Cardinal characteristics, projective wellorders and large continuum

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Abstract

We extend the work of [7] by presenting a method for controlling cardinal characteristics in the presence of a projective wellorder and $2^{\aleph_0} > \aleph_2$. This also answers a question of Harrington [11] by showing that the existence of a Δ^1_3 wellorder of the reals is consistent with Martin's axiom and $2^{\aleph_0} = \aleph_3$.

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1. Introduction

In [7] the present authors established the consistency of the existence of a Π^1_2 maximal almost disjoint family together with a lightface projective wellorder and $\mathfrak{b} = 2^{\aleph_0} = \aleph_3$. As the argument used there was only suitable for handling countable objects, it left open the problem of obtaining projective wellorders with $2^{\aleph_0}$ greater than $\aleph_2$ while simultaneously controlling cardinal characteristics of prominent interest. We solve this problem in the present paper, using an iteration based on the specialization and branching of Suslin trees. As an application we obtain the consistency of $\mathfrak{p} = \mathfrak{b} = \aleph_2 < \mathfrak{a} = \mathfrak{s} = 2^{\aleph_0} = \aleph_3$ with a lightface Δ^1_3 wellorder.

A consequence of our work is the consistency of Martin's Axiom with a lightface Δ^1_3 wellorder and $2^{\aleph_0} = \aleph_3$. This improves a result of [9], where $2^{\aleph_0} = \aleph_2$ was obtained, and also answers a question of Harrington from [11], where he obtained the same result with a boldface Δ^1_3 wellorder.

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2. Martin's Axiom, Projective Wellorders and Large Continuum

We work over the constructible universe L. Fix a canonical sequence $\langle S_\alpha : 1 < \alpha < \omega_3\rangle$ of stationary subsets of $\omega_2 \cap \mathrm{cof}(\omega_1)$ and a nicely definable almost disjoint family $\vec{B} = \langle B_\xi : \xi \in \omega_2\rangle$ of subsets of ω_1 (see [7]). For each $\alpha < \omega_3$, let W_α be the L-least subset of ω_2 which codes α. Say that a transitive ZF^- model $\mathcal{M}$ is *suitable* if $\omega_2^{\mathcal{M}}$ exists and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$. From this it follows, of course, that $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$.

We will define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_3, \beta < \omega_3\rangle$ such that in $L^{\mathbb{P}_{\omega_3}}$, MA holds, $2^\omega = \omega_3$, and there is a Δ^1_3-definable wellorder of the reals. The construction can be thought of as a preliminary stage followed by a coding stage. In the preliminary stage we provide the necessary apparatus, in order to force a Δ^1_3 definition of our wellorder of the reals.

Preliminary Stage: For each $0 < \alpha < \omega_3$ and $n \in \omega$, let $\mathbb{K}^0_{\omega\cdot\alpha+n}$ be the poset for adding a Suslin tree $T_{\omega\cdot\alpha+n}$ with countable conditions, see [12, Theorem 15.23]. Let $\mathbb{K}_{0,\alpha} = \prod_{n\in\omega} \mathbb{K}^0_{\omega\cdot\alpha+n}$ with full support. Then $\mathbb{K}_{0,\alpha}$ is countably closed and has size 2^ω. In particular, it does not collapse cardinals provided that CH holds in the ground model.

In what follows we shall identify the T_α's with subsets of ω_1 using the L-least bijection between $\omega^{<\omega_1}$ and ω_1. And vice versa, the phrase "$A \subset \omega_1$ is an ω_1-tree" means throughout the paper that the preimage of A under the L-least bijection between $\omega^{<\omega_1}$ of L and ω_1 is an ω_1-tree. (We can consider such a preimage only in models of $\omega_1 = \omega_1^L$, which is the case in suitable models.)

In $L^{\mathbb{K}_{0,\alpha}}$, code $T_{\omega\cdot\alpha+n}$ via a stationary kill of $S_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma}$ for $\gamma \in T_{\omega\cdot\alpha+n}$. More precisely, for every $1 \leq \alpha < \omega_3$ let $\mathbb{K}_{1,\alpha,n} = \prod_{\gamma\in\omega_1} \mathbb{K}^1_{\alpha,n,\gamma}$ with full support where for $\gamma \in T_{\omega\cdot\alpha+n}$, $\mathbb{K}^1_{\alpha,n,\gamma}$ adds a closed unbounded subset $C_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma}$ of ω_2 disjoint from $S_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma}$ and for $\gamma \notin T_{\omega\cdot\alpha+n}$, $\mathbb{K}^1_{\alpha,n,\gamma}$ is the trivial poset. Then $\mathbb{K}_{1,\alpha} = \prod_{n\in\omega} \mathbb{K}_{1,\alpha,n}$ with full support is countably closed, ω_2-distributive, and ω_3-c.c. provided that GCH holds in the ground model [2].

Next, we shall introduce some auxiliary notation. For a set X of ordinals we denote by $0(X)$, $I(X)$, and $II(X)$ the sets $\{\eta : 3\eta \in X\}$, $\{\eta : 3\eta + 1 \in X\}$ and $\{\eta : 3\eta + 2 \in X\}$, respectively. Let $Even(X)$ be the set of even ordinals in X and $Odd(X)$ be the set of odd ordinals in X.

In the following we treat 0 as a limit ordinal. Let $D_{\omega\cdot\alpha+n}$ be a subset of ω_2 coding $W_{\omega\cdot\alpha+n}$, $W_{\omega\cdot\alpha}$, and the sequence $\langle C_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma} : \gamma \in T_{\omega\cdot\alpha+n}\rangle$. More precisely, $0(D_{\omega\cdot\alpha+n}) =$

[2] A more general fact will be proven later after we define the final poset

$W_{\omega\cdot\alpha+n}$, $I(D_{\omega\cdot\alpha+n}) = W_{\omega\cdot\alpha}$, and $II(D_{\omega\cdot\alpha+n})$ equals

$$\chi(\{\langle\gamma,\eta\rangle : \gamma \in T_{\omega\cdot\alpha+n}, \eta \in C_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma}\}),$$

where $\chi : \omega_1 \times \omega_2 \to \omega_2$ is some some nicely definable bijection. Let $E_{\omega\cdot\alpha+n}$ be the club in ω_2 of intersections with ω_2 of elementary submodels of $L_{(\omega\cdot\alpha+n)+\omega_2}[D_{\omega\cdot\alpha+n}]$ which contain $\omega_1 \cup \{D_{\omega\cdot\alpha+n}\}$ as a subset. (These elementary submodels form an ω_2-chain.) Now choose $Z_{\omega\cdot\alpha+n}$ to be a subset of ω_2 such that $Even(Z_{\omega\cdot\alpha+n}) = D_{\omega\cdot\alpha+n}$, and if $\beta < \omega_2$ is $\omega_2^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $Z_{\omega\cdot\alpha+n} \cap \beta \in \mathcal{M}$, then β belongs to $E_{\omega\cdot\alpha+n} \cap E_{\omega\cdot\alpha}$. (This is easily done by placing in $Z_{\omega\cdot\alpha+n}$ a code for a bijection $\phi : \beta_1 \to \omega_1$ on the interval $(\beta_0, \beta_0 + \omega_1)$ for each adjacent pair $\beta_0 < \beta_1$ from $E_{\omega\cdot\alpha+n} \cap E_{\omega\cdot\alpha}$.) Using the same argument as in [7] we have:

$(*)_{\alpha,n}$: If $\beta < \omega_2$ and $\mathcal{M}$ is any suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $Z_{\omega\cdot\alpha+n} \cap \beta, Z_{\omega\cdot\alpha} \cap \beta, T_{\omega\cdot\alpha+n} \in \mathcal{M}$, then $\mathcal{M} \vDash \psi(\omega_1, \omega_2, Z_{\omega\cdot\alpha+n} \cap \beta, T_{\omega\cdot\alpha+n}, Z_{\omega\cdot\alpha} \cap \beta)$, where $\psi(\omega_1, \omega_2, Z, T, Z')$ is the formula

"$0(Even(Z))$ and $I(Even(Z)) = I(Even(Z'))$ are the L-least codes for ordinals $\omega \cdot \tilde{\alpha} + n$ and $\omega \cdot \tilde{\alpha}$ for some $n \in \omega$, respectively, and $\chi^{-1}[II(Even(Z))] = \{\langle\gamma,\eta\rangle : \gamma \in T, \eta \in \bar{C}_\gamma\}$, where T is an ω_1-tree and $\bar{C}_\gamma$ is a closed unbounded subset of ω_2 disjoint from $S_{\omega_1\cdot(\omega\cdot\tilde{\alpha}+n)+\gamma}$ for all $\gamma \in T$".

In $L^{\mathbb{K}_{0,\alpha} * \mathbb{K}_{1,\alpha}}$ let $\mathbb{K}^2_{\alpha,n}$ add a subset $X_{\alpha,n}$ of ω_1 which almost disjointly codes $Z_{\omega\cdot\alpha+n}$. More precisely, let $\mathbb{K}^2_{\alpha,n}$ be the poset of all pairs $\langle s, s^*\rangle \in [\omega_1]^{<\omega_1} \times [Z_{\omega\cdot\alpha+n}]^{<\omega_1}$, where a pair $\langle t, t^*\rangle$ extends $\langle s, s^*\rangle$ if and only if t end-extends s and $t \setminus s \cap B_\xi = \emptyset$ for every $\xi \in s^*$. Let $\mathbb{K}_{2,\alpha} = \prod_{n\in\omega} \mathbb{K}^2_{\alpha,n}$ with full support. Then $\mathbb{K}_{2,\alpha}$ is countably closed and ω_2-c.c. provided that CH holds in the ground model.

As a result of this manipulation we get the following:

$(**)_{\alpha,n}$: If $\beta < \omega_2$ and $\mathcal{M}$ is any suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $X_{\omega\cdot\alpha+n}, X_{\omega\cdot\alpha}, T_{\omega\cdot\alpha+n} \in \mathcal{M}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_{\omega\cdot\alpha+n}, T_{\omega\cdot\alpha+n}, X_{\omega\cdot\alpha})$, where $\phi(\omega_1, \omega_2, X, T, X')$ is the following formula:

"Using the sequence $\vec{B}$, the sets X, X' almost disjointly code subsets Z, Z' of ω_2 such that $\psi(\omega_1, \omega_2, Z, T.Z')$ holds".

Fix ϕ as above and consider the following poset:

Definition 2.1. Let $X, X', T \subseteq \omega_1$, be such that $\phi(\omega_1, \omega_2, X, T, X')$ holds in any suitable model $\mathcal{M}$ containing X, X', T as elements and such that $\omega_1^L = \omega_1^{\mathcal{M}}$. Denote by $\mathcal{L}(X, T, X')$ the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that:

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$,
2. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$,
3. if $\gamma \leq |r|$, $\mathcal{M}$ is a suitable model containing $r \restriction \gamma$ as an element, then $\mathcal{M} \models \phi(\omega_1, \omega_2, X \cap \gamma, T \cap \gamma, X' \cap \gamma)$.

The extension relation is end-extension.

Set $\mathbb{K}^3_{\alpha,m} = \mathcal{L}(X_{\omega\cdot\alpha+m}, T_{\omega\cdot\alpha+m}, X_{\omega\cdot\alpha})$ for every $\alpha \in \omega_3\setminus\{0\}$, $m \in \omega$, and set $\mathbb{K}^3_{0,m}$ to be the trivial poset for every $m \in \omega$. Let $\mathbb{K}_{3,\alpha} = \prod_{m\in\omega} \mathbb{K}^3_{\alpha,m}$ with full support. If $\alpha \in \omega_3\setminus\{0\}$, $m \in \omega$, then $\mathbb{K}^3_{\alpha,m}$ adds a function $Y_{\omega\cdot\alpha+m} : \omega_1 \to 2$ such that for every suitable model $\mathcal{M}$ such that $Y_{\omega\cdot\alpha+m} \restriction \eta$ and $T_{\omega\cdot\alpha+m} \cap \eta$ are in $\mathcal{M}$, we have $\mathcal{M} \models \phi(\omega_1, \omega_2, X_{\omega\cdot\alpha+m} \cap \eta, T_{\omega\cdot\alpha+m} \cap \eta, X_{\omega\cdot\alpha} \cap \eta)$.

Let $\mathbb{K}_\alpha = \mathbb{K}_{0,\alpha} * \mathbb{K}_{1,\alpha} * \mathbb{K}_{2,\alpha} * \mathbb{K}_{3,\alpha}$. We shall consider only $p = \langle p_i \rangle_{i\leq 3} \in \mathbb{K}_\alpha$ with the property that $\mathbb{K}_\alpha \restriction i$ forces (i.e., the maximal condition in $\mathbb{K}_\alpha \restriction i$ forces) $p_i \in \dot{\mathbb{K}}_{i,\alpha}$, where $\mathbb{K}_\alpha \restriction i$ is of course the iteration of $\mathbb{K}_{j,\alpha}$'s for $j < i$. This entails no loss of generality since for every $p \in \mathbb{K}_\alpha$ we can find an equivalent condition p' with the property above. In its turn, each p_i is a sequence $\langle p_{i,m} : m \in \omega \rangle$, where $p_{i,m}$ is forced by $\mathbb{K}_\alpha \restriction i$ to be an element of $\dot{\mathbb{K}}^i_{\alpha,m}$. And finally, $p_{1,\alpha,m}$ can be written as a sequence $\langle p_{1,\alpha,m,\zeta} : \zeta \in \omega_1 \rangle$, where $p_{1,\alpha,m,\zeta}$ is forced by $\mathbb{K}_{0,\alpha}$ to be an element of $\dot{\mathbb{K}}^1_{\alpha,m,\zeta}$.

For every $i \leq 3$ the poset $\mathbb{K}_\alpha \restriction i$ is countably closed, and hence the set $\mathbb{D}_\alpha$ of such $p \in \mathbb{K}_\alpha$ that p_i is (the canonical $\mathbb{K}_\alpha \restriction i$-name for) an element of L_{ω_1} for all $i \in \{0, 2, 3\}$ is dense in $\mathbb{K}_\alpha$.

Let $I \subseteq \omega_3$ and $p \in \prod_{\alpha\in I} \mathbb{K}_\alpha$. Denote by $\mathrm{supp}_\omega(p)$ and $\mathrm{supp}_{\omega_1}(p)$ the sets $\{\langle i, \alpha \rangle : i \in \{0, 2, 3\}, \alpha \in I, p_{i,\alpha}$ is not the maximal condition in $\mathbb{K}_{i,\alpha}\}$ and $\{\langle 1, \alpha, m, \zeta \rangle : \alpha \in I, m \in \omega, \zeta \in \omega_1, p_{1,\alpha,m,\zeta}$ is not the maximal condition in $\mathbb{K}^1_{\alpha,m,\zeta}\}$, respectively. We say that $p \in \prod_{\alpha\in I} \mathbb{K}_\alpha$ is a condition with *mixed support* if $|\mathrm{supp}_\omega(p)| = \omega$ and $|\mathrm{supp}_{\omega_1}(p)| = \omega_1$. Let $\mathbb{P}_0$ be the suborder of $\prod_{\alpha<\omega_3} \mathbb{K}_\alpha$ consisting of all conditions with mixed support and $\mathbb{D} = \mathbb{P}_0 \cap \prod_{\alpha<\omega_3} \mathbb{D}_\alpha$. It follows from the above that $\mathbb{D}$ is a dense subset of $\mathbb{P}_0$.

The following proposition resembles [7, Lemma 1].

Proposition 2.2. *$\mathbb{P}_0$ is ω-distributive.*

Proof. Given a condition $p_0 \in \mathbb{P}_0$ and a collection $\{O_n\}_{n\in\omega}$ of open dense subsets of $\mathbb{P}_0$, choose the least countable elementary submodel $\mathcal{N}$ of some large L_θ (θ regular) such that $\{p_0\} \cup \{\mathbb{P}_0\} \cup \{O_n\}_{n\in\omega} \subset \mathcal{N}$. Build a subfilter g of $\mathbb{P}_0 \cap \mathcal{N}$, below p_0, which hits all dense subsets of $\mathbb{P}_0$ which belong to $\mathcal{N}$. Let g_α be a $\mathbb{K}_\alpha$-generic filter over L such that $g \subset \prod_{\alpha\in\omega_3} g_\alpha$. Write g_α in the form $g_{0,\alpha} * g_{1,\alpha} * g_{2,\alpha} * g_{3,\alpha}$, where $g_{i,\alpha}$ is a $\mathbb{K}_{i,\alpha}$-generic over $L[*_{j<i} g_{j,\alpha}]$.

Now for every $\alpha \in \mathcal{N} \cap \omega_3$ the filter $g_{0,\alpha} * g_{1,\alpha} * g_{2,\alpha}$ has a greatest lower bound $p_{0,\alpha} * p_{1,\alpha} * p_{2,\alpha}$ because the forcing $\mathbb{K}_{0,\alpha} * \mathbb{K}_{1,\alpha} * \mathbb{K}_{2,\alpha}$ is ω-closed. The condition $\langle p_{0,\alpha}, p_{1,\alpha}, p_{2,\alpha} \rangle$ is obviously $(\mathcal{N}, \mathbb{K}_{0,\alpha} * \mathbb{K}_{1,\alpha} * \mathbb{K}_{2,\alpha})$-generic.

On each component $\alpha \in \mathcal{N} \cap \omega_3$ and $m \in \omega$ define $p_{3,\alpha,m} = \bigcup g_{3,\alpha,m}$.[3] It suffices to verify that $p_{3,\alpha,m}$ is a condition in $\mathbb{K}^3_{\alpha,m}$, for this will give us a condition in $\mathbb{P}_0$ which meets each of the O_n's.

Let $G := G_{0,\alpha,0} * G_{0,\alpha,m} * G_{1,\alpha,0} * G_{1,\alpha,m} * G_{2,\alpha,0} * G_{2,\alpha,m}$ be a $\mathbb{K}^0_{\alpha,0} * \mathbb{K}^0_{\alpha,m} * \mathbb{K}^1_{\alpha,0} * \mathbb{K}^1_{\alpha,m} * \mathbb{K}^2_{\alpha,0} * \mathbb{K}^2_{\alpha,m}$-generic filter over L containing

$$\langle p_{0,\alpha,0}, p_{0,\alpha,m}, p_{1,\alpha,0}, p_{1,\alpha,m}, p_{2,\alpha,0}, p_{2,\alpha,m} \rangle.$$

Since the latter is a $(\mathcal{N}, \mathbb{K}^0_{\alpha,0} * \mathbb{K}^0_{\alpha,m} * \mathbb{K}^1_{\alpha,0} * \mathbb{K}^1_{\alpha,m} * \mathbb{K}^2_{\alpha,0} * \mathbb{K}^2_{\alpha,m})$- generic condition, the isomorphism π of the transitive collapse $\bar{\mathcal{N}}$ of $\mathcal{N}$ onto $\mathcal{N}$ extends to an elementary embedding from

$$\bar{\mathcal{N}}_0 := \bar{\mathcal{N}}[\bar{g}_{0,\bar{\alpha},0} * \bar{g}_{0,\bar{\alpha},m} * \bar{g}_{1,\bar{\alpha},0} * \bar{g}_{1,\bar{\alpha},m} * \bar{g}_{2,\bar{\alpha},0} * \bar{g}_{2,\bar{\alpha},m}]$$

into $L_\theta[G]$. Here $\bar{g}_{i,\bar{\alpha},j} = \pi^{-1}(g_{i,\alpha,j})$, where $i \in 2$ and $j \in \{0, m\}$, and $\bar{\xi} = \pi^{-1}(\xi)$ for all $\xi \in \mathcal{N} \cap \mathrm{Ord}$. By the genericity of G we know that, letting $X_{\omega\cdot\alpha} = \bigcup G_{2,\alpha,0}$ and $X_{\omega\cdot\alpha+m} = \bigcup G_{2,\alpha,m}$, the property $(**)_{\alpha,m}$ holds. By elementarity, $\bar{\mathcal{N}}_0$ is a suitable model and $\bar{\mathcal{N}}_0 \vDash \phi(\omega_1, \omega_2, x_{\omega\cdot\bar{\alpha}+m}, t_{\omega\cdot\bar{\alpha}+m}, x_{\omega\cdot\bar{\alpha}})$, where $x_{\omega\cdot\bar{\alpha}} = \pi^{-1}(\bigcup g_{2,\alpha,0}) = \bigcup \bar{g}_{2,\bar{\alpha},0}$, $x_{\omega\cdot\bar{\alpha}+m} = \pi^{-1}(\bigcup g_{2,\alpha,m}) = \bigcup \bar{g}_{2,\bar{\alpha},m}$, and $t_{\omega\cdot\bar{\alpha}+m} = \pi^{-1}(\bigcup g_{0,\alpha,m}) = \bigcup \bar{g}_{0,\bar{\alpha},m}$. By the construction of $\mathbb{P}_0$ and elementarity, $\bar{\mathcal{N}}_0 = \bar{\mathcal{N}}[x_{\omega\cdot\bar{\alpha}}, x_{\omega\cdot\bar{\alpha}+m}]$ and hence

$$\bar{\mathcal{N}}[x_{\omega\cdot\bar{\alpha}}, x_{\omega\cdot\bar{\alpha}+m}] \vDash \phi(\omega_1, \omega_2, x_{\omega\cdot\bar{\alpha}+m}, t_{\omega\cdot\bar{\alpha}+m}, x_{\omega\cdot\bar{\alpha}}).$$

Let ξ be such that $\bar{\mathcal{N}} = L_\xi$ and let $\mathcal{M}$ be any suitable model containing $p_{3,\alpha}(m)$, and such that $\omega_1^{\mathcal{M}} = \omega_1 \cap \mathcal{N} (= \mathrm{dom} p_{3,\alpha}(m))$. We have to show that $\mathcal{M} \vDash \phi(\omega_1, \omega_2, x_{\omega\cdot\bar{\alpha}+m}, t_{\omega\cdot\bar{\alpha}+m}, x_{\omega\cdot\bar{\alpha}})$. Set $\eta = \mathcal{M} \cap \mathrm{Ord}$ and consider the suitable model $\mathcal{M}_2 \subseteq \mathcal{M}$, $\mathcal{M}_2 = L_\eta[x_{\omega\cdot\bar{\alpha}}, x_{\omega\cdot\bar{\alpha}+m}]$.

Three cases are possible.

Case a). $\eta > \xi$. Since $\mathcal{N}$ was chosen to be the least countable elementary submodel of L_θ containing the initial condition, the poset and the sequence of dense sets, it follows that ξ (and therefore also $\omega_1^{\bar{\mathcal{N}}}$) is collapsed to ω in $L_{\xi+2}$, and hence this case cannot happen.

Case b). $\eta = \xi$. In this case $\mathcal{M}_2 \vDash \phi(\omega_1, \omega_2, x_{\omega\cdot\bar{\alpha}+m}, t_{\omega\cdot\bar{\alpha}+m}, x_{\omega\cdot\bar{\alpha}})$. (Indeed, $\mathcal{M}_2 = L_\eta[x_{\omega\cdot\bar{\alpha}}, x_{\omega\cdot\bar{\alpha}+m}] = \bar{\mathcal{N}}[x_{\omega\cdot\bar{\alpha}}, x_{\omega\cdot\bar{\alpha}+m}] = \bar{\mathcal{N}}_0$.) Since ϕ is a Σ_1-formula, $\omega_1^{\mathcal{M}_2} = \omega_1^{\mathcal{M}}$ and $\omega_2^{\mathcal{M}_2} = \omega_2^{\mathcal{M}}$, we have $\mathcal{M} \vDash \phi(\omega_1, \omega_2, x_{\omega\cdot\bar{\alpha}+m}, t_{\omega\cdot\bar{\alpha}+m}, x_{\omega\cdot\bar{\alpha}})$.

[3]Formally this is $\bigcup\{r_{3,\alpha,m} : r_{3,\alpha} \in g_{3,\alpha} \text{ and } \langle r_{i,\alpha}\rangle_{i\leq 3} \in \mathbb{D}_\alpha\}$.

Case c). $\eta < \xi$. In this case $\mathcal{M}_2$ is an element of $\bar{N}[x_{\omega\cdot\bar\alpha}, x_{\omega\cdot\bar\alpha+m}]$. Since $L_\theta[G]$ satisfies $(**)_{\alpha,m}$, by elementarity so does the model $\bar{N}[x_{\omega\cdot\bar\alpha}, x_{\omega\cdot\bar\alpha+m}]$ with $X_{\omega\cdot\alpha}$, $X_{\omega\cdot\alpha+m}$, $T_{\omega\cdot\alpha}$, $T_{\omega\cdot\alpha+m}$ replaced by $x_{\omega\cdot\bar\alpha}$, $x_{\omega\cdot\bar\alpha+m}$, $t_{\omega\cdot\bar\alpha}$, $t_{\omega\cdot\bar\alpha+m}$, respectively. In particular, $\mathcal{M}_2 \models \phi(\omega_1, \omega_2, x_{\omega\cdot\bar\alpha+m}, t_{\omega\cdot\bar\alpha+m}, x_{\omega\cdot\bar\alpha})$. Since ϕ is a Σ_1-formula, $\omega_1^{\mathcal{M}_2} = \omega_1^{\mathcal{M}}$, $\omega_2^{\mathcal{M}_2} = \omega_2^{\mathcal{M}}$, we have $\mathcal{M} \models \phi(\omega_1, \omega_2, x_{\omega\cdot\bar\alpha+m}, t_{\omega\cdot\bar\alpha+m},$
$x_{\omega\cdot\bar\alpha})$, which finishes our proof. □

We say that $q \leq^* p$ if $q \leq p$, $\mathrm{supp}_\omega(p) = \mathrm{supp}_\omega(q)$, and $p_{l,\alpha} = q_{l,\alpha}$ for all $\langle l, \alpha\rangle \in \mathrm{supp}_\omega(p)$.

The proof of the following statement resembles that of [14, Proposition 3.7] and its idea seems to be often used in the context of mixed support iterations.

Proposition 2.3. *If $\gamma \notin T_{\omega\cdot\alpha_0+n}$ for some $\alpha_0 < \omega_3$ and $n \in \omega$, then $S_{\omega_1\cdot(\omega\cdot\alpha_0+n)+\gamma}$ is stationary in $L^{\mathbb{P}_0}$. In particular, $\mathbb{P}_0$ does not collapse ω_2.*

Proof. Let $p \in \mathbb{D}$ be such that $p \Vdash \gamma \notin T_{\omega\cdot\alpha_0+n}$ for some $\alpha_0 < \omega_3$ and $n \in \omega$, and $\dot{C}$ be a $\mathbb{P}_0$-name for a club. We shall construct a condition $q \leq p$ which forces $\dot{C} \cap S_{\omega_1\cdot(\omega\cdot\alpha_0+n)+\gamma} \neq \emptyset$.

Let us construct an increasing chain $\langle M_i : i < \omega_2\rangle$ of elementary submodels of L_θ, where θ is big enough, such that

(*i*) $M_i \supset [M_i]^\omega$ for all $i \in \omega_2$;

(*ii*) $M_i = \bigcup_{j<i} M_j$ for all $i \in \omega_2$ of cofinality ω_1; and

(*iii*) $\omega_1 \cup \{p, \mathbb{P}_0, \dot{C}, \alpha, \ldots\} \subset M_0$.

Now a standard Fodor argument yields $i \in \omega_2$ such that $i = M_i \cap \omega_2 \in S_{\omega_1\cdot(\omega\cdot\alpha_0+n)+\gamma}$ and $i \notin S_\beta$ for any $\beta \in M_i \setminus \{\omega_1 \cdot (\omega \cdot \alpha_0 + n) + \gamma\}$. Let also $\langle O_\xi : \xi < \omega_1\rangle \in M_i^{\omega_1}$ be the sequence in which all $\leq^*$-dense subsets of $\mathbb{P}_0$ which are elements of M_i appear cofinally often. Construct by induction on ξ a $\leq^*$-decreasing sequence $\langle q^\xi : \xi < \omega_1\rangle \in (\mathbb{D} \cap M_i)^{\omega_1}$ such that $q^0 = p$ and $q^\xi \in O_\xi$ for all $\xi < \omega_1$. Let $q \in \prod_{\alpha<\omega_3} \mathbb{K}_\alpha$ be such that $\mathrm{supp}(q) = \bigcup_{\xi<\omega_1} \mathrm{supp}(q^\xi)$, $q_{l,\alpha} = p_{l,\alpha}$ for all $\langle l, \alpha\rangle \in \mathrm{supp}_\omega(p)$, and $q_{0,\alpha} \Vdash q_{1,\alpha,m,\zeta} = \bigcup_{\xi<\omega_1} q^\xi_{1,\alpha,m,\zeta} \cup \{i\}$ for all $\langle 1, \alpha, m, \zeta\rangle \in \mathrm{supp}_{\omega_1}(q)$.

Claim 2.4. $q \in \mathbb{P}_0$.

Proof. Since $\omega_1 \subset M_i$ and $q^\xi \in M_i$ for all $\xi < \omega_1$, we conclude that $\mathrm{supp}(q^\xi) \subset M_i$ and $q^\xi_\nu \in M_i$ for all $\nu \in \mathrm{supp}(q^\xi)$. Let us fix any $\langle 1, \alpha, m, \zeta\rangle \in \mathrm{supp}_{\omega_1}(q)$ and find ξ_0 such that $\langle 1, \alpha, m, \zeta\rangle \in \mathrm{supp}_{\omega_1}(q^{\xi_0})$. For every $j < i$ the set O of those conditions $r \in \mathbb{P}_0$ such that $r_{0,\alpha} \Vdash_{\mathbb{K}_{0,\alpha}} \max r_{1,\alpha,m,\zeta} > j$ is $\leq^*$-dense and belongs to M_i, consequently $O = O_\xi$ for some $\xi > \xi_0$, which implies that $q_{0,\alpha} = q^\xi_{0,\alpha} \Vdash_{\mathbb{K}_{0,\alpha}} \max q^\xi_{1,\alpha,m,\zeta} > j$. Therefore

$q_{0,\alpha} \Vdash_{\mathbb{K}_{0,\alpha}} i > \max q^{\xi}_{1,\alpha,m,\zeta} > j$, consequently $q_{0,\alpha} \Vdash_{\mathbb{K}_{0,\alpha}} i = \sup \bigcup_{\xi<\omega_1} q^{\xi}_{1,\alpha,m,\zeta}$. It follows from the above that $\omega_1 \cdot (\omega \cdot \alpha + m) + \zeta \in M_i$ and $\omega_1 \cdot (\omega \cdot \alpha + m) + \zeta \neq \omega_1 \cdot (\omega \cdot \alpha_0 + n) + \gamma$, and the choice of i was made to ensure $i \notin S_\beta$ for all $\beta \in S_i \setminus \{\omega_1 \cdot (\omega \cdot \alpha_0 + n) + \gamma\}$. Thus $q_{0,\alpha}$ forces that $q_{1,\alpha,m,\zeta} = \bigcup_{\xi<\omega_1} q^{\xi}_{1,\alpha,m,\zeta} \cup \{i\}$ is a closed bounded subset of ω_2 disjoint from $S_{\omega_1\cdot(\omega\cdot\alpha+m)+\zeta}$ which completes our proof. □

Claim 2.5. *For every open dense subset $E \in M_i$ of $\mathbb{P}_0$ and $r \leq q$ there exists $r_1 \in E \cap M_i$ such that r and r_1 are compatible. In other words, q is an $\langle M_i, \mathbb{P}_0\rangle$-generic condition.*

Proof. Fix E, r as above and set $K = \mathrm{supp}_\omega(r) \cap M_i$. Without loss of generality, $r \in \mathbb{D}$. Then $K \in M_i$ and $r_{k,\alpha} \in M_i$ for all $\langle k, \alpha\rangle \in K$ because $M_i \supset [M_i]^\omega$. Let O be the set of $u \in \mathbb{P}_0$ such that either u is $\leq^*$-incompatible with p, or $u \leq^* p$ and there exists $\mathbb{D} \cap E \ni z \leq u$ with the following properties:

(1) $K \subset \mathrm{supp}(z)$, and for all $\langle k, \alpha\rangle \in K$ we have $r_{k,\alpha} \leq z_{k,\alpha}$;

(2) $z_{0,\alpha} \Vdash z_{1,\alpha,m,\zeta} = u_{1,\alpha,m,\zeta}$ for all $\langle k, \alpha\rangle \in K$ and $\zeta \in \omega_1$.

It is easy to see that $O \in M_i$. We claim that O is a $\leq^*$-dense subset of $\mathbb{P}_0$. So let us fix $s \in \mathbb{P}_0$. If s is $\leq^*$ incompatible with p, then $s \in O$. Otherwise there exists $t \leq^* s, p$. Let $w \in \mathbb{P}_0$ be such that $\mathrm{supp}_\omega(w) = K$, $w \restriction K = r \restriction K$, $\mathrm{supp}_{\omega_1}(w) = \mathrm{supp}_{\omega_1}(t)$, and $w \restriction \mathrm{supp}_{\omega_1}(w) = t \restriction \mathrm{supp}_{\omega_1}(t)$. Since $t \leq^* p$ and $r \leq q \leq p$, w is a condition in $\mathbb{D}$ and $w \leq t$. Extend w to a condition $z \in E \cap \mathbb{D}$ and let u be such that $\mathrm{supp}_\omega(u) = \mathrm{supp}_\omega(p)$, $u \restriction \mathrm{supp}_\omega(p) = p \restriction \mathrm{supp}_\omega(p)$, $\mathrm{supp}_{\omega_1}(u) = \mathrm{supp}_{\omega_1}(z)$, and $u \restriction \mathrm{supp}_{\omega_1}(z) = z \restriction \mathrm{supp}_{\omega_1}(z)$. Since $z \in \mathbb{D}$ we conclude that $u \in \mathbb{P}_0$ and hence $u \leq^* p$. By the definition we also have that $z \leq u$, and $z \leq w$ together with the definition of w imply that z satisfies (1). Thus z witnesses that $u \in O$. Moreover, $z \leq w \leq t$ implies $u \leq^* t$, and therefore $u \leq^* s$. This completes the proof that O is $\leq^*$-dense.

Let $\xi < \omega_1$ be such that $O = O_\xi$. Then $r \leq q \leq^* q^\xi \leq^* p$ and there exists z witnessing that $q^\xi \in O$, i.e., $\mathbb{D} \cap E \ni z \leq q^\xi$ and z satisfies (1), (2) with q^ξ instead of u. Moreover, since all relevant objects are elements of M_i, we can additionally assume that $z \in M_i$. Therefore $\mathrm{supp}(z) \subset M_i$, which together with (1), (2) implies that $\mathrm{supp}_\omega(z) \cap \mathrm{supp}_\omega(r) = K$ and $\mathrm{supp}_{\omega_1}(z) = \mathrm{supp}_{\omega_1}(q^\xi) \subset \mathrm{supp}_{\omega_1}(r)$. Define y as follows: $\mathrm{supp}_\omega(y) = \mathrm{supp}_\omega(r) \cup \mathrm{supp}_\omega(z)$, $\mathrm{supp}_{\omega_1}(y) = \mathrm{supp}_{\omega_1}(r)$, $y_{k,\alpha} = z_{k,\alpha}$ for $\langle k, \alpha\rangle \in \mathrm{supp}_\omega(z)$, $y_{k,\alpha} = r_{k,\alpha}$ for $\langle k, \alpha\rangle \in \mathrm{supp}_\omega(r) \setminus \mathrm{supp}_\omega(z) = \mathrm{supp}_\omega(r) \setminus M_i$, and $y \restriction \mathrm{supp}_{\omega_1}(y) = r \restriction \mathrm{supp}_{\omega_1}(r)$. A direct verification shows that $y \in \mathbb{P}_0$ and $y \leq r, z$, which completes the proof of the claim. □

Finally, we shall show that q forces $\dot{C} \cap S_{\omega_1\cdot(\omega\cdot\alpha_0+n)+\gamma} \neq \emptyset$. For this it suffices to prove that $q \Vdash i \in \dot{C}$. Suppose to the contrary that $r \Vdash \dot{C} \cap (j, i) = \emptyset$ for some $r \leq p$ and $j < i$. Let E be the set of those conditions $z \in \mathbb{P}_0$ such that there exists $\beta > j$ with the property $z \Vdash \beta \in \dot{C}$. E is an open dense subset of $\mathbb{P}_0$ and $E \in M_i$. Therefore there exists

$z \in E \cap M_i$ and $y \in \mathbb{P}_0$ such that $y \le z, r$. Since $z, j, \dot{C}, \mathbb{P}_0 \in M_i$ and there exists $\beta > j$ such that $z \Vdash \beta \in \dot{C}$, there exists such a $\beta \in M_i$, which means that $\beta \in (j, i)$. Therefore $y \Vdash \beta \in \dot{C}$ for some $\beta \in (j, i)$, which together with $y \le r$ and our choice of r leads to a contradiction. □

A simple Δ-system argument gives the following

Proposition 2.6. $\mathbb{P}_0$ *has the* ω_3*-chain condition.*

Combining Propositions 2.2, 2.3, and 2.6 we conclude that $\mathbb{P}_0$ preserves cardinals.

Coding stage. We define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \le \omega_3, \beta < \omega_3 \rangle$ of c.c.c. posets such that in $L^{\mathbb{P}_{\omega_3}}$, Martin's axiom holds and there is a Δ^1_3 definable wellorder of the reals. Let $\mathbb{P}_0$ be the poset defined above and let $F : \omega_3 \backslash \{0\} \to L_{\omega_3}$ be a bookkeeping function such that for all $a \in L_{\omega_3}$, the preimage $F^{-1}(a)$ is cofinal in both $\mathrm{Succ}(\omega_3)$ and $\mathrm{Lim}(\omega_3)$. At limit stages of our iteration we will introduce the wellorder of the reals and at successor stages of the iteration we will take care of all instances of Martin's axiom. Fix a nicely definable sequence of almost disjoint subsets of ω, $\vec{C} = \langle C_{(\xi,\eta)} : \xi \in \omega_1, \eta \in \omega \cdot 3 \rangle$. We will assume that all names for reals are nice. Recall that an $\mathbb{H}$-name $\dot{f}$ for a real is called *nice* if $\dot{f} = \bigcup_{i\in\omega} \{\langle\langle i, j^i_p \rangle, p\rangle : p \in \mathcal{A}_i(\dot{f})\}$ where for all $i \in \omega$, $\mathcal{A}_i(\dot{f})$ is a maximal antichain in $\mathbb{H}$, $j^i_p \in \omega$ and for all $p \in \mathcal{A}_i(\dot{f})$, $p \Vdash \dot{f}(i) = j^i_p$. If $\alpha < \beta < \omega_3$, we can assume that all $\mathbb{P}_\alpha$-names precede in the canonical wellorder $<_L$ of L all $\mathbb{P}_\beta$-names for reals which are not $\mathbb{P}_\alpha$-names. For x, y reals in $L[G_\alpha]$, where G_α is $\mathbb{P}_\alpha$-generic, let σ^α_x be the $<_L$-least $\mathbb{P}_\gamma$-name for x, where $\gamma \le \alpha$ is least so that x has a $\mathbb{P}_\gamma$-name. Then clearly $<_\alpha$ is an initial segment of $<_\beta$, for $\alpha < \beta$. Now if G is a $\mathbb{P}_{\omega_3}$-generic filter, then $<^G = \bigcup_{\alpha<\omega_3} \{\dot{<}^G_\alpha : \alpha < \omega_3\}$ where $\dot{<}_\alpha$ is a $\mathbb{P}_\alpha$-name for $<_\alpha$, is the desired wellorder of the reals. For any pair of reals x, y in $L[G]$ such that $x <_\alpha y$, let $x * y = \{2n : n \in x\} \cup \{2n+1 : n \in y\}$ and let $\Delta(x*y) = \{2n : n \in x*y\} \cup \{2n+1 : n \notin x*y\}$.

We proceed with the inductive definition of $\mathbb{P}_{\omega_3}$. Suppose $\mathbb{P}_\alpha$ has been defined.

If $\alpha = \omega \cdot \beta + n$ *is a successor:* Suppose that $F(\alpha) = \sigma$. If σ is a $\mathbb{P}_\alpha$-name for a c.c.c. poset which involves only conditions $p \in \mathbb{P}_\alpha$ such that $p(0)(\eta)$ is the trivial condition in $\mathbb{K}_\eta$ for all $\eta > \beta$, let $\dot{\mathbb{Q}}_\alpha = \sigma$. Otherwise, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the trivial poset.

If α *is a limit:* If $\alpha = 0$ let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the trivial poset. If $\alpha \in \mathit{Lim}(\omega_3) \backslash \{0\}$, $\alpha = \omega \cdot \beta$, then let $\mathbb{Q}_\alpha$ be the two stage iteration $\mathbb{Q}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$ defined as follows. First note that:

Claim 2.7. $\{T_{\alpha+n} : n \in \omega\}$ *is a sequence of Suslin trees.*

Proof. Let $\mathbb{P}_{0,<\alpha}$ and $\mathbb{P}_{0,\ge\alpha}$ be the suborders of $\prod_{\gamma<\alpha} \mathbb{K}_\gamma$ and $\prod_{\gamma\ge\alpha} \mathbb{K}_\gamma$ respectively, of all conditions with mixed supports. Let $\bar{\mathbb{P}}_\alpha$ be the factor poset $\mathbb{P}_\alpha / \mathbb{P}_0$.

By definition of the finite support iteration, not only $\bar{\mathbb{P}}_\alpha \in L^{\mathbb{P}_0}$, but in fact $\bar{\mathbb{P}}_\alpha \in L^{\mathbb{P}_{0,<\alpha}}$. Then identifying $\bar{\mathbb{P}}_\alpha$ with its $\mathbb{P}_0$-name we have

$$\mathbb{P}_\alpha = \mathbb{P}_0 * \bar{\mathbb{P}}_\alpha = (\mathbb{P}_{0,<\alpha} \times \mathbb{P}_{0,\geq\alpha}) * \bar{\mathbb{P}}_\alpha = (\mathbb{P}_{0,<\alpha} * \bar{\mathbb{P}}_\alpha) \times \mathbb{P}_{0,\geq\alpha}.$$

Thus in particular, for every $n \in \omega$, $T_{\alpha+n}$ is generic over $L^{\mathbb{P}_{0,<\alpha}*\bar{\mathbb{P}}_\alpha}$ and so $T_{\alpha+n}$ remains a Suslin tree in $L^{\mathbb{P}_\alpha}$. □

Recall that if T is an $\aleph_1$-tree, then the poset consisting of all finite partial functions p from T to ω such that if $p(s) = p(t)$ then s and t are comparable with extension relation superset, adds a specializing function for T. We will be referring to this poset as a forcing notion for specializing T. By a result of Baumgartner, if T has no ω_1-branch then this poset has the countable chain condition (see [2, Theorem 8.2]).

If $F(\alpha)$ is not a pair $\{\sigma^\alpha_x, \sigma^\alpha_y\}$ for some reals x, y in $L^{\mathbb{P}_\alpha}$ which involve only conditions $p \in \mathbb{P}_\alpha$ such that $p(0)(\eta)$ is the trivial condition in $\mathbb{K}_\eta$ for all $\eta > \beta$, let $\mathbb{Q}_\alpha$ be a $\mathbb{P}_\alpha$-name for the finite support iteration $\langle \mathbb{P}^{0,\alpha}_n, \dot{\mathbb{Q}}^{0,\alpha}_n : n \in \omega\rangle$, where $\dot{\mathbb{Q}}^{0,\alpha}_n$ is a $\mathbb{P}^{0,\alpha}_n$-name for specializing $T_{\alpha+n}$ for all $n \in \omega$. Otherwise, let $x = (\sigma^\alpha_x)^{G_\alpha}$, $y = (\sigma^\alpha_y)^{G_\alpha}$. In $L^{\mathbb{P}_\alpha}$ define $\mathbb{Q}^0_\alpha$ to be the finite support iteration $\langle \mathbb{P}^{0,\alpha}_n, \dot{\mathbb{Q}}^{0,\alpha}_n : n \in \omega\rangle$ where if $n \in \Delta(x * y)$ then $\dot{\mathbb{Q}}^{0,\alpha}_n$ is a $\mathbb{P}^{0,\alpha}_n$-name for specializing $T_{\alpha+n}$; otherwise let $\dot{\mathbb{Q}}^{0,\alpha}_n$ be a $\mathbb{P}^{0,\alpha}_n$-name for $T_{\alpha+n}$. For every $n \in \omega$ let $A_{\alpha+n}$ be the generic subset of ω_1 added by Q^0_n.

Then let $\mathbb{Q}^1_\alpha$ almost disjoint code the sequences $\langle A_{\alpha+n} : n \in \omega\rangle$, $\langle Y_{\alpha+n} : n \in \omega\rangle$ and $\langle T_{\alpha+n} : n \in \omega\rangle$. More precisely, in $L^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha}$ let $\mathbb{Q}^1_\alpha$ be the poset of all pairs $\langle s, s^*\rangle$ where $s \in [\omega]^{<\omega}$ and $s^* \in [\langle\mu, n\rangle : n \in \omega, \mu \in Y_{\alpha+n}]^{<\omega} \cup [\langle\mu,\eta\rangle : \eta \in [\omega, \omega\cdot 2), \mu \in A_{\alpha+n}]^{<\omega} \cup [\langle\mu,\eta\rangle : \eta \in [\omega\cdot 2, \omega\cdot 3), \mu \in T_{\alpha+n}]^{<\omega}$. The extension relation is $\langle t, t^*\rangle \leq \langle s, s^*\rangle$ if and only if t end extends s and $(t\backslash s) \cap C_{\mu,\eta} = \emptyset$ for all $(\mu,\eta) \in s^*$. Let R_α be the generic real added by $\mathbb{Q}^1_\alpha$ and let $\mathbb{Q}_\alpha = \mathbb{Q}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$.

With this the inductive construction of $\mathbb{P}_{\omega_3}$ is complete. Clearly in $L^{\mathbb{P}_{\omega_3}}$, MA holds and $\mathfrak{c} = \omega_3$. We will see that the wellorder $<^G$, where G is $\mathbb{P}_{\omega_3}$-generic, has a Δ^1_3-definition.

Lemma 2.8. *$x < y$ if and only if there is a real R such that for every countable suitable model $\mathcal{M}$ such that $R \in \mathcal{M}$, there is a limit ordinal $\tilde{\alpha} \in [\omega\cdot 2, \omega_3^{\mathcal{M}})$ such that for every $n \in \omega$ the set $\{\gamma \in \omega_1 : S_{\omega_1\cdot(\tilde{\alpha}+n)+\gamma}$ is not stationary$\}$ is an ω_1-tree, which is specialized for $n \in \Delta(x * y)$ and has a branch for $n \notin \Delta(x * y)$.*

Proof. Let G be $\mathbb{P}_{\omega_3}$-generic and let x, y be reals in $L[G]$. Suppose $x < y$. Then there is $0 < \alpha < \omega_3$, a limit ordinal, $\alpha = \omega\cdot\beta$, such that $F(\alpha) = \{\sigma^\alpha_x, \sigma^\alpha_y\}$ and $\sigma^\alpha_x, \sigma^\alpha_y$ involve only conditions $p \in \mathbb{P}_\alpha$ such that $p(0)(\eta)$ is the trivial condition in $\mathbb{K}_\eta$ for all $\eta > \beta$. Let R_α be the real added by $\mathbb{Q}^1_\alpha$ and let $\mathcal{M}$ be a suitable model containing R_α. Then the sequences $\langle Y_{\alpha+n} \cap \eta : n \in \omega\rangle$, $\langle T_{\alpha+n} \cap \eta : n \in \omega\rangle$, $\langle A_{\alpha+n} \cap \eta : n \in \omega\rangle$ also belong to $\mathcal{M}$. Fix n. Since

$X_{\alpha+n}\cap\eta$, $X_\alpha\cap\eta$ are in $\mathcal{M}$, we have that $\mathcal{M}\vDash\phi(\omega_1,\omega_2,X_{\alpha+n}\cap\eta,T_{\alpha+n}\cap\eta,X_\alpha\cap\eta)$. This means that $\mathcal{M}$ models the following statement:
Using the sequence $\vec{B}$, the sets $X_{\alpha+n}\cap\eta, X_\alpha\cap\eta$ almost disjointly code subsets Z_n, Z of ω_2, respectively, such that $0(Even(Z_n))$ and $I(Even(Z_n)) = I(Even(Z))$ are the L-least codes for ordinals $\tilde{\alpha}_n+n$ and $\tilde{\alpha}_n$ for some limit $\tilde{\alpha}_n<\omega_3$, and $\chi^{-1}[II(Even(Z))] = \{\langle\gamma,\zeta\rangle : \gamma\in T_{\alpha+n}\cap\eta, \zeta\in\bar{C}_\gamma\}$, where $T_{\alpha+n}\cap\eta$ is an ω_1-tree and $\bar{C}_\gamma$ is a closed unbounded subset of ω_2 disjoint from $S_{\omega_1\cdot(\tilde{\alpha}_n+n)+\gamma}$ for all $\gamma\in T_{\alpha+n}\cap\gamma$.

Since Z does not depend on n, we conclude that all $\tilde{\alpha}_n$'s coincide and we shall denote them simply by $\tilde{\alpha}$. Let us also note that $A_{\alpha+n}\cap\eta\in\mathcal{M}$ is a specializing function for (resp. a branch through) $T_{\alpha+n}\cap\eta$ provided so is $A_{\alpha+n}$ with respect to $T_{\alpha+n}$.

Thus in $\mathcal{M}$ there is a limit ordinal $\tilde{\alpha}\in[\omega\cdot 2,\omega_3)$ such that for every $n\in\omega$ the set $T_{\alpha+n}\cap\eta = \{\gamma\in\omega_1 : S_{\omega_1\cdot(\tilde{\alpha}+n)+\gamma}$ is not stationary$\}$ is a ω_1-tree, which is specialized for $n\in\Delta(x*y)$ and has a branch for $n\notin\Delta(x*y)$.

To see the other implication, suppose x,y are reals in $L[G]$ and there is a real R such as in the formulation. By Löwenheim-Skolem theorem, the same property holds for $\mathcal{M}=L_{\omega_4}$. This means that in L_{ω_4} (and hence also in L) there is a limit ordinal $\tilde{\alpha}\in[\omega\cdot 2,\omega_3)$ such that for every $n\in\omega$ the set $I_n=\{\gamma\in\omega_1 : S_{\omega_1\cdot(\tilde{\alpha}+n)+\gamma}$ is not stationary$\}$ is an ω_1-tree, which is specialized for $n\in\Delta(x*y)$ and has a branch for $n\notin\Delta(x*y)$. By the definition of $\mathbb{P}_0$ and Proposition 2.3 we have that $I_n=T_{\tilde{\alpha}+n}$. Thus for some $n\in\omega$ there exists a branch through $T_{\tilde{\alpha}+n}$, which means that $F(\tilde{\alpha})$ is a pair $\{\sigma^\alpha_a,\sigma^\alpha_b\}$ for some reals $a<b$ in $L^{\mathbb{P}_{\tilde{\alpha}}}$, $\dot{\mathbb{Q}}^{0,\tilde{\alpha}}_n$ is a $\mathbb{P}^{0,\tilde{\alpha}}_n$-name for specializing $T_{\tilde{\alpha}+n}$ for all $n\in\Delta(a*b)$, and $\dot{\mathbb{Q}}^{0,\tilde{\alpha}}_n$ is a $\mathbb{P}^{0,\tilde{\alpha}}_n$-name for $T_{\tilde{\alpha}+n}$ otherwise. It follows from the above that $\Delta(x*y)=\Delta(a*b)$, consequently $x=a$ and $y=b$, and hence $x<y$. □

Thus we have obtained the following.

Theorem 2.9. *The existence of a Δ^1_3-definable wellorder of the reals is consistent with Martin's axiom and $\mathfrak{c}=\omega_3$.*

3. Cardinal characteristics, projective wellorders and large continuum

We will conclude by pointing out that the model constructed above can be easily modified to obtain the consistency of $\mathfrak{c}=\omega_3$, the existence of a Δ^1_3-definable wellorder of the reals and certain inequalities between some of the cardinal characteristics of the real line. An excellent exposition of the subject of cardinal characteristics of the real line can be found in [4].

Let κ be a regular uncountable cardinal. In [5, Theorem 3.1], Brendle shows that if V is a model of $\mathfrak{c}=\kappa$, $2^\kappa=\kappa^+$, $\mathcal{H}=\langle f_\alpha : \alpha<\kappa\rangle$ is an unbounded, $<^*$-wellordered sequence of strictly increasing functions in ${}^\omega\omega$ and $\mathcal{A}$ is a maximal almost disjoint family, then in V there is a ccc poset $\mathbb{P}(\mathcal{A}, \mathcal{H})$ of size κ which preserves the unboundedness of $\mathcal{H}$ and destroys the maximality of $\mathcal{A}$. A similar result concerning the bounding and the splitting numbers, was obtained by Fischer and Steprāns. In [8, Lemma 6.2] they show that if V is a model of $\forall\lambda < \kappa(2^\lambda \leq \kappa)$, $\mathcal{H}$ is an unbounded $<^*$-directed family in ${}^\omega\omega$ and $\mathrm{cov}(\mathcal{M}) = \kappa$, then there is a ccc poset $\mathbb{P}(\mathcal{H})$ of size κ which preserves the unboundedness of $\mathcal{H}$ and adds a real not split by $V \cap [\omega]^\omega$. Thus if V is a model of $\forall\lambda < \kappa(2^\lambda \leq \kappa)$, $\mathcal{H}$ is an unbounded $<^*$-directed family in ${}^\omega\omega$, then there is a ccc poset $\mathbb{P}(\mathcal{H})$ which preserves the unboundedness of $\mathcal{H}$ and adds a real not split by $V \cap [\omega]^\omega$ (just take $\mathbb{C}_\kappa * \mathbb{P}(\mathcal{H})$ where $\mathbb{C}_\kappa$ is the poset for adding κ many Cohen reals).

Also, recall that if $\mathcal{H}$ is an unbounded directed family of reals such that each countable subfamily is dominated by an element of the family, then in order to preserve the unboundedness of $\mathcal{H}$ along a finite support iteration of ccc posets, it is sufficient to preserve its unboundedness at each successor stage of the iteration (see [13]). Note also that the unboundedness of unbounded directed families of reals is preserved by posets of size smaller than the size of the family (see [1]). It remains to point out that the posets used to provide the Δ^1_3 definition of the wellorder on $\mathbb{R}$ are of size $\aleph_1$. Therefore subject to an appropriate bookkeeping function we can obtain the following.

Corollary 3.1. *There is a generic extension of the constructible universe L in which there is a Δ^1_3-definable wellorder of the reals and*

$$\mathfrak{b} = \aleph_2 < \mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \aleph_3.$$

If in addition at successor stages along the iteration we force with all σ-centered posets of size $\leq \omega_1$, then in the final generic extension we will have $\mathrm{MA}_{<\omega_2}(\sigma\text{-centered})$. However by Bell's theorem $\mathfrak{m}(\sigma\text{-centered}) = \mathfrak{p}$, where $\mathfrak{m}(\sigma\text{-centered})$ is the least cardinal κ for which $MA_\kappa(\sigma\text{-centered})$ fails (see [3] or [4, Theorem 7.12]). Therefore in the final extension we will have $\mathfrak{p} = \omega_2$ and so we obtain the following result.

Corollary 3.2. *There is a generic extension of the constructible universe L in which there is a Δ^1_3-definable wellorder of the reals and*

$$\mathfrak{p} = \mathfrak{b} = \aleph_2 < \mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \aleph_3.$$

The authors expect that similar methods can be used to establish the results of the paper for $2^{\aleph_0} = \aleph_n$ where $n \in \omega$. The following question remains of interest.

Question. Is there a generic extension of the constructible universe L in which $2^{\aleph_0} = \kappa$, Martin's axiom holds and there is a projective wellorder of the reals, where κ is the least L-cardinal of uncountable L-cofinality such that L_κ satisfies ϕ for some sentence ϕ?

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