Exercise 1. Let \( \langle \kappa_i : i \in \alpha \rangle \) be a sequence of cardinals. We define the infinite sum of cardinals to be:
\[
\sum_{i \in \alpha} \kappa_i = |\bigcup_{i \in \alpha} X_i|,
\]
where \( \{X_i : i \in \alpha\} \) is a disjoint family of sets such that \( |X_i| = \kappa_i \) for each \( i \in \alpha \). Show that this definition makes sense (using AC) and show that for an infinite cardinal \( \lambda \) the following equation holds: \( \sum_{i \in \lambda} \kappa_i = \lambda \cdot \sup_{i \in \lambda} \kappa_i \).

Exercise 2. Let \( \langle \kappa_i : i \in \alpha \rangle \) be as above. We define the infinite product of cardinals as follows:
\[
\prod_{i \in \alpha} \kappa_i = |\prod_{i \in \alpha} X_i|,
\]
where the \( X_i \)'s are such that \( |X_i| = \kappa_i \) for each \( i \in \alpha \) and \( \prod_{i \in \alpha} X_i := \{f : f \text{ is a function, } \text{dom}(f) = \alpha, \text{ and } \forall i \in \alpha(f(i) \in X_i)\} \). Show that this is a well-defined notion and show that for an infinite cardinal \( \lambda \) and a non-decreasing sequence \( \langle \kappa_i : i \in \lambda \rangle \) of cardinals the following equation holds: \( \prod_{i \in \lambda} \kappa_i = (\sup_{i \in \lambda} \kappa_i)^\lambda \).

Exercise 3. Prove that if \( \langle \kappa_i : i \in \alpha \rangle \) and \( \langle \lambda_i : i \in \alpha \rangle \) are two sequences of cardinals such that for each \( i \in \alpha \), \( \kappa_i < \lambda_i \), then \( \sum_{i \in \alpha} \kappa_i < \prod_{i \in \alpha} \lambda_i \). Use this to prove Koenig’s Theorem (Kunen Theorem 1.13.12.).

Exercise 4. Prove that \( |\beth_\omega| = \prod_{n \in \omega} \beth_n = \beth_{\omega + 1} \).

Exercise 5. Let \( \kappa \) be an infinite cardinal and \( \alpha < \kappa^+ \). Prove that there exist \( X_n \subseteq \alpha, n \in \omega, \) such that \( \text{o.t.}(X_n) < \kappa^n \) (here we consider ordinal exponentiation) and \( \alpha = \bigcup_{n \in \omega} X_n \).

The last fact is known as “Milner-Rado Paradox”.

Exercise 6. Let \( \kappa \) be an infinite cardinal and \( \prec \) be a well-order on \( \kappa \). Prove that there exists \( X \in [\kappa]^\kappa \) such that \( \prec \cap X^2 = \in \cap X^2 \), i.e., \( \prec \) and \( \in \) coincide on \( X \).
Exercise 1 (Kunen I.13.34). Let $W$ be a vector space over some field $F$, and let $W^* = \text{Hom}(W, F)$ be the dual vector space. Consider $W$ as a subspace of $W^{**}$ as usually ($x \in W$ is identified with the map $\phi \mapsto \phi(x)$ in $W^{**}$). Let $W_0 = W$ and $W_{n+1} = W_n^{**}$, so that $W_n \subset W_{n+1}$. Let $W_\omega = \bigcup_{n<\omega} W_n$.

Prove that if $|F| < 2^\omega$ and $\omega \leq \dim(W) < 2^\omega$, then $|W_\omega| = \dim(W_\omega) = 2^\omega$.

Exercise 2 (Kunen I.13.36). Assume CH. Prove that $\omega_n = \omega_n$ for all $n < \omega$.

Exercise 3 (Kunen I.13.39). Suppose that $\kappa$ is an infinite cardinal, $\alpha = \bigcup_{n<\epsilon} X_n$ for some $c < \omega$, and the order type of each $X_n$ is less than $\kappa^\omega$ (ordinal exponentiation!). Show that $\alpha < \kappa^\omega$.

Exercise 4 (Kunen 1.15.10). Let $\mathfrak{B}$ be any structure for $L$ such that

$$\max\{|L|, \omega\} \leq \kappa \leq |B|$$

for some infinite cardinal $\kappa$. Suppose that $S \subset B$ has size $|S| \leq \kappa$. Show that there exists an elementary submodel $\mathfrak{A}$ of $\mathfrak{B}$ such that $S \subset A$ and $|A| = \kappa$.

**Hint:** Use I.13.22 and I.13.21, or just look it up in some model theory book.

Exercise 5. Prove that $(\mathbb{Q}, <)$ is an elementary substructure of $(\mathbb{R}, <)$.

**Hint:** Use the previous exercise to find countable $X \supset \mathbb{Q}$ such that $(X, <)$ is an elementary substructure of $(\mathbb{R}, <)$. Then construct a monotone bijection $\phi : \mathbb{Q} \to X$ (this is the famous Cantor’s back and forth argument which you may find in many books or just reinvent!), and argue that it may be extended to a monotone bijection $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ by the completeness of $\mathbb{R}$.

There are of course other approaches.
Übungen für 02.04.2014

Exercise 1 (Kunen I.16.6). (ZF\(^{-}\)). Let \(\text{pow}(x, y)\) be \(\forall z (z \subseteq x \rightarrow z \in y)\). Let \(\gamma\) be a limit ordinal and \(a, b \in R(\gamma)\). Prove that \(R(\gamma) \models \text{pow}(a, b)\) iff \(b = \mathcal{P}(a)\), i.e., \(R(\gamma) \prec_{\text{pow}} V\).

Exercise 2 (Kunen I.16.8). (ZFC\(^{-}\)). Assume that \(0 < \gamma < \delta\) are ordinals and \(R(\gamma) < R(\delta)\). Prove that \(R(\gamma) \models ZFC\), and hence also \(R(\delta) \models ZFC\). You may use the fact that \(R(\gamma) \models ZC\) for any limit \(\gamma\).

Exercise 3 (Kunen I.16.9). Assume that \(ZFC \not\vdash \exists \gamma [R(\gamma) \models ZFC]\). Show that \(ZFC\) is inconsistent.

Exercise 4 (Kunen I.16.10). Show how to modify Definition I.15.5 to give a correct definition of \((V, \in) \models \varphi(\delta)\) in the case of \(\Delta_0\) formulas.

Exercise 5 (Kunen I.16.17). Describe a two-element non-transitive \(M\) that is isomorphic to \(\{0, 1\}\), such that \(\cap^M\) is defined but \(\cap\) is not absolute for \(M\), and such that \(\subseteq\) is not absolute for \(M\).
Work in ZF unless otherwise indicated.

**Exercise 1.** Which axioms of ZF are true in ON?

**Exercise 2.** (AC). For \( \kappa > \omega \), show that \( |H(\kappa)| = 2^{<\kappa} \).

**Exercise 3.** (AC). For \( \kappa > \omega \), show that \( H(\kappa) = R(\kappa) \) iff \( \kappa = \beth_\omega \).

**Exercise 4.** Show that in \( R(\omega + \omega) \), it is not true that every well-ordering is isomorphic to an ordinal.

*Hint.* Consider \( 2 \times \omega \), ordered lexicographically. Track down the specific instance of Replacement which fails in \( R(\omega + \omega) \).

**Exercise 5.** (AC.) Recall that Zermelo set theory, Z, is ZF without Replacement. Show that for all \( \kappa > \omega \), \( H(\kappa) \) is a model for \( Z - P \). Show that the Power Set Axiom is true in \( H(\kappa) \) iff \( \kappa = \beth_\gamma \), for some limit \( \gamma \). Show that Replacement fails in \( H(\beth_\omega) \).
Exercise 1 (II.4.8). Prove that the notions “$R$ well-orders $A$” and “$R$ is well-founded on $A$” are absolute for $R(\gamma)$ for any limit $\gamma$.

Why can’t we use here II.4.7 directly?

Exercise 2 (II.4.6). Let $\gamma$ be a limit ordinal such that $\forall \alpha < \gamma [\alpha^2 < \gamma]$. Show that ordinal sum and product are defined in $R(\gamma)$ and are absolute for $R(\gamma)$.

Exercise 3 (II.4.9). (ZFC). Prove that $R(\gamma) \models AC^+$ and $H(\kappa) \models AC^+$ for any limit $\gamma$ and regular $\kappa$.

Exercise 4 (II.4.21). Let AI be “our standard” Axiom of Infinity, and let $AU$ denote the Axiom des Undendlichen of Zermello: $\exists x (\emptyset \in x \land \forall y \in x (\{y\} \in x))$. Work in ZFC and produce transitive models for $ZC+\neg AU$ and for $ZC-\text{Inf}+AU+\neg AI$.

Exercise 5 (II.4.22). Find a transitive $M \models ZC-P$ in which $\omega \times \omega$ and $\omega^* := \{\{n\} : n \in \omega\}$ do not exist.

There are hints to almost all of these exercises in the book. Feel free to use them!
Übungen für 7.05.2014

Exercise 1 (II.4.26). Let $M$ be a transitive class, and assume that the axioms of Extensionality, Comprehension, Pairing, Union, and Infinity hold in $M$. Prove that $\omega \in M$.

Exercise 2 (II.4.29). Let $M$ be a transitive model for ZF$\neg$P. Let $\ast, \ast \in M$ be two group operations on $\omega$. Prove that the statement $(\omega, \ast) \cong (\omega, \ast)$ is absolute for $M$.

Exercise 3 (II.5.6). Assume AC. Find a formula $\phi$ such that every transitive $M$ satisfying $M \prec_\phi V$ is of the form $R(\gamma)$ for some ordinal $\gamma = \beth_\gamma$.

Exercise 4 (II.5.12). Work in ZFC plus the assumption that $R(\gamma) \models \text{ZFC}$ for some $\gamma$. Prove that the minimal such $\gamma$ has cofinality $\omega$.

Exercise 5 (II.5.13). Show that there is a finite set $\Lambda$ of instances of the Comprehension axiom such that $\Lambda$ together with the axioms of ZF other than Comprehension, proves all instances of Comprehension.

There are hints in the book simplifying these exercises greatly!
Übungen für 14.05.2014. Mengenlehre 1

Exercise 1 (II.6.30). Convince yourself\(^1\) that the class \(L[A]\) defined in II.6.29 is a transitive model of ZFC if \(A\) consists of ordinals. Find the place in the argument where the fact that \(A \subseteq ON\) is used! Prove that \(L[A] \vDash GCH\) for \(A \subseteq \omega\).

Exercise 2 (II.6.31). Suppose that \(V = L[A]\) for some \(A \subseteq \omega_1\). Prove that \(GCH\) holds in \(L[A]\).

Later we shall show that \(V = L[A]\) is essential in the above exercise.

Exercise 3 (II.6.33). Assume \(V = L\) and prove that \(L(\alpha) = R(\alpha)\) iff \(\alpha = \aleph_\alpha\).

Exercise 4 (III.2.7). Let \(\kappa\) be singular. Show that there is a family \(A\) of \(\kappa\) two-element subsets of \(\kappa\) such that no \(B \in [A]^\kappa\) forms a delta system.

Exercise 5 (Folklore). Let \(A\) be an uncountable collection of finite subsets of \(\omega_1\) and \(M\) an elementary submodel of \(H(\omega_2)\) containing \(A\) as an element. Let \(A \in A \setminus M\) and \(D = A \cap M\). Prove that there exists an uncountable delta system \(B \subseteq A, B \in M\), with kernel \(D\).

Hint: pick in \(M\) a maximal delta subsystem of \(A\) with the kernel \(D\) and show that it is uncountable. Use the fact that if \(|X| = \omega\) and \(X \in M\) then \(X \subseteq M\).

The same ideas as in the above exercise allow to prove also more general instances of the delta system lemma.

\(^1\)I will not ask you to present this near blackboard because this is analogous to the case of \(L\) and lengthy.
Recall from III.3.23 that a subset $C$ of a poset $P$ is centered, if for any $n \in \omega$ and all $p_1, \ldots, p_n \in C$ there exists $q \in P$ such that $q \leq p_i$ for all $i \leq n$. If, moreover, $q$ may be found in $C$, then $C$ is called a filter. A poset $P$ is called $\sigma$-centered if it can be written as a countable union of its centered subsets.

**Exercise 1** (III.3.27(part 1)). If $X$ is a compact Hausdorff space, then $X$ is separable if $O_X$ is $\sigma$-centered if $O_X$ is a countable union of filters. Here $O_X$ is ordered by inclusion, i.e., $U \leq V$ means $U \subset V$.

The standard base for the topology on $2^A$ consists of sets $[s]$, $s \in Fn(A, 2)$, where $[s] = \{x \in 2^A : x \upharpoonright \text{dom}(s) = s\}$. Thus $U \subset 2^A$ is open iff it is a union of a collection of sets of the form $[s]$.

**Exercise 2** (III.3.27(part 2)). Let $\kappa$ be a cardinal and $X = 2^\kappa$. Show that $O_X$ is ccc. Show that $O_X$ is $\sigma$-centered iff $\kappa \leq 2^\omega$.

Hint: If $\kappa \leq 2^\omega$, then take any metrizable separable topology on $\kappa$ (e.g., via some bijection with a subset of $\mathbb{R}$), fix a countable base $B$ for this topology, and look at characteristic functions of finite unions of elements of $B$. For the case $\kappa > 2^\omega$ show that a separable space cannot have more than $2^\omega$ mutually different clopen subsets.

**Exercise 3** (IV.2.8). Let $\tau = \{\langle \emptyset, p\rangle, \{\langle \emptyset, q\rangle\}, r\}$. Compute $\tau_G$ for each of the 8 possibilities for $p, q, r$ being $\in$ or $\notin G$.

**Exercise 4** (IV.2.16). Using the notation of Lemma IV.2.15, replace the definition of $\pi$ by: $\pi = \{\langle v, p\rangle : \exists \langle \sigma, q\rangle \in \tau \exists r[\langle v, r\rangle \in \sigma \land p \leq r \land p \leq q]\}$. Let $b = \pi_G$ and show that $\cup a = b$.

**Exercise 5** (IV.2.28). Let $M$ be a ctm for ZFC. Find a poset $P$ and a sentence $\psi \in \mathcal{FL}_P \cap M$ and two different generic filters $G, H$ with $M[G] = M[H]$ and $M[G] \models \psi$ and $M[H] \not\models \psi$ because some $\tau_G$ differs from $\tau_H$. 


Übungen für 28.05.2014. Mengenlehre 1

**Exercise 1.** Let $M$ be a ctm for ZFC and $\mathbb{P} \in M$ be a poset. Let also $G \subseteq \mathbb{P}$ be a filter. Show that the following conditions are equivalent:

1. $G \cap D \neq \emptyset$, whenever $D \in M$ and $D$ is dense in $\mathbb{P}$;
2. $G \cap A \neq \emptyset$, whenever $A \in M$ and $A$ is a maximal antichain in $\mathbb{P}$;
3. $G \cap E \neq \emptyset$, whenever $E \in M$ and for every $p \in \mathbb{P}$ there exists $q \in E$ such that $p$ and $q$ are compatible.

Furthermore, show that in all these items, if we assume that $G$ is just a centered subset of $\mathbb{P}$, then it is automatically a filter.

**Exercise 2** (IV.2.46). Assume that $M$ is a ctm for ZFC, and let $\mathbb{P} = Fn(\omega, 2)$. Then there is a filter $G$ on $\mathbb{P}$ such that there is no transitive $N \supset M$ such that $G \in N$, $N \modelsZF - P$, and $\omega(N) = \omega(M)$.

**Exercise 3.** Let $M$ be a ctm for ZFC and $\mathbb{P} \in M$ be a poset. Suppose that $\tau \in M^\mathbb{P}$ and $\text{dom}(\tau) \subset \{\bar{n} : n \in \omega\}$. Let

$$\sigma = \{\langle \bar{n}, p \rangle : \forall q \in \mathbb{P}(\langle \bar{n}, q \rangle \in \tau \rightarrow p \perp q\}.$$ 

Show that $\sigma_G = \omega \setminus \tau_G$, where $G$ is a $\mathbb{P}$-generic over $M$.

**Exercise 4.** Let $M$ be a ctm for ZFC and $\mathbb{P} = (2^{<\omega_1})_M$, where $p \leq q$ means that $p$ is an extension of $q$. Let $G$ be a $\mathbb{P}$-generic over $M$. Show that in $M[G]$ there exists a bijection between $(\omega_1)_M$ and $(2^{<\omega})_M$.

**Hint:** Look at the restrictions of $\bigcup G$ to intervals $[\alpha, \alpha + \omega)$ for $\alpha < (\omega_1)_M$.

**Exercise 5** (IV.2.47). Assume that $M$ is a ctm for ZFC. Give an example of $\mathbb{P} \in M$ and a (non-generic) filter $G$ on $\mathbb{P}$ for which $\mathbb{P} \setminus G \notin M[G]$. 
UBungen fr 4.06.2014. Mengenlehre 1

In the following, unless we state otherwise: \( M \) represents a c.t.m. for ZFC, \( \mathbb{P} \in M \) is a p.o., and \( G \) is a filter which is \( \mathbb{P} \)-generic over \( M \).

**Exercise 1.** Assume that \( \mathbb{P} \) doesn’t have the largest element. For an element \( x \in M \) redefine the name \( \dot{x} \) so that \( \dot{x}_G = x \).

**Exercise 2.** Suppose \( \langle \mathbb{P}, \leq \rangle \) is a partial order in \( M \) which may or may not have a largest element. In \( M \), fix \( 1 \notin \mathbb{P} \), and define the p.o. \( \langle \mathbb{Q}, \leq, 1 \rangle \) by: \( \mathbb{Q} = \mathbb{P} \cup \{1\} \) where \( \mathbb{P} \) retains the same order and \( \forall p \in \mathbb{P}(p < 1) \). Show that if \( G \subset \mathbb{P} \) is a filter, \( G \) is \( \mathbb{P} \)-generic over \( M \) iff \( G \cup \{1\} \) is \( \mathbb{Q} \)-generic over \( M \), and \( M[G] \) (defined as a \( \mathbb{P} \)-extension) is the same as \( M[G \cup \{1\}] \) (defined as a \( \mathbb{Q} \)-extension).

**Exercise 3.** Assume \( f : A \to M \) and \( f \in M[G] \). Show that there is a set \( B \in M \) such that \( f : A \to B \).

Hint. Let \( B = \{ b : \exists p \in \mathbb{P}(p \models b \in \text{ran}(\tau)) \} \), where \( f = \tau_G \).

**Exercise 4.** Assume \( \alpha \) is a cardinal of \( M \). Show that the following are equivalent.

1. Whenever \( B \in M \), \( ^\alpha B \cap M = ^\alpha B \cap M[G] \);
2. \( ^\alpha M \cap M = ^\alpha M \cap M[G] \);
3. In \( M \): The intersection of \( \alpha \) many dense open subsets of \( \mathbb{P} \) is dense.

Recall that a subset \( O \) of \( \mathbb{P} \) is open if for every \( p \in O \) and \( q \leq p \) we have \( q \in O \) (i.e., \( O \) is downwards closed).

A p.o. satisfying (3) is called \( \alpha^+ \)-Baire. \( \kappa \)-Baire means that the intersection of less than \( \kappa \) dense open sets is dense.

**Exercise 5.** Let \( \mathbb{P} \in M \) be non-atomic. Let \( M = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots \) (\( n \in \omega \)) be such that \( M_{n+1} = M_n[G_n] \) for some \( G_n \) which is \( \mathbb{P} \)-generic over \( M_n \). Show that \( \bigcup_{n \in \omega} M_n \) cannot satisfy the Power Set Axiom. Furthermore, show that the \( G_n \) may be chosen so that there is no c.t.m. \( N \) for ZFC with \( \langle G_n : n \in \omega \rangle \in N \) and \( o(N) = o(M) \).

Hint, \( \{ n : p \in G_n \} \) can code \( o(M) \).
In the following, unless we state otherwise: $M$ represents a c.t.m. for ZFC, $P \in M$ is a p.o., and $G$ is a filter which is $P$-generic over $M$.

A poset $P$ is called $\lambda$-closed, where $\lambda$ is a cardinal, if every decreasing sequence $\langle p_\xi : \xi < \alpha \rangle$ of elements of $P$ of length $\alpha < \lambda$ has a lower bound.

**Exercise 1.** Prove that every $\lambda$-closed poset is $\lambda$-Baire (see the definition on the previous exercise sheet). Show that if $P$ is $\lambda$-closed and $\lambda$ is singular then $P$ is $\lambda^+$-closed.

**Exercise 2.** Suppose that $P$ is countable and non-atomic. Show that there is a dense embedding from $\{ p \in Fn(\omega,\omega) : \text{dom}(p) \in \omega \}$ into $P$.

*Hint.* Map $\{ p : \text{dom}(p) = 1 \}$ onto an infinite antichain in $P$, now handle $\{ p : \text{dom}(p) = 2 \}$, etc.

It follows from the exercise above that all countable non-atomic posets yield the same generic extensions.

Observe that for every forcing poset $P$, each map $i : P \to P$ gives rise to a natural map $i^* : M^P \to M^P$ defined as follows: $i^*(\tau) = \{ \langle i^*(\sigma), i(p) \rangle : (\sigma, p) \in \tau \}$.

**Exercise 3.** If $P$ (i.e., $\langle P, \leq, 1_P \rangle$) is a p.o., an automorphism of $P$ is a 1-1 map $i$ from $P$ onto $P$ which preserves $\leq$ and satisfies $i(1_P) = 1_P$; thus also $i^*(\bar{x}) = \bar{x}$ for each $x$. $P$ is called almost homogeneous iff for all $p,q \in P$, there is an automorphism $i$ of $P$ such that $i(p)$ and $q$ are compatible. Suppose that $P \in M$ and $P$ is almost homogeneous in $M$. Show that if $p \Vdash \phi(\bar{x}_1,\ldots,\bar{x}_n)$, then $1_P \Vdash \phi(\bar{x}_1,\ldots,\bar{x}_n)$; thus, either $1_P \Vdash \phi(\bar{x}_1,\ldots,\bar{x}_n)$ or $1_P \Vdash \neg \phi(\bar{x}_1,\ldots,\bar{x}_n)$.

**Exercise 4.** Show that any $Fn(I,J,k)$ is almost homogeneous.

For $P = Fn(\omega,2)$ give an example showing that the conclusion of the previous exercise is not any more true for arbitrary names.

**Exercise 5.** Let $\kappa$ be a cardinal of uncountable cofinality and $f : \kappa \to \kappa$. Show that there exists a closed and unbounded $C \subset \kappa$ such that for all $\alpha \in C$ and $\beta \in \alpha$ we have that $f(\beta) \in \alpha$ (i.e., range($f \upharpoonright \alpha$) $\subset \alpha$).
Übungen für 18.06.2014.

In the following, unless we state otherwise: $M$ represents a c.t.m. for ZFC, $\mathbb{P} \in M$ is a p.o., and $G$ is a filter which is $\mathbb{P}$-generic over $M$.

**Exercise 1.** $\kappa$ is called strongly Mahlo iff $\kappa$ is strongly inaccessible and \( \{ \alpha < \kappa : \alpha \text{ is regular} \} \) is stationary in $\kappa$. Show that for such $\kappa$, \( \{ \alpha < \kappa : \alpha \text{ is strongly inaccessible} \} \) is stationary in $\kappa$.

**Exercise 2.** Let \( (\mathbb{P} = Fn(I, 2, \omega_1))^M \), where \( (|I| \geq \omega_1)^M \). Show that $M[G]$ satisfies CH, regardless of whether $M$ does.

**Exercise 3.** Suppose, in $M$, $\omega = \text{cf}(\lambda) < \lambda$. Show that $Fn(\lambda, 2, \lambda)^M$ adds a map from $\omega$ onto $\lambda^+$.

**Exercise 4.** Assume in $M$ that $\kappa > \omega$, $\kappa$ is regular, and $\mathbb{P}$ has the $\kappa$-c.c. In $M[G]$, let $C \subset \kappa$ be c.u.b. Show that there exists $C' \subset C$ such that $C' \in M$ and $C'$ is c.u.b. in $\kappa$.

**Exercise 5.** Suppose that in $M$: $S \subset \omega_1$ is stationary and $\mathbb{P}$ is either c.c.c. or $\omega_1$-closed. Show that $S$ remains stationary in $M[G]$. 