b\, \exists \gamma < b \\ & (Q_{g(i),f(\gamma)}(g(x)) = Q_{g(i)}(g(x)) \neq Q_{g(j)}(g(x)) = Q_{g(j),f(\gamma)}(g(x))).\end{aligned}$$

Proof. We only prove the direction from left to right.

Suppose $\forall j < i\,(Q_{g(i)} \neq Q_{g(j)})$. Then

$$\forall j < i\, \exists x < \hat{\alpha}\, \exists \gamma < \hat{\alpha}\, (Q_{g(i),f(\gamma)}(g(x)) = Q_{g(i)}(g(x)) \neq Q_{g(j)}(g(x)) = Q_{g(j),f(\gamma)}(g(x))).$$

Since the matrix of the above formula is Σ_2, Lemma 3.6 provides a $b < \hat{\alpha}$ such that the right hand side of (3.6) holds. □

By Lemma 3.9, the greatest common length is measured within $\hat{\alpha}$ through the map g. One advantage of this measure has to do with the regularity. That is, $(g^{-1} \restriction W) \cap \delta$ is α-finite for any α-r.e. set W and $\delta < \hat{\alpha}$, since $\hat{\alpha} = t\sigma 2p\,(\alpha) \leq \sigma 1p\,(\alpha)$. An arbitrary α-r.e. set W, however, need not be regular.

Suppose $e, \eta < \alpha$. The *greatest common length with respect to e though g at stage η* is defined as

$$c_g(\eta, e) = \begin{cases} \max\{b < \min(\hat{\alpha}, \eta) : \exists j < g_\eta^{-1}(e)(Q_{e,\eta} \restriction \operatorname{ran}(g_\eta \restriction b) = Q_{g_\eta(j),\eta} \restriction \operatorname{ran}(g_\eta \restriction b))\} \\ \qquad\qquad \text{if } e < \eta \text{ and } e \in \operatorname{ran}(g_\eta \restriction \min\{\hat{\alpha}, \eta\}), \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Note that c_g is an α-recursive function.

The index e is said to be *the least index for $\{e' : Q_{e'} = Q_e\}$ via g at stage η*, if

$$\exists \delta < \eta\, \forall \rho\, (\delta \leq \rho \leq \eta \rightarrow c_g(\rho, e) = c_g(\eta, e) < \hat{\alpha}),$$

and the characteristic function of this relation is denoted by $l'_{Q,g}(\eta, e)$ (or $l'(\eta, e)$ for short). Notice that $l'_{Q,g}(\eta, e)$ (or $l'(\eta, e)$) is α-recursive.

Lemma 3.10. $l(e) = 1 \leftrightarrow \lim_{\eta \to \alpha} l'(\eta, e) = 1$.

Proof. Let $i = g^{-1}(e) < \hat{\alpha}$.

Suppose $l(e) = 1$. Then $\forall j < i\,(Q_{g(i)} \neq Q_{g(j)})$. As in Lemma 3.9, there is a $b_0 < \hat{\alpha}$ such that

$$\forall j < i\, \exists x < b_0\, \exists \gamma < b_0 (Q_{e,f(\gamma)}(g(x)) = Q_e(g(x)) \neq Q_{g(j)}(g(x)) = Q_{g(j),f(\gamma)}(g(x))).$$

Thus,

$$\forall j < i\, \forall \eta > f(b_0)\, (Q_{e,\eta} \restriction \operatorname{ran}(g \restriction b_0)) \neq Q_{g(j),\eta} \restriction (\operatorname{ran}(g \restriction b_0))).$$

Let η_0 be a stage such that

$$\forall \eta > \eta_0\, (g_\eta \restriction (\max\{i, b_0\} + 1) = g \restriction (\max\{i, b_0\} + 1)).$$

Also, it follow easily from $t\sigma 2p\,(\alpha) \leq \sigma 1p\,(\alpha)$ and Theorem 3.1 that there is an η_1 such that

$$\forall j \leq i\, (Q_{g(j)} \restriction \operatorname{ran}(g \restriction b_0) = Q_{g(j),\eta_1} \restriction \operatorname{ran}(g \restriction b_0)).$$

Then for any $\eta > \max\{\eta_0, \eta_1, f(b_0)\}$,

$$\begin{aligned} c_g(\eta, e) &= \max\{b < \min(\hat{\alpha}, \eta) : \exists j < i\, (Q_{e,\eta} \restriction \operatorname{ran}(g_\eta \restriction b) = Q_{g_\eta(j),\eta} \restriction \operatorname{ran}(g_\eta \restriction b))\} \\ &= \max\{b < b_0 : \exists j < i\, (Q_e \restriction \operatorname{ran}(g \restriction b) = Q_{g(j)} \restriction \operatorname{ran}(g \restriction b))\} \end{aligned}$$

is a constant less than $\hat{\alpha}$, and so $\lim_{\eta \to \alpha} l'(\eta, e) = 1$.

Now assume δ is a stage such that $\forall \eta > \delta\, (l'(\eta, e) = 1)$. Then $\forall \eta > \delta\, (c_g(\eta, e) = c_g(\delta, e) < \hat{\alpha})$. For the sake of contradiction, suppose $j < i$ and $Q_{g(j)} = Q_{g(i)} = Q_e$.

Similar to the existence of η_0 and η_1 above, there is a stage $\eta_2 > c_g(\delta, e) + 1$ such that

$$\begin{aligned} \forall \eta > \eta_2\, [g_\eta \restriction (\max\{i, c_g(\delta, e)\} + 1) &= g \restriction (\max\{i, c_g(\delta, e)\} + 1) \\ \wedge\ Q_{g(j)} \restriction \operatorname{ran}(g \restriction (c_g(\delta, e) + 1)) &= Q_{g(j),\eta} \restriction \operatorname{ran}(g \restriction (c_g(\delta, e) + 1)) \\ \wedge\ Q_e \restriction \operatorname{ran}(g \restriction (c_g(\delta, e) + 1)) &= Q_{e,\eta} \restriction \operatorname{ran}(g \restriction (c_g(\delta, e) + 1))]. \end{aligned}$$

Thus, $c_g(\eta, e) \geq c_g(\delta, e) + 1$ for each $\eta > \eta_2$, a contradiction. □

Construction. At each stage η, the construction below is carried out in four steps as described earlier. Two α-recursive functions $F_P(\eta, e)$ and $F_Q(\eta, e)$ are defined to denote the follower of P_e at stage η and the follower of Q_e at stage η respectively. During the construction, ρ is said to be *unused* if ρ has not been in the range of F_P and F_Q defined so far.

The construction proceeds as follows.

At stage η. **Step One.** For each $e < \alpha$,

Case 1.1: $e \geq \eta$ or $l'(\eta, e) = 0$. Set $F_Q(\eta, e) = -1$.

Case 1.2: Case 1.1 fails and either η is a limit ordinal such that $\lim_{\gamma \to \eta} F_Q(\gamma, e) \neq -1$ exists or $\eta = \eta' + 1$ is a successor ordinal such that $F_Q(\eta', e) \geq 0$. Then let

$$F_Q(\eta, e) = \begin{cases} \lim_{\gamma \to \eta} F_Q(\gamma, e) & \text{if } \eta \text{ is a limit ordinal,} \\ F_Q(\eta', e) & \text{if } \eta = \eta' + 1. \end{cases}$$

Case 1.3: Case 1.1 and Case 1.2 fail. Let $e_0 < e_1 < \ldots < e_\zeta < \ldots$ be a list of all e's of Case 1.3 and $\rho_0 < \rho_1 < \ldots < \rho_\zeta < \ldots$ be a list of all unused ρ. Let $F_Q(\eta, e_\zeta) = \rho_\zeta$ for each e_ζ.

Step Two. For any $e < \alpha$,

Case 2.1. Either η is a limit ordinal such that $\lim_{\gamma \to \eta} F_P(\gamma, e) \neq -1$ exists or $\eta = \eta' + 1$ is a successor ordinal such that $F_P(\eta', e) \geq 0$. Then set

$$F_P(\eta, e) = \begin{cases} \lim_{\gamma \to \eta} F_P(\gamma, e) & \text{if } \eta \text{ is a limit ordinal,} \\ F_P(\eta', e) & \text{if } \eta = \eta' + 1. \end{cases}$$

Case 2.2. Case 2.1 fails and $e < \eta$. Similar to Case 1.3, define $F_P(\eta, e)$ to be ρ'_ζ, whenever e is the ζ^{th} ordinal in Case 2.2 and ρ'_ζ is the ζ^{th} unused ρ by the end of step one.

Case 2.3. Case 2.1 and Case 2.2 fail. $F_P(\eta, e)$ will be defined in step three.

Step Three. Let $\rho''_0 < \rho''_1 < \ldots < \rho''_\zeta < \ldots$ be a list of ρ's such that

(i) The ordinal ρ is not $F_P(\eta, e)$ and not $F_Q(\eta, e')$ for any defined $F_P(\eta, e)$ and $F_Q(\eta, e')$, and

(ii) Either $\rho < \eta$ or $\rho \in \{F_P(\delta, d) : d < \alpha, \delta < \eta\} \cup \{F_Q(\delta, d) : d < \alpha, \delta < \eta\}$.

Now recursively define

$$\begin{aligned} e''_\zeta = {} & \text{the first enumerated } e > \sup_{\zeta' < \zeta} e''_{\zeta'} \text{ such that } P_e \supseteq \textstyle\bigcup_{\delta < \eta} A_{\rho''_\zeta, \delta} \\ & \text{and that } F_P(\eta, e) \text{ is undefined by the end of step two.} \end{aligned}$$

Define $F_P(\eta, e''_\zeta)$ to be ρ''_ζ.

Finally, for $F_P(\eta, e)$ still undefined, let $F_P(\eta, e) = -1$.

Step Four. For any $\rho < \alpha$, if $\rho = F_P(\eta, e)$, then let $A_{\rho,\eta} = (\bigcup_{\zeta<\eta} A_{\rho,\zeta}) \cup P_{e,\eta}$; if $\rho = F_Q(\eta, e)$, then let $A_{\rho,\eta} = (\bigcup_{\zeta<\eta} A_{\rho,\zeta}) \cup Q_{e,\eta}$. Otherwise, let $A_{\rho,\eta} = \bigcup_{\zeta<\eta} A_{\rho,\zeta}$.

Verification. Clause (iii) of the next lemma implies that the above construction is α-recursive.

Lemma 3.11. *Assume $\eta < \alpha$.*

(i) For all $e < \eta$, $F_P(\eta, e) \geq 0$ and $(F_Q(\eta, e) \geq 0 \leftrightarrow l'(\eta, e) = 1)$;

(ii) $\eta \subseteq \mathrm{ran}(F_P \restriction (\{\eta\} \times \alpha)) \cup \mathrm{ran}(F_Q \restriction (\{\eta\} \times \alpha))$;

(iii) $\{e : F_P(\eta, e) \neq -1\}$, $\{e : F_Q(\eta, e) \neq -1\}$, $\mathrm{ran}(F_P \restriction (\{\eta\} \times \alpha)) \setminus \{-1\}$ and $\mathrm{ran}(F_Q \restriction (\{\eta\} \times \alpha)) \setminus \{-1\}$ are α-finite;

(iv) $\forall e, e' \, (F_P(\eta, e), F_Q(\eta, e') \geq 0 \rightarrow F_P(\eta, e) \neq F_Q(\eta, e'))$ and $\forall e, e' \, ((F_P(\eta, e) = F_P(\eta, e') \geq 0) \vee (F_Q(\eta, e) = F_Q(\eta, e') \geq 0) \rightarrow e = e')$. In other words, at stage η, the assignment of followers is one-one;

(v) $\forall e \, (F_P(\eta, e) \geq 0 \rightarrow \forall \delta > \eta \, F_P(\delta, e) = F_P(\eta, e))$, i.e. P_e never releases its follower for any e;

(vi) $\forall e \, (\eta > e \wedge \forall \delta \geq \eta \, (l'(\delta, e) = 1) \rightarrow \forall \delta > \eta \, (F_Q(\delta, e) = F_Q(\eta, e)))$, i.e. Q_e never release its follower after stage η if e is thought to be the least index via g from stage η onwards;

(vii) $A_{\rho,\eta}$ is equal to $P_{e,\eta}$ if $F_P(\eta, e) = \rho$, and is equal to $Q_{e,\eta}$ if $F_Q(\eta, e) = \rho$.

Proof. By induction on η and δ (δ is as in Clause (v)-(vi)). □

Define $F_P^*, F_Q^* : \alpha \to \alpha \cup \{-1\}$ by

$$F_P^*(e) = \lim_{\eta \to \alpha} F_P(\eta, e), \quad F_Q^*(e) = \lim_{\eta \to \alpha} F_Q(\eta, e).$$

That is, $F_P^*(e)$ is the permeant follower of P_e; and $F_Q^*(e)$, if defined, is the permanent follower of Q_e.

Part (i), (v) and (vi) of Lemma 3.11 together imply that

$$\forall e \, (F_P^*(e) \downarrow \neq -1), \quad \forall e \, (l(e) = 1 \rightarrow F_Q^*(e) \downarrow \neq -1).$$

For $e < \alpha$ such that $l(e) = 0$, Lemma 3.10 implies that there are cofinally many stages η satisfying $l'(\eta, e) = 0$, and so there are cofinally many stages η such that $F_Q(\eta, e) = -1$. Thus,

$$\forall e \, (l(e) = 0 \rightarrow F_Q^*(e) \uparrow \vee F_Q^*(e) = -1).$$

By (iv), the assignment of permanent followers is one-one, i.e.

$$\text{(3.7)} \quad \forall e, e' \, [(l(e') = 1 \rightarrow F_P^*(e) \neq F_Q^*(e')) \wedge (e \neq e' \rightarrow F_P^*(e) \neq F_P^*(e')) \\ \wedge (e \neq e' \wedge l(e) = l(e') = 1 \rightarrow F_Q^*(e) \neq F_Q^*(e'))].$$

According to (vii),

$$\forall e \, (P_e = A_{F_P^*(e)}), \text{ and } \forall e \, (l(e) = 1 \rightarrow Q_e = A_{F_Q^*(e)}).$$

Consequently, $\{A_e\}_{e<\alpha}$ is a universal numbering of all α-r.e. sets. To show that $\{A_e\}_{e<\alpha}$ is a Friedberg numbering, it is only necessary to show that $\{A_e\}_{e<\alpha}$ is one-one. Observe that by (3.7), $\{A_e\}_{e<\alpha}$ being one-one is immediate once $\alpha \subseteq \mathrm{ran}(F_P^*) \cup \mathrm{ran}(F_Q^*)$ has been proved.

Let $\rho < \alpha$. By (v), if $\rho = F_P(\eta, e)$ for some η, e, then $\rho = F_P^*(e) \in \mathrm{ran}(F_P^*)$. Now suppose $\rho \neq F_P(\eta, e)$ for all η and e. Then at stage $\rho+1$, according to (ii), $\rho = F_Q(\rho+1, e')$. Moreover, $\forall \eta > \rho + 1 \, (\rho = F_Q(\eta, e'))$. Otherwise, at the least stage $\eta > \rho + 1$ with

$\rho \neq F_Q(\eta, e')$, it is defined in step three that $F_P(\eta, e'') = \rho$ for some e'', yielding a contradiction. Since $\forall \eta > \rho\, (\rho = F_Q(\eta, e'))$, we immediately get $l(e') = 1$ and $\rho = F^*_Q(e')$.

3.4. **Pseudostability.** Through out this section of pseudostability, we make the assumption that $\sigma 1p\,(\alpha) > \omega$. Under this assumption, we introduce the notion of pseudostability and generalize some properties of α-stable ordinals to pseudostable ordinals. In Section 3.5, pseudostability will be used to show the nonexistence of a Friedberg numbering when $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$.

Suppose $\{A_e\}_{e<\alpha}$ is an arbitrary numbering. As noticed in Section 3.2, α-stable ordinals are used to obtain, roughly speaking, an upper bound of the least differences between a given α-finite set C and α-finitely many α-r.e. sets of the numbering. That idea succeeds mainly because of the following property: for any ζ and α-finite set C, if $\delta < \sigma 1p\,(\alpha)$ and β is a large enough α-stable ordinal, then

$$\forall e < \delta\, (A_e \supseteq C \cup [\zeta, \beta) \leftrightarrow A_{e,\beta} \supseteq C \cup [\zeta, \beta)).$$

Pseudostable ordinals are defined mainly by this property.

Lemma 3.12. *Suppose $\{A_e\}_{e<\alpha}$ is a numbering. Then there exists an α-recursive function $h : \alpha^5 \to \alpha$, such that: for any $\gamma, \zeta < \alpha$, α-finite set $C \subset \alpha$, and α-finite (partial) function $p : \alpha \xrightarrow{\text{one-one}} \alpha$ satisfying $|\operatorname{dom}(p)|_\alpha < \sigma 1p\,(\alpha)$, we have*

(i) For each $\eta < \alpha$, $h(\eta, \gamma, \zeta, C, p) \leq \eta$ is defined.
(ii) The sequence $\{h(\eta, \gamma, \zeta, C, p)\}_{\eta<\alpha}$ is nondecreasing.
(iii) There is a $\beta < \alpha$ such that

$$\beta = \lim_{\eta \to \alpha} h(\eta, \gamma, \zeta, C, p) = h(\beta, \gamma, \zeta, C, p) > \max\{\gamma, \zeta, C, p, \sup C\},$$

and

$$\forall e \in \operatorname{ran}(p)\, (A_e \supseteq C \cup [\zeta, \beta) \leftrightarrow A_{e,\beta} \supseteq C \cup [\zeta, \beta)). \tag{3.8}$$

The rest of this section will be devoted to the proof of Lemma 3.12.

An ordinal $\beta < \alpha$ is said to be *pseudostable relative to the numbering* $\{A_e\}_{e<\alpha}$, if $\beta = \lim_{\eta\to\alpha} h(\eta, \gamma, \zeta,\, C, p)$ for some h, γ, ζ, C, p satisfying all the requirements in Lemma 3.12. Immediately from the definition, for any $\zeta < \alpha$, and for any C, p as in Lemma 3.12, pseudostable ordinals $\{\lim_{\eta\to\alpha} h(\eta, \gamma, \zeta, C,\, p) : \gamma < \alpha\}$ are cofinal in α.

In the construction given in Section 3.5, Lemma 3.12 is applied as follows: the function p is an initial segment of the graph of a tame Σ_2 projection from $t\sigma 2p\,(\alpha)$ to α, γ is a stage such that all approximations related to the initial segment have reached their final limit, C is an initial segments of the set to be constructed, and $\zeta = \sup C$.

The method of proof of Lemma 3.12 consists of a Skolem hull argument below α with respect to the property (3.8) and, roughly speaking, coding the approximation of the Skolem hull construction into the enumeration of an α-r.e. subset with α-cardinality less than $\sigma 1p\,(\alpha)$. By Theorem 3.1, the α-r.e. set is α-finite. Thus, its enumeration terminates before α. Consequently, the Skolem hull is also below α.

Skolem hull argument. From now on, γ, ζ, C, p are as in Lemma 3.12 and fixed. For each $n < \omega$, define the Skolem function

$$\begin{aligned} z_0(\gamma, \zeta, C, p) &= \max\{\gamma, \zeta, C, p, \sup C\} + 1, \\ z_{n+1}(\gamma, \zeta, C, p) &= \mu z \geq z_n(\gamma, \zeta, C, p)(\forall e \in \operatorname{ran}(p)\, (A_e \supseteq C \cup [\zeta, z_n(\gamma, \zeta, C, p)) \to \\ &\qquad A_{e,z} \supseteq C \cup [\zeta, z_n(\gamma, \zeta, C, p))). \end{aligned}$$

Note that $z_n(\gamma, \zeta, C, p)$ is defined uniformly from the parameters γ, ζ, C, p. To simply the notation, we suppress these parameters unless the possibility of confusion arises.

Lemma 3.13. $\{z_n : n < \omega\} \subseteq \alpha$.

Proof. Since p is one-one, $|\operatorname{ran}(p)|_\alpha = |\operatorname{dom}(p)|_\alpha < \sigma 1p\,(\alpha)$. Thus, any α-r.e. subset of $\operatorname{ran}(p)$ is α-finite, by Theorem 3.1.

By induction on n, if $z_n < \alpha$, the set $\{e \in \operatorname{ran}(p) : A_e \supseteq C \cup [\zeta, z_n)\}$ is α-finite. Hence $z_{n+1} < \alpha$ by Σ_1 replacement. It follows that $\{z_n : n < \omega\} \subseteq \alpha$. □

Lemma 3.14. $\forall n\, \forall \eta \geq z_{n+1}\, \forall e \in \operatorname{ran}(p)\, (A_e \supseteq C \cup [\zeta, z_n) \leftrightarrow A_{e,\eta} \supseteq C \cup [\zeta, z_n))$.

Proof. By the definition of z_{n+1} and the fact that $\{A_e\}_{e<\alpha}$ are α-r.e. sets. □

Let $\beta(\gamma, \zeta, C, p) = \max_{n<\omega} z_n(\gamma, \zeta, C, p)$. Again, note that $\beta(\gamma, \zeta, C, p)$ is defined uniformly in terms of the parameters γ, ζ, C, p. Therefore, we suppress these parameters for simplicity.

Lemma 3.15. $\forall e \in \operatorname{ran}(p)\, (A_e \supseteq C \cup [\zeta, \beta) \leftrightarrow A_{e,\beta} \supseteq C \cup [\zeta, \beta))$.

Proof. For any $e \in \operatorname{ran}(p)$,

$$\begin{aligned} & A_e \supseteq C \cup [\zeta, \beta) \\ \leftrightarrow\ & \forall n < \omega\, (A_e \supseteq C \cup [\zeta, z_n)) \\ \leftrightarrow\ & \forall n < \omega\, (A_{e,\beta} \supseteq C \cup [\zeta, z_n)) \quad \text{by Lemma 3.14} \\ \leftrightarrow\ & A_{e,\beta} \supseteq C \cup [\zeta, \beta). \end{aligned}$$

□

It will be shown later that $\beta < \alpha$. For the moment assume that this is true. To prove Lemma 3.12, it remains to define h by the approximation of $\{z_n\}_{n<\omega}$, so that $\beta = \lim_{\eta\to\alpha} h(\eta, \gamma, \zeta, C, p)$.

At stage η, define the approximation of $\{z_n\}_{n<\omega}$ by induction on $n < \omega$ as follows:

$$z_{0,\eta} = \min\{z_0, \eta\},$$

$$\begin{aligned} z_{n+1,\eta} = \max\{&\max_{\eta'<\eta} z_{n+1,\eta'}, \mu z \leq \eta[(z \geq z_{n,\eta}) \wedge \\ &\forall e \in \operatorname{ran}(p)\, (A_{e,\eta} \supseteq C \cup [\zeta, z_{n,\eta}) \to A_{e,z} \supseteq C \cup [\zeta, z_{n,\eta}))]\}. \end{aligned}$$

In the definition of $z_{n+1,\eta}$, "$\max_{\eta'<\eta} z_{n+1,\eta'}$" ensures that $z_{n+1,\eta}$ is nondecreasing with respect to η, and "$A_{e,\eta} \supseteq C \cup [\zeta, z_{n,\eta}) \to A_{e,z} \supseteq C \cup [\zeta, z_{n,\eta})$" is a Skolem hull construction.

Lemma 3.16. *Suppose $n < \omega$. Then*

(i) $\{z_{n,\eta}\}_{\eta<\alpha}$ is a nondecreasing sequence;
(ii) $\forall \eta\, (z_{n,\eta} \leq \min\{z_n, \eta\})$;
(iii) $\forall \eta \geq z_n\, (z_{n,\eta} = z_n)$.

Proof. Clause (i) is immediate from the definition of $z_{n,\eta}$. Also from the definition of $z_{n,\eta}$, an induction on η shows $\forall \eta\, \forall n\, (z_{n,\eta} \leq \eta)$. Hence $\forall \eta < z_n\, \forall n\, (z_{n,\eta} \leq \min\{z_n, \eta\})$. Therefore, to prove (ii), only (iii) needs to be shown.

Clause (iii) is proved by induction on n and η. We omit the details. □

For any $\eta < \alpha$, define

$$h(\eta, \gamma, \zeta, C, p) = \max_{n<\omega} z_{n,\eta}.$$

By Lemma 3.16,

$$\forall \eta \geq \beta\, (h(\eta, \gamma, \zeta, C, p) = \beta).$$

It is easy to check (i)-(iii) of Lemma 3.12. To complete the proof of Lemma 3.12, it remains only to verify that $\max_{n<\omega} z_n < \alpha$, i.e. $\beta < \alpha$. The following lemma deals with a special case and is straightforward to verify.

Lemma 3.17. *If $z_{n+1} = z_n$, then $\forall m < \omega\, (m > n \to z_m = z_n)$.*

Lemma 3.17 suggests that if $z_n = z_{n+1}$, for some $n < \omega$, then $\beta = \max_{m<\omega} z_m = z_n < \alpha$. Thus, to show $\beta < \alpha$ in general, we only need to check the case when $\{z_n\}_{n<\omega}$ is strictly increasing. That case will be addressed in the coding part below.

Coding. Let γ, ζ, C, p be given and $\{z_n\}_{n<\omega}$ be defined as in previous part of Skolem hull argument. In this part, we always assume that $\{z_n\}_{n<\omega}$ is strictly increasing. Then it is immediate from the definition of z_n that

$$\forall n < \omega\, \exists e \in \mathrm{ran}(p)\, (A_e \supseteq C \cup [\zeta, z_n)). \tag{3.9}$$

With the above formula in mind, it is straightforward to code the approximation of β by enumerating (n, e) such that, (n, e) is enumerated at stage η if $A_{e,\eta} \supseteq C \cup [\zeta, z_{n,\eta})$. It is tempting to assume (mistakenly) that $z_{n+1,\eta} = z_{n+1}$ if and only if the (n, e)'s have completed their enumeration at stage η. Nevertheless, in that event, the enumeration of (n, e)'s may terminate before the enumeration of some (m, e'), $m < n$, due to the approximation of z_n and z_m, $m < n$. The trick to cover this possibility is to incorporate the enumeration of the (m, e')'s, for all $m < n$, in the enumeration of the (n, e)'s: Suppose at stage η, $A_{e,\eta} \supseteq C \cup [\zeta, z_{n,\eta})$. Then $(n, e_0, e_1, \dots, e_n)$ is enumerated if $(n-1, e_0, e_1, \dots, e_{n-1})$ is enumerated by stage η. Then for $n > 0$, the enumeration of the $(n, e_0, e_1, \dots, e_n)$'s does not terminate whenever some $(n-1, e'_0, e'_1, \dots, e'_{n-1})$ is yet to be enumerated.

More precisely, define an α-r.e. set $D \subseteq \bigcup_{n<\omega}(\{n\} \times \mathrm{ran}(p)^{n+1})$ as follows, where

$$\bigcup_{n<\omega}(\{n\} \times \mathrm{ran}(p)^{n+1}) = \{(n, e_0, e_1, \dots, e_n) : n < \omega, e_0, e_1, \dots, e_n \in \mathrm{ran}(p)\}.$$

Suppose $\eta < z_0$. Then let $D_\eta = \emptyset$.

At stage $\eta \geq z_0$, the enumeration of D_η is carried out in ω steps. Let

$$D_{\eta,0} = (\textstyle\bigcup_{\eta'<\eta} D_{\eta'}) \cup \{(0, e) : e \in \mathrm{ran}(p) \ \wedge\ A_{e,\eta} \supseteq C \cup [\zeta, z_0)\},$$

and if $n > 0$,

$$D_{\eta,n} = (\textstyle\bigcup_{m<n} D_{\eta,m}) \cup \{(n, e_0, e_1, \dots, e_n) : e_0, e_1, \dots, e_n \in \mathrm{ran}(p) \wedge \\ A_{e_n,\eta} \supseteq C \cup [\zeta, z_{n,\eta}) \ \wedge\ (n-1, e_0, e_1, \dots, e_{n-1}) \in D_{\eta,n-1}\}.$$

$$D_\eta = \textstyle\bigcup_{n<\omega} D_{\eta,n}.$$

Then let $D = \bigcup_{\eta<\alpha} D_\eta$.

Lemma 3.18. *If $n > 0$, $(n-1, e_0, e_1, \dots, e_{n-1}) \in D$ and $A_{e_n} \supseteq C \cup [\zeta, z_n)$, then $(n, e_0, e_1, \dots, e_n) \in D$.*

Proof. Let $\eta > z_n$ be large enough such that $(n-1, e_0, e_1, \dots, e_{n-1}) \in D_\eta$, and $C \cup [\zeta, z_n) \subseteq A_{e_n,\eta}$. Since $\eta > z_n$, we have $z_{n,\eta} = z_n$. Thus, $(n, e_0, e_1, \dots, e_n) \in D_\eta$. □

Lemma 3.19. *For any $n < \omega$ and $\eta < \alpha$,*

$$\eta \geq z_{n+1} \leftrightarrow D_\eta \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}) = D \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}). \tag{3.10}$$

Proof. The lemma is proved by induction on n.

Let $n = 0$. By the definition of D_η, for all $e \in \mathrm{ran}(p)$ and $\eta < \alpha$,

$$(0, e) \in D_\eta \leftrightarrow (\eta \geq z_0 \ \wedge\ A_{e,\eta} \supseteq C \cup [\zeta, z_0)).$$

Thus, for any $e \in \mathrm{ran}(p)$,

$$(0, e) \in D \leftrightarrow A_e \supseteq C \cup [\zeta, z_0),$$

According to (3.9), $D \restriction (\{0\} \times \mathrm{ran}(p)) \neq \emptyset$. Therefore,

$$D \restriction (\{0\} \times \mathrm{ran}(p)) = D_\eta \restriction (\{0\} \times \mathrm{ran}(p)) \to \eta \geq z_1.$$

The other direction of (3.10) for $n = 0$ is immediate from the definition of z_1.

With the intention of showing (3.10) when $n > 0$, assume that (3.10) is true for $0, \dots, n-1$. Pick any $\eta < \alpha$. We consider three cases.

Case 1. $\eta < z_n$. Since (3.10) is true for $n-1$, let $(n-1, e_0, e_1, \dots, e_{n-1}) \in D \setminus D_\eta$. Let $e_n \in \mathrm{dom}(p)$ be any index such that $A_{e_n} \supseteq C \cup [\zeta, z_n)$. Then $(n, e_0, e_1, \dots, e_n) \in D \setminus D_\eta$. Hence $D_\eta \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}) \neq D \restriction (\{n\} \times \mathrm{ran}(p)^{n+1})$.

Case 2. $z_n \leq \eta < z_{n+1}$. Then $z_{n,\eta} = z_n$ and by the definition of z_{n+1}, there is some $e_n \in \mathrm{ran}(p)$ such that $A_{e_n} \supseteq C \cup [\zeta, z_n)$ but $A_{e_n,\eta} \not\supseteq C \cup [\zeta, z_n)$. Let $x \in C \cup [\zeta, z_n) \setminus A_{e_n,\eta}$. Since $z_n > \sup C$, we have $x < z_n$.

Subcase 2.1. there exists $(n-1, e_0, e_1, \dots, e_{n-1}) \in D \setminus \bigcup_{\eta' < z_n} D_{\eta'}$. Then

1. Since $(n-1, e_0, e_1, \dots, e_{n-1}) \notin \bigcup_{\eta' < z_n} D_{\eta'}$, $(n, e_0, e_1, \dots, e_n) \notin \bigcup_{\eta' < z_n} D_{\eta'}$;
2. For any δ such that $z_n \leq \delta \leq \eta$, we have $(n, e_0, e_1, \dots, e_n) \notin D_\delta \setminus \bigcup_{\delta' < \delta} D_{\delta'}$, as $A_{e_n,\delta} \not\supseteq C \cup [\zeta, z_n)$ and $z_n = z_{n,\delta}$.

Thus, $(n, e_0, e_1, \dots, e_n) \in D \setminus D_\eta$. Hence $D_\eta \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}) \neq D \restriction (\{n\} \times \mathrm{ran}(p)^{n+1})$.

Subcase 2.2. Subcase 2.1 fails. Then we claim that $\max_{\eta' < z_n} z_{n,\eta'} = z_n$. It will be proved in a moment. For now assume the claim and let $\eta' < z_n$ be such that $z_{n,\eta'} > x$. Since $\eta' < z_n$, there is $(n-1, e_0, e_1, \dots, e_{n-1}) \in D \setminus D_{\eta'}$. Therefore,

1. If $\delta \leq \eta'$, then $(n, e_0, e_1, \dots, e_n) \notin D_\delta$ since $(n-1, e_0, e_1, \dots, e_{n-1}) \notin D_\delta$;
2. If $\eta' < \delta \leq \eta$, then $z_{n,\delta} > x$ and $x \in C \cup [\zeta, z_{n,\delta}) \setminus A_{e_n,\delta}$. Therefore, $A_{e_n,\delta} \not\supseteq C \cup [\zeta, z_{n,\delta})$ and $(n, e_0, e_1, \dots, e_n) \notin D_\delta \setminus \bigcup_{\delta' < \delta} D_{\delta'}$.

Thus, $(n, e_0, e_1, \dots, e_n) \in D \setminus D_\eta$. Hence $D_\eta \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}) \neq D \restriction (\{n\} \times \mathrm{ran}(p)^{n+1})$.

Case 3. $\eta \geq z_{n+1}$. One can see immediately that $z_{n,\eta} = z_n = z_{n,z_{n+1}}$. Suppose $A_{e_n,\eta} \supseteq C \cup [\zeta, z_{n,\eta})$ and $(n-1, e_0, e_1, \dots, e_{n-1}) \in D_{\eta,n-1}$. Then, by the definition of z_{n+1}, $A_{e_n,z_{n+1}} \supseteq C \cup [\zeta, z_{n,z_{n+1}})$ and by (3.10) for $n-1$, $(n-1, e_0, e_1, \dots, e_{n-1}) \in D_{z_{n+1}}$. Thus, $(n, e_0, e_1, \dots, e_n) \in D_{z_{n+1}}$. Hence $D_\eta \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}) = D_{z_{n+1}} \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}) = D \restriction (\{n\} \times \mathrm{ran}(p)^{n+1})$.

Finally, in Subcase 2.2, to see $\max_{\eta' < z_n} z_{n,\eta'} = z_n$, assume for a contradiction that $M = \max\{z_{n-1}, \max_{\eta' < z_n} z_{n,\eta'}\} < z_n$. Then there exists $(n-1, e_0^*, e_1^*, \dots, e_{n-1}^*) \in D \setminus D_M$. Let δ be the first stage that $(n-1, e_0^*, e_1^*, \dots, e_{n-1}^*)$ is enumerated into D. Then by (3.10) for $n-1$ and the assumption of Subcase 2.2, we conclude $M < \delta < z_n$. Now

(a) If $n = 1$, then $z_0 \leq M < \delta < z_1$ and $(0, e_0^*) \in D_\delta \setminus D_M$. Since $(0, e_0^*) \in D_\delta$, by the definition of D_δ, $A_{e_0^*,\delta} \supseteq C \cup [\zeta, z_0)$. Then by the definition of $z_{1,\delta}$, $A_{e_0^*,z_{1,\delta}} \supseteq C \cup [\zeta, z_0) = C \cup [\zeta, z_{0,\delta})$. Therefore, $A_{e_0^*,M} \supseteq C \cup [\zeta, z_0)$ and $(0, e_0^*) \in D_M$, a contradiction.

(b) If $n \geq 2$, then $z_{n-1} \leq M < \delta < z_n$ and $(n-1, e_0^*, e_1^*, \dots, e_{n-1}^*) \in D_\delta \setminus \bigcup_{\delta' < \delta} D_{\delta'}$. By definition of D_δ, $A_{e_{n-1}^*,\delta} \supseteq C \cup [\zeta, z_{n-1,\delta}) = C \cup [\zeta, z_{n-1}) = C \cup [\zeta, z_{n-1,M})$ and $(n-2, e_0^*, e_1^*, \dots, e_{n-2}^*) \in D_\delta$. Similar to the proof in (a), we have

$$A_{e_{n-1}^*,M} \supseteq C \cup [\zeta, z_{n-1,M}).$$

And by (3.10) for $n-2$, $(n-2, e_0^*, e_1^*, \dots, e_{n-2}^*) \in D_M$. Thus $(n-1, e_0^*, e_1^*, \dots, e_{n-1}^*)$ is in D_M, again a contradiction. □

Corollary 3.20. *For any $\eta < \alpha$,*

$$\eta \geq \beta \leftrightarrow D_\eta = D.$$

The next task to show that D is α-finite.

Lemma 3.21. *Every α-r.e. subset of $\bigcup_{n<\omega} (\{n\} \times \mathrm{ran}(p)^{n+1})$ is α-finite.*

Proof. Let $\kappa = \max\{|\operatorname{dom}(p)|_\alpha, \omega\}$. Since $|\operatorname{dom}(p)|_\alpha, \omega < \sigma 1p\,(\alpha)$, we have $\kappa < \sigma 1p\,(\alpha)$. Since p is one-one, it follows immediately that $|\operatorname{ran}(p)|_\alpha \leq \kappa$. Therefore, $|\{n\}\times\operatorname{ran}(p)^{n+1}|_\alpha \leq \kappa$ for all $n < \omega$. Furthermore, the α-finite bijections from $\{n\} \times \operatorname{ran}(p)^{n+1}$ to κ may be defined uniformly for all $n < \omega$. Hence $|\bigcup_{n<\omega}(\{n\} \times \operatorname{ran}(p)^{n+1})|_\alpha \leq \kappa < \sigma 1p\,(\alpha)$ and the lemma follows by Theorem 3.1. □

Lemma 3.20 and 3.21 combine to imply that D is α-finite. Hence

Lemma 3.22. $\max_{n<\omega} z_n < \alpha$, *i.e.* $\beta < \alpha$.

Observe at this point that Lemma 3.12 holds whenever $\sigma 1p\,(\alpha) > \omega$. Since no restriction on the numbering is required, if $\sigma 1p\,(\alpha) > \omega$, then Lemma 3.12 is applicable for any type of numberings. In particular, Lemma 3.12 is also true for a Gödel numbering when $\sigma 1p\,(\alpha) > \omega$. Notice that a Gödel numbering exists in L_α for all Σ_1 admissible ordinal α. Thus, in general, the nonexistence of a Friedberg numbering when $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$ (see Section 3.5) is not due to the existence of pseudostable ordinals.

3.5. When $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$.

In this section, we prove

Theorem 3.23. *If $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, then there is no Friedberg numbering of α-r.e. sets.*

Since $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, by Clause (1) of Corollary 3.5 , $\omega < \sigma 1p\,(\alpha)$ and $\sigma 2cf(\alpha) < \alpha$. Therefore, in this situation, the notion of pseudostability is applicable and Σ_2 replacement fails.

Let $\{A_e\}_{e<\alpha}$ be a one-one numbering, and let h be an α-recursive function satisfying Lemma 3.12. The objective is to construct an α-r.e. set X, so that $X \notin \{A_e\}_{e<\alpha}$. Thus, $\{A_e\}_{e<\alpha}$ is not a Friedberg numbering.

Fix the terminology as follows. Let

$$g : t\sigma 2p\,(\alpha) \xrightarrow[\text{onto}]{\text{one-one}} \alpha$$

be a tame Σ_2 projection, and according to Lemma 3.4 and Clause (2) of Corollary 3.5, let

$$f : \sigma 2cf(\alpha) \to t\sigma 2p\,(\alpha)$$

be a strictly increasing tame Σ_2 cofinal function so that $f(0) = 0$. Moreover, assume $f' : \alpha \times \sigma 2cf(\alpha) \to t\sigma 2p\,(\alpha)$, $g' : \alpha \times t\sigma 2p\,(\alpha) \to \alpha$ tamely generate f and g respectively. As in Section 3.3, f_η, g_η will be used to denote functions $\lambda x\,(f'(\eta, x))$ and $\lambda x\,(g'(\eta, x))$. Moreover, we assume that for all $\eta < \alpha$, f_η, g_η are nondecreasing and $\operatorname{ran}(f_\eta), \operatorname{ran}(g_\eta) \subseteq [0, \eta]$.

Strategy. As in Section 3.3, g makes it possible to arrange the indices of $\{A_e\}_{e<\alpha}$ on $t\sigma 2p\,(\alpha)$. The function f partitions $t\sigma 2p\,(\alpha)$ into $\sigma 2cf(\alpha)$ many blocks: $\{[f(i), f(i+1)) : i < \sigma 2cf(\alpha)\}$. $[f(i), f(i+1)))$ is said to be *the i^{th} block* (or *block i*) *of* $t\sigma 2p\,(\alpha)$. By *α-r.e. sets in the i^{th} block* (or *α-r.e. sets in block i*), we mean the α-r.e. sets are from the collection $\{A_e : g(e) \in [f(i), f(i+1))\}$. Since the numbering $\{A_e\}_{e<\alpha}$ is one-one, each α-r.e. set is in at most one block. The set X is constructed by diagonalizing against α-r.e. sets in each block.

Suppose $i < \sigma 2cf(\alpha)$, $\gamma < \alpha$, $C \subset \alpha$ is an α-finite set, and $\beta = \beta(\gamma, \sup C, C, g \restriction f(i))$ is the pseudostable ordinal obtained in Lemma 3.12 when $\zeta = \sup C$ and $p = g \restriction f(i)$, i.e.

$$\beta = \lim_{\eta\to\alpha} h(\eta, \gamma, \sup C, C, g \restriction f(i)),$$

and $X \restriction \beta = C \cup [\sup C, \beta)$. Then it follows from Lemma 3.12 that

$$\forall e \in \operatorname{ran}(g \restriction f(i))\,(A_e \supseteq X \restriction \beta \leftrightarrow A_{e,\beta} \supseteq X \restriction \beta). \tag{3.11}$$

Since $\{A_e\}_{e<\alpha}$ is a one-one numbering, there is at most one e in the range of $g \restriction f(i)$ such that $A_e = X \restriction \beta$. Therefore, by (3.11), the set $\{e \in \mathrm{ran}(g \restriction f(i)) : A_e \supsetneq X \restriction \beta\}$ is α-finite. According to Σ_1 replacement, let $u \geq \beta$ be such that

$$\forall e \in \mathrm{ran}(g \restriction f(i))\,(A_e \supsetneq X \restriction \beta \leftrightarrow A_{e,u} \supsetneq X \restriction \beta). \tag{3.12}$$

Now suppose e is in the range of $g \restriction f(i)$, then

(i) if $A_e \not\supseteq X \restriction \beta$, then there is a least $w < \beta$ such that $A_e(w) \neq X(w)$;

(ii) if $A_e \supseteq X \restriction \beta$, then either $A_e = X \restriction \beta$ or $A_{e,u} \supsetneq X \restriction \beta$.

Thus, to diagonalize against A_e in block j for all $j < i$ (i.e. $e \in \mathrm{ran}(g \restriction f(i))$), it suffices to define $X \restriction (u+1) = C \cup [\sup C, \beta) \cup \{u\}$. In our construction, X is defined by iterating this strategy though $i < \sigma 2cf(\alpha)$.

This strategy may be converted to an effective one, largely because f, g and h are effectively and tamely approximated. The only difficulty concerns obtaining a nice recursive approximation of u in (3.12) (notice that the intention is to make $X \restriction [\beta, u) = \emptyset$). A recursive approximation of u requires information regarding $I_{\beta,i} = \{e \in \mathrm{ran}(g \restriction f(i)) : A_e \supsetneq X \restriction \beta\}$. Lemma 3.3 and Lemma 3.12 provide a way around this difficulty. Notice that a correct guess of the set $I'_{\beta,i} = \{e \in \mathrm{ran}(g \restriction f(i)) : A_e \supseteq X \restriction \beta\}$ is obtained from stage β onwards. Thus, only a coding of the existence of an A_e which is equal to $X \restriction \beta$, where e is in the range of $g \restriction f(i)$, is needed to determine $I_{\beta,i}$: if such an A_e exists, then $I_{\beta,i}$ is obtained by enumerating all $e \in I'_{\beta,i}$ such that $A_e \supsetneq X \restriction \beta$ until only one index in $I'_{\beta,i}$ remains to be enumerated; if no such A_e exists, then $I_{\beta,i} = I'_{\beta,i}$. As will be seen in a moment, the coding is tame Σ_2 and hence, by Lemma 3.3, is α-finite.

The above strategy is an analogue of that in $B\Sigma_2$ models. The difference between the two constructions mainly arises from the upper bound established in the constructions. In $B\Sigma_2$ models, it is an upper bound of the least differences between any pair of r.e. sets in some blocks; in L_α, since Σ_2 replacement fails, the upper bound is only for the least differences between X and the α-r.e. sets in some α-finite part of the numbering.

Construction. X is first constructed recursively in $\emptyset'$ by induction though $\sigma 2cf(\alpha)$ with the intention of coding the existence of A_e such that A_e is equal to $X \restriction \beta_i$, where e is in the range of $g \restriction f(i)$, $i < \sigma 2cf(\alpha)$, and β_i is a pseudostable ordinal specified below.

Let $i < \sigma 2cf(\alpha)$. Suppose for all $j < i$, the values of $\gamma_j, \beta_j, u_j, X[j]$ and $G(j)$ have been defined. For i, the values of $\gamma_i, \beta_i, u_i, X[i]$ and $G(i)$ are defined as follows.

Stage γ_i is defined to be a stage such that the approximation of f below $i+1$ and the approximation of g below $f(i)+1$ have reached their limits from stage γ_i onwards:

$$\gamma_i = \max\{\mu\zeta\,(\forall \zeta' \geq \zeta\,(f_{\zeta'} \restriction (i+1) = f \restriction (i+1))),\ \mu\zeta\,(\forall \zeta' \geq \zeta\,(g_{\zeta'} \restriction (f(i)+1) = g \restriction (f(i)+1))\}.$$

If $\bigcup_{j<i} X[j]$ is α-finite, then let β_i be the pseudostable ordinal obtained in Lemma 3.12 when $\gamma = \gamma_i$, $C = \bigcup_{j<i} X[j]$, $\zeta = \sup(\bigcup_{j<i} X[j])$ and $p = g \restriction f(i)$, i.e.

$$\beta_i = \lim_{\zeta \to \alpha} h(\zeta, \gamma_i, \sup(\textstyle\bigcup_{j<i} X[j]), \bigcup_{j<i} X[j], g \restriction f(i)).$$

The pseudostable ordinal β_i together with an upper bound u_i defined below will be applied to diagonalize A_e in block j for all $j < i$. Intuitively, the upper bound u_i is a stage at which all α-r.e. sets with indices in the range of $g \restriction f(i)$ containing $(\bigcup_{j<i} X_j) \cup [\sup(\bigcup_{j<i} X_j), \beta_i)$ as a proper

subset have been enumerated. More precisely, we define

$$u_i = \mu u \geq \beta_i\, [\forall e \in \mathrm{ran}(g \restriction f(i))\, (A_e \supsetneq (\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i) \\ \to A_{e,u} \supsetneq (\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i))].$$

$X[i]$ is defined to be an end extension of $\bigcup_{j<i} X[j]$ using β_i and u_i as parameters with the intention of diagonalizing A_e in block j for all $j < i$:

$$X[i] = (\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i) \cup \{u_i\}.$$

X succeeds in diagonalizing A_e in a block j for all $j < i$ if X is an end extension of $X[i]$, for the reason shown in the part of the strategy.

$G(i)$ is defined below to provide the desired code of the existence of A_e in a block $j < i$ such that A_e is identical with $(\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i))$, i.e.

$$G(i) = \begin{cases} 1 & \text{if } \exists e \in \mathrm{ran}(g \restriction f(i))\, (A_e = (\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i)), \\ 0 & \text{otherwise.} \end{cases}$$

$G(i)$ will be a parameter of the recursive approximation of u_i as shown in the section we described the strategy. We review the idea briefly in the following.

For the rest of this paragraph we only consider A_e's such that e is in the range of $g \restriction f(i)$. Also for simplicity, let Υ_i denote the α-finite set $(\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i))$. Since β_i is pseudostable, whether A_e contains Υ_i as a subset is determined at stage β_i. If $G(i) = 0$, then all A_e containing Υ_i as a subset will contain Υ_i as a proper subset. Therefore when $G(i) = 0$, to determine u_i, one only needs to wait until each A_e containing Υ_i as a subset at stage β has enumerated an element not in Υ_i. If $G(i) = 1$, then all but one A_e containing Υ_i as a subset would contain Υ_i as a proper subset. Thus when $G(i) = 1$, to determine u_i, one only needs to wait until all but one A_e containing Υ_i as a subset at stage β has enumerated an element not in Υ_i.

Lemma 3.24. *The function $q : i \mapsto (\gamma_i, \beta_i, u_i, X[i], G(i))$ is tame Σ_2 and has domain $\sigma 2cf(\alpha)$.*

Proof. Suppose $\delta = \mathrm{dom}(q) \leq \sigma 2cf(\alpha)$. Notice that

(1) f, g are tame Σ_2;
(2) For every $i < \delta$ and $\zeta \geq \beta_i$, $h(\zeta, \gamma_i, \sup(\bigcup_{j<i} X[j]), \bigcup_{j<i} X[j], g \restriction f(i)) = \beta_i$, i.e. the approximation to β_i reaches its limit at stage β_i and does not change thereafter;
(3) For every $i < \delta$, by Lemma 3.12 and definitions of β_i and u_i,

$$G(i) = 0 \leftrightarrow \forall e \in \mathrm{ran}(g \restriction f(i))(A_{e,\beta_i} \supseteq (\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i) \\ \to A_{e,u_i} \supsetneq (\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i)).$$

Now it is straightforward to verify that the function q is Σ_2.

Moreover, q can be viewed as a (partial) function on $\sigma 2cf(\alpha)$. Since Lemma 3.4 implies that q is tame Σ_2, we have $q \restriction a$ is α-finite, whenever $a \leq t\sigma 2p\,(\alpha)$.

For the sake of contradiction, assume $\delta < \sigma 2cf(\alpha)$. Then $q \restriction \delta$ and $\bigcup_{j<\delta} X[j]$ are α-finite. This implies that γ_δ and β_δ are defined. Since $f(\delta) < t\sigma 2p\,(\alpha) \leq \sigma 1p\,(\alpha)$ and

g is a tame Σ_2 one-one function, by Theorem 3.1, each α-r.e. subset of $\mathrm{ran}(g \restriction f(\delta))$ is α-finite. Hence

$$\{e \in \mathrm{ran}(g \restriction f(\delta)) : A_e \supsetneq (\bigcup_{j<\delta} X[j]) \cup [\sup(\bigcup_{j<\delta} X[j]), \beta_\delta)\}$$

is α-finite. Thus, u_δ is well defined by Σ_1 replacement, and so are $X[\delta]$ and $G(\delta)$, a contradiction. □

Lemma 3.25. $G : \sigma 2cf(\alpha) \to \{0, 1\}$ *is* α*-finite.*

Proof. By Lemma 3.24, $\{i < \sigma 2cf(\alpha) : G(i) = 1\}$ is tame Σ_2. Since $\sigma 2cf(\alpha) < t\sigma 2p(\alpha)$, according to Lemma 3.3, G is α-finite. □

Let

$$X = \bigcup_{i<\sigma 2cf(\alpha)} X[i].$$

Lemma 3.26. $X \notin \{A_e\}_{e<\alpha}$.

Proof. Assume $X \in \{A_e\}_{e<\alpha}$ for a contradiction. Since f is cofinal and g is onto, there is $i < \sigma 2cf(\alpha)$ and $e \in \mathrm{ran}(g \restriction f(i))$ such that $X = A_e$. Let Υ_i denote the α-finite set $(\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i)$ for simplicity.

Observe that for every $i' > i$, $X[i']$ is an end extension of $X[i]$. Hence

$$\Upsilon_i = X \restriction u_i = A_e \restriction u_i.$$

Since $X \supsetneq \Upsilon_i$, it follows from the definition of u_i that $A_{e,u_i} \supsetneq \Upsilon_i$. But notice that $A_{e,u_i} \subseteq A_e \restriction u_i = \Upsilon_i$, a contradiction. □

Verifying that X is α-r.e. Lemma 3.26 states that X is not in the numbering $\{A_e\}_{e<\alpha}$. To see that $\{A_e\}_{e<\alpha}$ is not a universal numbering, we only need to show that X is α-r.e. We will effectively reconstruct the set X as an α-r.e. set using the α-finite code G as a parameter.

Again, let Υ_i denote $(\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i)$ for simplicity. Note that by Lemma 3.12, for any $e \in \mathrm{ran}(g \restriction f(i))$, A_e contains Υ_i if and only if Υ_i is enumerated into A_e by stage β_i. Moreover, at most one $e \in \mathrm{ran}(g \restriction f(i))$ satisfies $A_e = \Upsilon_i$. And by the definition of G, such an e exists if and only if $G(i) = 1$. These observations yield an alternative definition of u_i with parameter G:

$$\begin{aligned} (3.13)\quad u_i = \mu u \geq \beta_i\, [G(i) = 0 \to \forall e \in \mathrm{ran}(g \restriction f(i))\, (A_{e,\beta_i} \supseteq \Upsilon_i \to A_{e,u} \supsetneq \Upsilon_i) \\ \wedge\ G(i) = 1 \to \forall^- e \in \mathrm{ran}(g \restriction f(i))(A_{e,\beta_i} \supseteq \Upsilon_i \to A_{e,u} \supsetneq \Upsilon_i)]. \end{aligned}$$

Here, by "$\forall^- e \in C$" where C is any α-finite set we mean "$\exists e_0 \in C\, \forall e \in C \setminus \{e_0\}$". Definition (3.13) implies that u_i is α-recursively defined by β_i, $g \restriction f(i)$ and $\bigcup_{j<i} X[j]$.

At each stage $\eta < \alpha$, the approximation of $\{X[i]\}_{i<\sigma 2cf(\alpha)}$ inductively for $i < \sigma 2cf(\alpha)$ is given as follows.

> Stage $\gamma_{i,\eta}$ is defined to be a stage not exceeding η such that the approximation of f below $i + 1$ and approximation of g below $f_\eta(i) + 1$ have attained their values at stage η and do not change thereafter until stage η:
>
> $$\begin{aligned} (3.14)\quad \gamma_{i,\eta} = \max\{\mu\zeta \leq \eta\, (\forall \zeta' \in [\zeta, \eta]\, (f_{\zeta'} \restriction (i + 1) = f_\eta \restriction (i + 1))), \\ \mu\zeta \leq \eta\, (\forall \zeta' \in [\zeta, \eta]\, (g_{\zeta'} \restriction (f_\eta(i) + 1) = g_\eta \restriction (f_\eta(i) + 1)))\}. \end{aligned}$$
>
> Pseudostable ordinal β_i is approximated via the function h, i.e.
>
> $$\beta_{i,\eta} = h(\eta, \gamma_{i,\eta}, \sup(\bigcup_{j<i} X_\eta[j]), \bigcup_{j<i} X_\eta[j], g_\eta \restriction f_\eta(i)).$$

An upper bound $u_{i,\eta} < \eta$ is defined by substituting $\beta_{i,\eta}, g_\eta, f_\eta, X_\eta[j]$ for $\beta_i, g, f, X[j]$ respectively and restricting u to the set $[\sup(\bigcup_{j<i} X_\eta[j]), \eta)$ in (3.13): Let $\Upsilon_{i,\eta}$ denote $(\bigcup_{j<i} X_\eta[j]) \cup [\sup(\bigcup_{j<i} X_\eta[j]), \beta_{i,\eta})$. Then

$$\begin{aligned} u_{i,\eta} = \mu u \geq \beta_{i,\eta}\, [\, & (u \geq \sup(\textstyle\bigcup_{j<i} X_\eta[j]))\ \wedge\ (u < \eta) \wedge \\ & G(i) = 0\ \to \forall e \in \mathrm{ran}(g_\eta \restriction f_\eta(i))\, (A_{e,\beta_{i,\eta}} \supseteq \Upsilon_{i,\eta} \to A_{e,u} \supsetneq \Upsilon_{i,\eta}) \wedge \\ & G(i) = 1\ \to \forall^- e \in \mathrm{ran}(g_\eta \restriction f_\eta(i))(A_{e,\beta_{i,\eta}} \supseteq \Upsilon_{i,\eta}\ \to\ A_{e,u} \supsetneq \Upsilon_{i,\eta})]. \end{aligned}$$

For some η, $u_{i,\eta}$ may be undefined.

Now define

$$X_\eta[i] = \begin{cases} \Upsilon_{i,\eta} \cup \{u_{i,\eta}\} & \text{if for each } j \leq i, u_{j,\eta} \text{ is defined,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma 3.27. $\lim_{\eta \to \alpha}(\gamma_{i,\eta}, \beta_{i,\eta}, u_{i,\eta}, X_\eta[i]) = (\gamma_{i,\eta_0}, \beta_{i,\eta_1}, u_{i,\eta_2}, X_{\eta_2}[i]) = (\gamma_i, \beta_i, u_i, X[i])$, *for every $i < \sigma 2cf(\alpha)$, $\eta_0 \geq \gamma_i$, $\eta_1 \geq \beta_i$, and $\eta_2 > u_i$.*

Proof. All equations are proved simultaneously by induction on i. Suppose Lemma 3.27 is proved for each $j < i$. Let $\eta < \alpha$ be a stage. We have

(1) If $\eta \geq \gamma_i$, then the definition of γ_i implies

$$f_\eta \restriction (i+1) = f \restriction (i+1), \quad g_\eta \restriction (f(i)+1) = g \restriction (f(i)+1).$$

Thus, according to (3.14), $\gamma_{i,\eta} = \gamma_i$.

(2) Let $\eta \geq \beta_i$. Then by their definitions and Lemma 3.12, $\beta_i > \max\{\gamma_i, \sup_{j<i} u_j\}$. According to the inductive hyphothesis and (1) of this proof, $(\gamma_{i,\eta}, \bigcup_{j<i} X_\eta[j], g_\eta \restriction f_\eta(i)) = (\gamma_i, \bigcup_{j<i} X[j], g \restriction f(i))$. By Lemma 3.12 again, $\beta_{i,\eta} = \beta_i$.

(3) Now suppose $\eta > u_i$. Since $u_i \geq \beta_i$, by (2) of the present proof, $(\beta_{i,\eta}, \gamma_{i,\eta}, \bigcup_{j<i} X_\eta[j], g_\eta \restriction f_\eta(i)) = (\beta_i, \gamma_i, \bigcup_{j<i} X[j], g \restriction f(i))$. So $u_{i,\eta}$ is defined and equal to u_i. Combine this with the inductive hypothesis, then we have $\forall j \leq i\, (u_{j,\eta} \downarrow= u_j)$. Hence $X_\eta[i] = X[i]$. □

Lemma 3.28. $X_\eta[i] \subseteq X$ *for all $\eta < \alpha$, $i < \sigma 2cf(\alpha)$.*

Proof. Fix a stage η. Lemma 3.12 implies that for every $i < \sigma 2cf(\alpha)$, we have

$$\beta_i > \max\{g \restriction (f(i)+1), \sup_{j<i} u_j\} \text{ and } u_i \geq \beta_i.$$

Thus, $\{\beta_i\}_{i<\sigma 2cf(\alpha)}$ and $\{u_i\}_{i<\sigma 2cf(\alpha)}$ are strictly increasing and cofinal in α.

Let $i^* < \sigma 2cf(\alpha)$ be the least i that $u_i \geq \eta$. Then for every $j < i^*$, $u_j < \eta$, and Lemma 3.27 implies that $X_\eta[j] = X[j] \subseteq X$.

Suppose $i \geq i^*$. $X_\eta[i] \subseteq X$ is trivially true if $X_\eta[i] = \emptyset$. Now assume $X_\eta[i] \neq \emptyset$. The by its definition, $X_\eta[i] \restriction \sup(\bigcup_{j<i^*} X[j]) = \bigcup_{j<i^*} X[j]$ and $X_\eta[i] \subseteq \eta$.

Case 1. $\eta \leq \beta_{i^*}$. By the definition of $\{X_\eta[j]\}_{j \leq i}$, we have

$$X_\eta[i] \subseteq \textstyle\bigcup_{j<i^*} X[j] \cup [\sup(\bigcup_{j<i^*} X[j]), \eta) \subseteq X[i^*].$$

Therefore, $X_\eta[i] \subseteq X$.

Case 2. $\beta_{i*} < \eta \leq u_{i^*}$. According to its definition, $u_{i^*,\eta}$ is undefined. Therefore $X_\eta[i] = \emptyset$, a contradiction. □

Note that the sequence $\{(\gamma_{i,\eta}, \beta_{i,\eta}, u_{i,\eta}, X_\eta[i])\}_{i<\sigma 2cf(\alpha), \eta<\alpha}$ is α-recursive. Thus, Lemma 3.27 and Lemma 3.28 combine to produce the following corollary.

Corollary 3.29. *X is α-r.e. Hence $\{A_e\}_{e<\alpha}$ is not a Friedberg numbering.*

Since $\{A_e\}_{e<\alpha}$ is an arbitrary one-one numbering, Corollary 3.29 implies that there is no Friedberg numbering when $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, proving Theorem 3.23.

The code G plays a significant role in the proof of Theorem 3.23. It exploits the property that the numbering $\{A_e\}_{e<\alpha}$ is one-one and makes the use of pseudostable ordinals to achieve the diagonalization against $\{A_e\}_{e<\alpha}$. Pseudostability is applicable for any numbering. Yet, if the numbering is not one-one in some blocks and the number of such repetitions is cofinal in $t\sigma 2p\,(\alpha)$ as the number of blocks increases, then a code such as G may not exist. Therefore, the diagonalization construction may not be applicable for other types of numberings.

Remark. As in Section 2.3, (2.7) is still a valid example of non-K-acceptable numbering in L_α for all Σ_1 admissible α, by simply replacing the notations appropriately.

If $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$, then the Friedberg numbering constructed in Section 3.2 is a natural example of a K_e-numbering (ref. Section 2.3 and 3.2). If $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, then there is no K_e-numberings as the situation of $B\Sigma_2$ models, but for a different reason: For the sake of contradiction assume $\{C_e\}_{e<\alpha}$ is a K_e-numbering. Then $\{e' < e : C_{e'} = C_e\}$ is Δ_2 for every $e < \alpha$. Therefore the least indices of $\{C_e\}_{e<\alpha}$ have an α-recursive approximation. Then the straightforward adaptation of the proof in [14] provides a Friedberg numbering in L_α. Hence, we have

Corollary 3.30. *If $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, there is no K_e-numbering in L_α.*

Corollary 3.31. *The following are equivalent:*

(1) $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$;
(2) There is a Friedberg numbering in L_α;
(3) There is a K_e-numbering in L_α.

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