\textbf{1. Introduction}

In this paper we prove the following result, which solves in the affirmative a question posed in [8].

**Theorem 1.1.** Let $\kappa$ be a successor cardinal in $L$.

1. There exists a GCH and cardinal preserving forcing poset $\mathbb{P} \in L$ such that in $L^\mathbb{P}$ the ideal $\text{NS}_\kappa$ of all non-stationary subsets of $\kappa$ is $\Delta^1_1$-definable over $H(\kappa^+)$. 

2. There exists a cardinal preserving forcing poset $\mathbb{P} \in L$ such that in $L^\mathbb{P}$ the ideal $\text{NS}_\kappa$ of all non-stationary subsets of $\kappa$ is $\Delta^1_1$-definable over $H(\kappa^+)$, and $2^\kappa = \kappa^{++}$. 

The motivation for Theorem 1.1 comes from generalized descriptive set theory, which, roughly speaking, is the study of “nice” subsets of $2^\kappa$ for $\kappa > \omega$. Descriptive set theory looks very different in this generalized setting compared to the classical case. For instance, the classical fact that $\Delta^1_1$ sets are Borel is not anymore true. And the non-stationary ideal on $\kappa$ (possibly restricted to certain stationary subset) considered in various forcing extensions is an important test space distinguishing various classes of “nice” subsets of $2^\kappa$, see, e.g., [7, Theorem 49] and references therein.

Theorem 1.1 is proved using almost disjoint coding followed by localization, a method invented by David in [3] and further developed in works of Friedman and collaborators. This is a new application of this method as the previous results regarding the definability of the ideal of non-stationary subsets of $\kappa$ were mainly achieved using combinatorics related to canary trees,
see [13] for the definition. For instance, Mekler and Shelah proved in [13] that $NS_{\omega_1}$ is $\Delta_1$-definable over $H(\omega_2)$ iﬀ there is a canary tree, and canary trees may or may not exist in models of GCH. The proof presented in [13] had some inaccuracies which were ﬁxed by Hyttinen and Rautiila in [10], where they also obtained the result that $NS_\kappa$ restricted to the ordinals of cofinality $\kappa$ can be $\Delta_1$-definable over $H(\kappa^+)$ for any regular $\kappa$. The results of [10] were further improved in [7], where it is also shown that $NS_\kappa$ is not $\Delta_1$-deﬁnable in $L$.

This topic also has connections with large cardinal theory: Using methods similar to those of [7], Friedman and Wu proved [8] that $NS_\kappa$ restricted to a measure 0 set can be $\Delta_1$-deﬁnable for a measurable $\kappa$. They also show that the unrestricted $NS_\kappa$ cannot be $\Delta_1$-deﬁnable for a weakly compact $\kappa$. Also note that $NS_\kappa$ is $\Delta_1$-deﬁnable if there exists a collection $S$ of stationary subsets of $\kappa$ such that $|S| = \kappa$ and each stationary subset of $\kappa$ contains some $S \in S$. For $\kappa = \omega_1$ this is consistent relative to the existence of inﬁnitely many Woodin cardinals, see [14, Section 6.2].

With the exception of the case $\kappa = \omega_1$, prior results on the $\Delta_1$-deﬁnability of $NS_\kappa$ are limited to restrictions of $NS_\kappa$. In the present paper our methods allow us to obtain the $\Delta_1$-deﬁnability of the full unrestricted $NS_\kappa$ for all successor $\kappa$.

Throughout this paper we work over the constructible universe $L$, thus unless otherwise speciﬁed $V = L$.

2. Proof of Theorem 1.1

Let $\gamma$ be the predecessor cardinal of $\kappa$, i.e., $\kappa = \gamma^+$. First we prove the ﬁrst part. At the end we shall indicate how to modify it in order to obtain the proof of the second part.

We say that a transitive ZF− model $M$ is suitable if $\gamma + 1 \subseteq M$, $(\gamma^{++})^M$ exists and $(\gamma^{++})^M = (\gamma^{++})^{L^M}$. From this it follows, of course, that $(\gamma^+)^M = (\gamma^+)^{L^M}$. We will need an appropriate sequence $\tilde{S} = \langle S_\alpha : \alpha < \kappa^+ \rangle$ of stationary subsets of $\kappa^+ \cap \text{Cof}(\kappa)$ such that $(\kappa^+ \cap \text{Cof}(\kappa)) \setminus \bigcup_{\alpha \in \kappa^+} S_\alpha$ is stationary. Let $\langle G_\xi : \xi \in \kappa^+ \cap \text{cof}(\kappa) \rangle$ be a $\mathcal{Q}_{\kappa^+}(\text{cof}(\kappa))$ sequence which is $\Sigma_1$ deﬁnable over $L_{\kappa^+}$. For every $\alpha < \kappa^+$ let us denote by $S_\alpha$ the set $\{\xi < \kappa^+ : G_\xi = \{\kappa \cdot (\alpha + 1)\}\}$. It follows from the above that $S_\alpha$’s are stationary subsets of $\text{cof}(\kappa) \cap \kappa^+$ which are mutually disjoint and the sequence $\tilde{S} = \langle S_\alpha : \alpha < \kappa^+ \rangle$ is $\Sigma_1$ deﬁnable over $L_{\kappa^+}$. Moreover, $\bigcup \{S_\alpha : \alpha < \kappa^+\}$ has fat complement because the set $S' = \{\xi < \kappa^+ : G_\xi = \{0\}\}$ is disjoint from the union considered above.

The idea of the proof will be to construct a poset $P$ such that in $V^P$ we will have the following $\Sigma_1$ deﬁnition of the complement of $NS_\kappa$: $S \subseteq \kappa$ is stationary iﬀ there exists $Y \in [\kappa]^\kappa$ such that for every suitable model $M$ of size $\gamma$ containing $Y \cap (\gamma^+)^M$, there is $\mu < (\gamma^{++})^M$ such that for all $\zeta \in T(S) \cap (\gamma^+)^M$ we have $M \models S_{\mu^+\zeta}$ is not stationary (where $T(S) =$
sequences\{2i + 1 : i \in S\} \cup \{2i : i \in \kappa \setminus S\} and \rho = \kappa + 3\). In the latter definition by \(S_{P, \mu + \zeta}\) we mean, of course, its \(M\)-version.

We shall force clubs disjoint from certain \(S_\alpha\)'s by initial segments. This forcing is well-studied and it is known (see, e.g., [2, Theorem 1]) that under GCH the poset consisting of closed bounded subsets of a stationary subset \(S \subset \lambda\), where \(\lambda\) is a successor cardinal, preserves cofinalities, introduces no bounded subsets of \(\lambda\), and creates a club subset of \(S\) if and only if \(S\) is fat in the sense that for every club \(C \subset \lambda\), \(C \cap S\) contains closed sets of ordinals of arbitrarily large order-types below \(\lambda\). Since Cof\((< \kappa) \cup S\) is easily seen to be fat for any stationary subset \(S \subset \text{Cof}(\kappa)\), the posets shooting clubs disjoint from \(S_\alpha\)'s will have all of these nice properties.

Similarly, but using this time the \((\kappa^+-\text{many})\) \(L\)-least codes for ordinals below \(\kappa^+\) and a \(\Sigma_1\) definable \(\check{\Omega}(\text{cof}(\gamma))\) sequence, we can obtain a \(\Sigma_1\) definable sequence \(\check{A} = \langle A_\zeta : \zeta < \kappa^+ \rangle\) of stationary subsets of cof\((\gamma) \cap \kappa\) which are mutually almost disjoint (that is, for all \(\zeta_0 \neq \zeta_1\) we have that \(A_{\zeta_0} \cap A_{\zeta_1}\) is bounded in \(\kappa\)).

Let us fix a function \(F : \kappa^+ \to L\) and set \(\rho = \kappa + 3\). Next, we shall define an iteration \(\langle \mathcal{P}_\xi, \check{Q}_\xi : \xi < \kappa^+\rangle\) depending\(^1\) on \(F\). Later we will choose a particular \(F\) such that the poset associated to it makes \(\text{NS}_\kappa\), the ideal of non-stationary subsets of \(\kappa\), \(\Delta_1\)-definable over \(H(\kappa^+)\). The choice of this \(F\) is done after Corollary 2.12.

Suppose that we have already defined \(\mathcal{P}_\xi\) for some \(\xi < \kappa^+\). Let us write \(\xi\) in the form \(\rho \cdot \alpha + \zeta\), where \(\zeta < \rho\), and suppose that together with \(\mathcal{P}_\xi\) we have also defined a sequence \(\langle Y_\beta : \beta < \alpha \rangle\) such that \(Y_\beta\) is a \(\mathcal{P}_{\rho(\beta + 1)}\)-name for a subset of \(\kappa\). If \(F(\alpha)\) is not a \(\mathcal{P}_{\rho \cdot \alpha}\)-name for a subset of \(\kappa\) then \(\check{Q}_\xi\) is trivial. Otherwise let \(G\) denote the \(\mathcal{P}_\xi\)-generic filter. If \(F(\alpha)^G\) is not stationary in \(V[G \upharpoonright \rho \cdot \alpha]\), then \(\check{Q}_\xi = \check{Q}_\xi^G\) is trivial. So suppose that \(F(\alpha)^G\) is stationary in \(V[G \upharpoonright \rho \cdot \alpha]\). Four cases are possible. Before passing to them we shall set the following notation: if \(A\) is a subset of \(\kappa\), then \(T(A) = \{2i + 1 : i \in A\} \cup \{2i : i \in \kappa \setminus A\}\).

Case 1. \(\zeta < \kappa\). If \(\zeta \notin T(F(\alpha)^G)\), then \(\check{Q}_\xi\) is the trivial poset. Otherwise \(\check{Q}_\xi\) is the standard poset shooting a club \(C_\xi\) disjoint from \(S_\xi\) via initial segments. The \(\mathcal{P}_\zeta\)-name of \(C_\xi\) will be denoted by \(\check{C}_\xi\).

Case 2. \(\zeta = \kappa\). Before defining \(\check{Q}_\xi\) we need to fix some notation and introduce some auxiliary objects. Given a set of ordinals \(X\), let \(\text{Even}(X)\) and \(\text{Odd}(X)\) be the sets of even and odd ordinals in \(X\), respectively. In the following we treat 0 as a limit ordinal. Let \(D_\alpha \subset \kappa^+\) be a set coding the sequences \(\langle Y_\beta^G : \beta < \alpha \rangle\) and \(\langle C_{\rho \cdot \alpha + \zeta} : \zeta < \kappa\rangle\). That is, letting \(\phi_l, \phi_r\) be the \(L\)-minimal injections of \(\alpha \times \kappa\) and \(\kappa \times \kappa^+\) into \(\text{Even}(\kappa^+)\) and \(\text{Odd}(\kappa^+)\),

\(^1\)Formally we should have written \(\langle \mathcal{P}_\xi^F, \check{Q}_\xi^F : \xi < \kappa^+\rangle\) instead of \(\langle \mathcal{P}_\xi, \check{Q}_\xi : \xi < \kappa^+\rangle\), but this would only burden the notation.
respectively, $D_\alpha$ is such that $Even(D_\alpha) = \phi_L(\{\langle \beta, i \rangle : \beta < \kappa, i \in Y^G_\beta \})$ and\footnote{Here we implicitly use that neither $\kappa$ nor $\kappa^+$ is collapsed by $P_\xi$. This will be proved in Lemmas 2.2 and 2.7. To be formally correct we should have presented this proof simultaneously with the inductive construction of $P$.}$Odd(D_\alpha) = \phi_L(\{\langle \zeta, \nu \rangle : \zeta \in T(F(\alpha)^G), \nu \in G_\zeta \})$. Then $Q_\xi$ adds a subset $X^\theta_\alpha$ of $\kappa$ which almost disjointly codes $D_\alpha$. More precisely, let $Q_\xi$ be the poset of all pairs $(s, s^*) \in [\kappa]^{\kappa} \times [D_\alpha]^{\kappa}$, where $(t, t^*)$ extends $(s, s^*)$ if and only if $t$ end-extends $s$ and $t \setminus s \cap A_\nu = \emptyset$ for every $\nu \in s^*$. Given a $Q_\xi$-generic filter $G(\xi)$ over $L[G]$, we set $X^\theta_\alpha = \bigcup \{s : \exists s^* (\langle s, s^* \rangle) \in G(\xi)\}$. By genericity we have that $D_\alpha = \{\nu : A_\nu \cap X^\theta_\alpha \text{ is bounded in } \kappa\}$.

Case 3. $\zeta = \kappa + 1$. Let us fix a strictly increasing continuous sequence $\langle N_\nu : \nu < \kappa^+ \rangle$ of elementary submodels of $L_\theta[X^\theta_\alpha]$ of size $\kappa$ which contain $\kappa \cup \{X^\theta_\alpha\}$ as a subset, where $\theta$ is a large enough cardinal. Denote by $E_\alpha$ the set $\{\langle \kappa^+ \rangle \cap N_\nu : \nu < \kappa^+\}$, where $N_\nu$ is the Mostowski collapse of $N_\nu$, and observe that $E_\alpha$ is a club in $\kappa^+$. Now choose $Z_\alpha$ to be a subset of $\kappa^+$ such that $Even(Z_\alpha) = D_\alpha$, and if $\beta < \kappa^+$ is $(\gamma^+)^M = (\gamma^+)^M$ for some suitable model $M$ such that $Z_\alpha \cap \beta \in M$, then $\beta$ belongs to $E_\alpha$. This is easily done by placing in $Z_\alpha$ a code for a bijection $\phi : \beta_1 \to \kappa$ on the odd part of the interval $(\beta_0, \beta_0 + \kappa)$ for each adjacent pair $\beta_0 < \beta_1$ from $E_\alpha$. Then $Q_\xi$ adds a subset $X^\theta_\alpha$ of $\kappa$ which almost disjointly codes $Z_\alpha$. More precisely, let $Q_\xi$ be the poset of all pairs $(s, s^*) \in [\kappa]^{\kappa} \times [Z_\alpha]^{\kappa}$, where $(t, t^*)$ extends $(s, s^*)$ if and only if $t$ end-extends $s$ and $t \setminus s \cap A_\nu = \emptyset$ for every $\nu \in s^*$. Given a $Q_\xi$-generic filter $G(\xi)$ over $L[G]$, we set $X^\theta_\alpha = \bigcup \{s : \exists s^* (\langle s, s^* \rangle) \in G(\xi)\}$. By genericity we have that $Z_\alpha = \{\nu : A_\nu \cap X^\theta_\alpha \text{ is bounded in } \kappa\}$.

As a result we have:

\begin{equation}
(\ast)_\alpha: \text{If } M \text{ is any suitable model such that } \kappa \cup \{X^\theta_\alpha, X^\theta_\alpha\} \subseteq M \text{ and } (\gamma^+)^M < \gamma^{++}, \text{ then }^3 M \models \psi(\gamma^+, \gamma^{++}, \alpha, F(\alpha)^G, X^\theta_\alpha), \text{ where } \psi(\gamma^+, \gamma^{++}, \mu, S, X) \text{ is the formula } \psi \text{ using the sequence } \mathcal{A}, \text{ the set } X \text{ almost disjointly codes a subset } D \text{ of } \gamma^{++} \text{ such that using } \phi_1 \text{ and } \phi_2, \text{ } \psi(\gamma^+, \gamma^{++}, \mu, S, X) \text{ codes }^4 \mu < \gamma^{++}, S \subseteq \gamma^+, \text{ and a sequence } \langle C_\xi : \xi \in T(S) \rangle, \text{ where } C_\xi \text{ is a club in } \gamma^{++} \text{ disjoint from } S_{\rho^M + \xi}.
\end{equation}

The proof of $(\ast)_\alpha$ is analogous to that of $(\ast)_\alpha$ in [4]. However, for the sake of completeness we shall present it. Given a suitable model $M$ with $(\gamma^+)^M = \beta$ and $\kappa \cup \{X^\theta_\alpha, X^\theta_\alpha\} \subseteq M$, observe that $Z_\alpha \cap \beta \in M$ because $Z_\alpha \cap \beta = \{\nu < \beta : \langle A_\nu \cap X^\theta_\alpha \rangle = \kappa\}$ and $\mathcal{A}^M = \mathcal{A}^L \upharpoonright \beta$, which yields $\beta \in E_\alpha$ by the construction of $Z_\alpha$. Also, $D_\alpha \cap \beta \in M$ because $D_\alpha = Even(Z_\alpha)$. Let $\nu < \kappa^+$ be such that $(\gamma^+)^N_\nu = \beta$. By the construction we have that $L_\theta[X^\theta_\alpha] \models \psi(\gamma^+, \gamma^{++}, \alpha, F(\alpha)^G, X^\theta_\alpha)$, and hence also $N_\nu \models \psi(\gamma^+, \gamma^{++}, \alpha, F(\alpha)^G, X^\theta_\alpha)$ by elementarity. Since the coding apparatus as well as stationary subsets involved into the formula $\psi$ are referring to $L$, for any 2 suitable models $M_0, M_1 \supseteq \{X\}$ we have that $M_0 \models \psi(\gamma^+, \gamma^{++}, \mu, S, X)$
iff \( M_1 \models \psi(\gamma^+, \gamma^{++}, \mu, S, X) \), provided that \((\gamma^{++})^{M_0} = (\gamma^{++})^{M_1}\). In particular, \( M \models \psi(\gamma^+, \gamma^{++}; \alpha, F(\alpha)^G, X_0^0) \) because \( \bar{N}_0 \models \psi(\gamma^+, \gamma^{++}, \alpha, F(\alpha)^G, X_0^0) \) and \((\gamma^{++})^{\bar{N}_0} = (\gamma^{++})^{\bar{M}} = \beta\), which completes the proof of \((*)_\alpha\).

**Case 4.** \( \zeta = \kappa + 2 \). In this case the poset \( Q_\xi \) localizes the property \((*)_\alpha\) of \( X_0^\alpha \) in the style of [3]. More precisely, \( Q_\xi \) consists of all functions \( r : |r| \to 2 \), where the domain \(|r|\) of \( r \) is a limit ordinal less than \( \kappa \), such that:

1. if \( \eta < |r| \) then \( \eta \in X_0^\alpha \) if \( r(3\eta + 1) = 1 \)
2. if \( \eta < |r| \) then \( \eta \in X_1^\alpha \) if \( r(3\eta + 2) = 1 \)
3. if \( \eta \leq |r| \), \( M \) is a suitable model of size \( \gamma \) containing \( r \upharpoonright \eta \) as an element and \( \eta = (\gamma^+)^M \), then \( M \models \psi(\gamma^+, \gamma^{++}, \mu, F(\alpha)^G \cap \eta, X_0^\alpha \cap \eta) \)
for some ordinal \( \mu \).

The order relation is given by extension. Observe that the poset \( Q_\xi \) produces a generic function from \( \kappa \) into \( 2 \), which is the characteristic function of a subset \( Y_\alpha \) of \( \kappa \) whose \( \mathbb{P}_\xi \)-name will be denoted by \( \hat{Y}_\alpha \).

Finally, assuming that \( (\mathbb{P}_\xi, \hat{Q}_\xi : \xi < \delta) \) has been defined for some limit \( \delta < \kappa^+ \), we define \( \mathbb{P}_\delta \) as follows. Let \( S_\delta \) be the set of all functions \( p \) with domain \( \delta \) such that \( p \upharpoonright \xi \in \mathbb{P}_\xi \) for all \( \xi < \delta \). For \( p \in S_\delta \) we shall denote the sets

\( \{ \xi < \delta : \xi \text{ is of the form } \rho \cdot \alpha + \zeta \text{ for some } \zeta < \kappa \text{ and } p(\xi) \neq 1_{Q_\xi} \} \)

and

\( \{ \xi < \delta : \xi \text{ is of the form } \rho \cdot \alpha + \zeta \text{ for some } \zeta \in \{ \kappa, \kappa+1, \kappa+2 \} \text{ and } p(\xi) \neq 1_{Q_\xi} \} \)

by \( \text{supp}_+ (p) \) and \( \text{supp}_- (p) \), respectively, and their union will be denoted by \( \text{supp}(p) \). The poset \( \mathbb{P}_\delta \) consists of those \( p \in S_\delta \) such that \( |\text{supp}_+ (p)| < \kappa \) and \( |\text{supp}_- (p)| < \kappa^+ \). This completes our definition of \( \mathbb{P} = \mathbb{P}_{\kappa^+} \) depending on the arbitrary bookkeeping function \( F \).

Even though the following remark has been used already, we isolate it here for future use.

**Remark 2.1.** Tracing back the statement of the formula \( \psi \) as well as the coding apparatus involved one can see that if \( N, M \) are suitable models such that \( (\gamma^+)^N = (\gamma^+)^M \), \( (\gamma^{++})^M = (\gamma^{++})^N \), and \( S, X \subset (\gamma^+)^M \) are elements of \( M \cap N \), then \( M \models \psi(\gamma^+, \gamma^{++}, \mu, S, X) \) iff \( N \models \psi(\gamma^+, \gamma^{++}, \mu, S, X) \) for any \( \mu < (\gamma^{++})^M \).

**Lemma 2.2.** The poset \( \mathbb{P} \) is \(< \kappa \) distributive.

Before passing to the proof of Lemma 2.2 we shall introduce some notation. Let \( D_\delta \) be the set of conditions \( p \in \mathbb{P}_\delta \) such that

- for all \( \xi \) of the form \( \rho \cdot \alpha + \zeta \), where \( \zeta \in \{ \kappa, \kappa+1 \} \), we have \( p(\xi) = \langle s_\xi, s_\xi^* \rangle \) for some \( s_\xi^* \in [\kappa^+]^{< \kappa} \) and \( s_\xi \in [\kappa]^{< \kappa} \);
- for all \( \xi \) of the form \( \rho \cdot \alpha + \kappa + 2 \) we have \( p(\xi) = \check{r} \) for some \( r : |r| \to 2 \);
- \( \models \xi p(\xi) \in \hat{Q}_\xi \) for all \( \xi \in \text{supp}(p) \).
If $Q$ is a poset, $q \in Q \in N$, then we say that $q$ is \emph{strongly $(N, Q)$-generic} if for every open dense subset $O$ of $Q$ which is an element of $N$ there exists $p \in O \cap N$ such that $q \leq p$.

\textbf{Proof of Lemma 2.2.} We shall prove by induction on $\xi < \kappa^+$ that $P_\xi$ has some property which is formally stronger than $(< \kappa)$ distributivity and that $D_\xi$ is dense in $P_\xi$. In order to formulate this property we shall introduce some auxiliary notions.

Let us fix some large enough regular cardinal $\theta$ and some large $n \in \omega$. Given a set $X \in L_\theta$, let $N_0$ be the least $\Sigma_n$-elementary submodel of $L_\theta$ such that $\{X\} \cup \{\gamma + 1\} \subset N_0$. The least means here that $N_0$ is the closure of $\{X\} \cup \{\gamma + 1\}$ with respect to all $\Sigma_n$ Skolem functions given by the well-ordering $<_L$ of $L_\theta$. Suppose that for some $\zeta < \kappa$ we have already constructed an increasing chain $(N_\zeta : \zeta < \xi)$ of $\Sigma_n$-elementary submodels of $L_\theta$. If $\zeta$ is limit then we set $N_\zeta = \bigcup_{\zeta < \xi} N_\zeta$. If $\zeta = \zeta_0 + 1$ let $N_\zeta$ be the minimal $\Sigma_n$-elementary submodel of $L_\theta$ such that $N_{\zeta_0} \in N_\zeta$. This completes the construction of the sequence $\langle N_\zeta : \zeta < \kappa \rangle$ which will be called the \textit{minimal sequence generated by $X$} throughout the proof.

By induction on $\xi < \kappa^+$ we shall show that $D_\xi$ is dense in $P_\xi$, and

$$(\hat{\xi}) \text{ for every } q \in P_\xi \text{ and } X \in L_\theta \text{ there exists a condition } q' \leq q \text{ which is strongly } (N_\zeta, P_\xi)\text{-generic for all limit } \zeta \leq \gamma, \text{ where } \langle N_\zeta : \zeta < \kappa \rangle \text{ is the minimal sequence generated by } \{q, X\}.$$ 

Notice that if $X = \langle B_\zeta : \zeta < \gamma \rangle$ is a sequence of open dense subsets of $P_\xi$, then it follows from the above that $q' \in \bigcap_{\zeta < \gamma} B_\zeta$, and hence $(\hat{\xi})$ implies the $(< \kappa)$ distributivity of $P_\xi$.

$(\hat{\xi})_0$ is vacuously true. So let us consider three non-trivial cases: $\xi$ is a successor ordinal, $\xi$ is limit of cofinality at most $\gamma$, and $\xi$ is limit of cofinality $\kappa$. The latter two cases will be addressed on pages 9 and 10, respectively.

1. $\xi = \zeta_0 + 1$. Let us write $\xi$ in the form $\rho \cdot \alpha + \iota$ for some $\iota < \rho$. If $\iota \leq \kappa + 1$ then $Q_{\xi_0}$ is a $P_{\xi_0}$-name for a $(< \kappa)$ closed poset, which makes this case straightforward. So let us assume that $\iota = \kappa + 2$, i.e., $\xi = \rho \cdot \alpha + \kappa + 2$.

First we shall prove that $P_\xi$ is $(< \kappa)$ distributive. Let us denote by $\mu$ the ordinal $\rho \cdot \alpha + \kappa$ and fix a collection $X = \{Q_{\xi_1} : \zeta < \gamma\}$ of open dense subsets of $P_\xi$ and a condition $q \in P_\xi$. Let also $\langle N_\zeta : \zeta < \kappa \rangle$ be the minimal sequence generated by $\{q, X\}$. We shall show that $1_{P_\mu}$ forces the poset

$$\hat{Q}_\mu := \hat{Q}_\mu * \hat{Q}_{\mu + 1} * \hat{Q}_{\mu + 2} = \hat{Q}_{\rho \cdot \alpha + \kappa} * \hat{Q}_{\rho \cdot \alpha + \kappa + 1} * \hat{Q}_{\rho \cdot \alpha + \kappa + 2}$$

to be $(< \kappa)$ distributive.

Using the inductive assumption we can find a condition $q' \in P_\mu$ such that $q' \leq q \upharpoonright \mu$ and $q'$ is strongly $(N_\zeta, P_\mu)$-generic for all limit $\zeta \leq \gamma$. Let $G$ denote a $P_\mu$-generic filter containing $q'$ and note that $N_\zeta[G] \cap \kappa = N_\zeta \cap \kappa$.

\textsuperscript{5}In this proof we will only use the first $\gamma + 1$ elements of minimal sequences. Longer initial segments of minimal sequences will be considered in the proof of Lemma 2.5.

\textsuperscript{6}Here we have $\xi \in N_0$ because $q \in N_0$ and $\xi$ is the domain of $q$. 

for all limit $\zeta < \gamma$. For every (not necessary limit) $\zeta \leq \gamma$ we shall denote the intersection $N_\zeta \cap \kappa$ by $\kappa_\zeta$. Since $X, \gamma \in N_0$, there exists an enumeration $\langle O_{\zeta+1}^\zeta : \zeta < \gamma \rangle \in N_0$ of $X$. We shall denote $Q_\mu^G$ by $Q_\mu$ and the $\mu$-names $Q_{\mu+1}^G$ and $Q_{\mu+2}^G$ by $Q_{\mu+1}$ and $Q_{\mu+2}$, respectively.

For every $\zeta < \gamma$ we denote by $O_{\zeta+1}^\zeta$ the open dense subset $\{r^G : \exists u \in G \text{ such that } \langle u, \tau \rangle \in O_{\zeta+1}^\zeta \}$ of $\hat{Q}_\mu = \hat{Q}_\mu^G$. Observe that $\langle O_{\eta+1}^\eta : \eta + 1 \leq \zeta \rangle \in N_0[G]$ for all $\zeta < \gamma$. The $(< \kappa)$ distributivity of $\mathbb{P}_\mu$ combined with the $(< \kappa)$ closure of $Q_\mu, Q_{\mu+1}$ implies that the set $U$ of conditions $r \in Q_\mu$ such that $r(\mu), r(\mu+1), r(\mu + 2)$ are of the form $\bar{a}$ for some set $a \in L$ of size $< \kappa$, is dense in $\hat{Q}_\mu$.

Set $p_0 = (q \upharpoonright \{\mu, \mu + 1, \mu + 2\})^G$. From now on we shall work in $L[G]$. The sequence $\langle N_\zeta[G] : \zeta < \gamma \rangle$ will guide our inductive construction of a decreasing sequence $\langle p_\zeta : \zeta < \gamma \rangle$ of conditions in $U$ such that $p_\zeta \in N_{\gamma+1}[G]$ belongs to all $O_{\zeta+1}^\zeta$'s as follows. Let $<_G$ be the canonical wellordering of $L[G]$: $x <_G y$ iff $\tau_x <_L \tau_y$, where $\tau_x, \tau_y$ are the $<_L$-minimal $\mathbb{P}_\mu$-names such that $\tau_x^G = x$ and $\tau_y^G = y$. Suppose that a condition $p_\zeta \in N_{\zeta+1} \cap U$ has been already constructed. Since $Q_\mu \ast Q_{\mu+1}$ is $(< \kappa)$ closed, we can inductively extend $\langle p_\zeta(\mu), p_\zeta(\mu + 1) \rangle$ to a strongly $(N_{\zeta+1}[G], Q_\mu \ast Q_{\mu+1})$-generic in $L[G]$ condition $\langle p_\zeta(\mu, \mu + 1) \rangle \in Q_\mu \ast Q_{\mu+1}$. We shall additionally assume that $\langle p_\zeta(\mu), p_\zeta(\mu + 1) \rangle$ is the $<_G$-minimal condition in $Q_\mu \ast Q_{\mu+1}$ with the properties described above. It follows that we can find $r \in N_{\zeta+1}[G]$ such that $\langle p_\zeta(\mu), p_\zeta(\mu + 1), r \rangle \in O_{\zeta+1}^\zeta$. In addition, we shall assume that $r$ is the $<_G$-minimal extension of $r$ with domain $\kappa_{\zeta+1}$ and such that $r_{\zeta+1} \upharpoonright \{3\eta : \eta < \zeta\} \cap [r, |r| + \gamma[)$ codes a bijection between $\kappa_{\zeta+1}$ and $\gamma$. Letting $p_{\zeta+1}$ be the condition $\langle p_\zeta(\mu), p_\zeta(\mu + 1), r_{\zeta+1} \rangle$, by the construction above we conclude that $p_{\zeta+1} \in N_{\zeta+2}[G] \cap U \cap O_{\zeta+1}^\zeta$.

If $\zeta$ is limit, then we set

$$p_\zeta(\mu) = \langle \bigcup_{\eta < \zeta} s_{\mu, \eta}, \bigcup_{\eta < \zeta} s^*_{\mu, \eta} \rangle \text{ and } p_\zeta(\mu + 1) = \langle \bigcup_{\eta < \zeta} s_{\mu+1, \eta}, \bigcup_{\eta < \zeta} s^*_{\mu+1, \eta} \rangle,$$

where $p_\eta(\mu) = \langle s_{\mu, \eta}, s^*_{\mu, \eta} \rangle$ and $p_\eta(\mu + 1) = \langle s_{\mu+1, \eta}, s^*_{\mu+1, \eta} \rangle$ for all $\eta < \zeta$. In addition, we set $p_\zeta(\mu + 2) = \bigcup_{\eta < \zeta} r_{\eta}$, where $r_{\eta} = p_\eta(\mu + 2)$ for all $\eta < \zeta$. Since $p_\eta$ for $\eta < \zeta$ have been constructed by choosing $<_G$-minimal conditions fulfilling certain requirements, the sequence $\langle p_\eta : \eta < \zeta \rangle$ is an element of $N_{\zeta+1}[G]$, and hence $p_\zeta \in N_{\zeta+1}[G]$ as well.

We claim that $p_\zeta \in \hat{Q}_\mu$. Observe that $\langle p_\zeta(\mu), p_\zeta(\mu + 1) \rangle \in Q_\mu \ast Q_{\mu+1}$ by the $(< \kappa)$ closeness of the latter poset. It suffices to show that $\langle p_\zeta(\mu), p_\zeta(\mu + 1) \rangle \Vdash p_\zeta(\mu + 2) \in Q_{\mu+2}$. Let $p_\zeta(\mu) = \langle s_{\mu, \zeta}, s^*_{\mu, \zeta} \rangle$, $p_\zeta(\mu + 1) = \langle s_{\mu+1, \zeta}, s^*_{\mu+1, \zeta} \rangle$, and $p_\zeta(\mu + 2) = r_{\zeta}$. Notice that the condition $\langle p_\zeta(\mu), p_\zeta(\mu + 1) \rangle$ is strongly $(N_\zeta[G], Q_\mu \ast Q_{\mu+1})$-generic in $L[G]$. This means that if $H := H(\mu) \ast H(\mu + 1)$ is a $Q_\mu \ast Q_{\mu+1}$-generic filter over $L[G]$ containing $\langle p_\zeta(\mu), p_\zeta(\mu + 1) \rangle$, then the isomorphism $\pi$ of the transitive collapse $\tilde{N}_\zeta[g]$ of $N_\zeta[G]$, onto $N_\zeta[G]$, extends
to an elementary embedding from

$$
\bar{N}_\zeta := \bar{N}_\zeta[\bar{g} \ast \bar{h}(\bar{\mu}) \ast \bar{h}(\bar{\mu} + 1)]
$$

into $L_\theta[G][H]$. Here $\bar{\mu} = \pi^{-1}(\mu)$, $\bar{h}(\bar{\mu})$ is the $\pi^{-1}(\mathbb{Q}_\mu)$-generic filter over $\bar{N}_\zeta[\bar{g}]$ determined by $p_\zeta(\mu)$, i.e., $\bar{h}(\bar{\mu})$ consists of the images under $\pi^{-1}$ of all conditions in $\mathbb{Q}_\mu$, which are weaker than $p_\zeta(\mu)$ and belong to $\bar{N}_\zeta[G]$. $\bar{h}(\bar{\mu} + 1)$ is defined in the same way.

By the genericity of $H$ we know that, letting $X_0^0$ and $X_\alpha^1$ be the unions of the first coordinates of elements of $H(\mu)$ and $H(\mu + 1)$, respectively, property $(\ast)_\alpha$ holds. By elementarity we have that $N_\zeta$ is a suitable model and $\bar{N}_\zeta \models \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), \pi^{-1}(F(\alpha)^G), x_0^0)$, where $x_0^0$ and $x_\alpha^1$ are the unions of the first coordinates of elements of $\bar{h}(\bar{\mu})$ and $\bar{h}(\bar{\mu} + 1)$ (equivalently, the first coordinates of $p_\zeta(\mu)$ and $p_\zeta(\mu + 1)$, respectively). Observe that by the construction of $\mathbb{P}$ we have $\bar{N}_\zeta = \bar{N}_\zeta[\bar{g}, x_0^0, x_\alpha^1]$ and hence $\bar{N}_\zeta[\bar{g}, x_0, x_1] \models \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), \pi^{-1}(F(\alpha)^G), x_0^0)$.

Let $M$ be any suitable model containing $\pi_\zeta$ and such that $(\gamma^+)_M = |\pi_\zeta|$, which is equal to $\kappa \cap N_\zeta = \kappa_\zeta$. We have to show that $M \models \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), F(\alpha)^G \cap \kappa_\zeta, x_0^0)$. Let us denote by $\nu$ and $\lambda$ the intersection $M \cap \text{Ord}$ and $\bar{N}_\zeta \cap \text{Ord}$, respectively. Three cases are possible.

**Case a.** $\nu > \lambda$. Since $N_\zeta$ was chosen to be the least sufficiently elementary submodel of $L_\theta[G]$ containing certain objects, it follows that $\kappa_\zeta = (\gamma^+)_M$ is collapsed to $\gamma$ in $L_\nu$, and hence this case cannot happen.

More precisely, $L_\nu$ can compute (and hence contains) the sequence $\langle \pi^{-1}[N_\eta] : \eta < \zeta \rangle$. Indeed, $\bar{N}_\zeta \in L_\nu$ since $\bar{N}_\zeta = L_\xi$, $\pi^{-1}[N_\eta] = \bigcup_{\eta' < \eta} \pi^{-1}[N_{\eta'}]$ for limit $\eta < \zeta$, and $\pi^{-1}[N_{\eta+1}]$ is the closure of $\{\pi^{-1}[N_\eta] \}_{\eta < \zeta}$ under the $\Sigma_n$ Skolem functions of $L_\xi$, and these are elements of $L_\nu$. Therefore $L_\nu$ contains the sequence $\langle N_\eta : \eta < \zeta \rangle$, where $N_\eta$ is the Mostowski collapse of $N_\eta$ (the Mostowski collapse of $N_\eta$ coincides with that of $\pi^{-1}[N_\eta]$), and hence $\langle \kappa_\eta = (\gamma^+)^{N_\eta} : \eta < \zeta \rangle \in L_\nu$, whereas the latter sequence is cofinal in $\kappa_\zeta$.

**Case b.** $\nu = \xi$. Since $(\gamma^+)^{N_\eta}[\bar{g}, x_0^0, x_\alpha^1] = (\gamma^+)_M$ and $(\gamma^{++})^{N_\eta}[\bar{g}, x_0^0, x_\alpha^1] = (\gamma^{++})_M$, we conclude that $M \models \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), F(\alpha)^G \cap \kappa_\zeta, x_0^0)$, see Remark 2.1.

**Case c.** $\nu < \xi$. In this case $M_1 := L_\nu[x_0^0, x_\alpha^1]$ is an element of $\bar{N}_\zeta[\bar{g}, x_0^0, x_\alpha^1]$. Since $L_\theta[G][H]$ satisfies $(\ast)_\alpha$, by elementarity so does the model $\bar{N}_\zeta[\bar{g}, x_0^0, x_\alpha^1]$, with $X_0^0$, $X_\alpha^1$, and $\alpha$ replaced by $x_0^0$, $x_\alpha^1$, and $\pi^{-1}(\alpha)$, respectively. In particular, $M_1 \models \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), F(\alpha)^G \cap \kappa_\zeta, x_\alpha^1)$. Applying Remark 2.1 we conclude that $M \models \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), F(\alpha)^G \cap \kappa_\zeta, x_\alpha^1)$, which finishes our proof of $p_\zeta \in \bar{\mathbb{Q}}_\mu$ and hence completes the construction of the sequence $\langle p_\zeta : \zeta \leq \gamma \rangle$.

By the construction we have $p_\gamma \in \bigcap_{\zeta < \gamma} \text{Ord}^{\zeta+1} \cap N_{\gamma+1}[G]$, and hence $\bar{\mathbb{Q}}_\mu$ as well as $\mathbb{P}_\zeta$ are $(< \kappa)$-distributive. Let $\tau$ be a $\mathbb{P}_\mu$-name such that $\tau^G = p_\gamma$ and for every $\zeta < \gamma$ let $q_\zeta \in G$ be such that $q_\zeta \leq q \restriction \mu$ and $\langle q_\zeta, \tau \rangle \in O_{\zeta+1}$.
Since $\mathbb{P}_\mu$ is $(< \kappa)$ distributive, there exists $q'' \in G$ such that $q'' \leq q_\zeta$ for all $\zeta$. In addition, we can assume that $q'' \in D_\mu$ and it forces all coordinates of $\tau$ to be equal to certain ground model objects. It follows from the above that $q \geq (q'', \tau) \in \cap_{\xi \leq \gamma} O_{\xi+1} \cap D_\xi,$ and hence $D_\xi$ is dense in $\mathbb{P}_\xi.$ Combined with the following claim this implies $(\dot{\tau}_\xi)$ and thus completes the successor case.

**Claim 2.3.** Let $\beta < \kappa^+.$ If $\mathbb{P}_\beta$ is $(< \kappa)$ distributive and $D_\beta$ is dense, then $(\dot{\tau}_\beta)$ holds.

**Proof.** Let $q \in \mathbb{P}_\beta,$ $X \in L_0,$ and $(N_\zeta : \zeta < \kappa)$ be the minimal sequence generated by $\{q, X\}.$ We need to find a condition $q' \leq q$ which is strongly $(N_\zeta, \mathbb{P}_\beta)$-generic for all limit $\zeta \leq \gamma.$

Set $p_0 = q$ and assume that conditions $(p_\eta : \eta < \zeta)$ have already been defined for some $\eta \leq \gamma$ so that $p_\eta \in N_{\eta+1} \cap D_\beta$ for all $\eta < \zeta.$ If $\zeta = \eta + 1,$ then $p_\eta$ is the $<_{L}$-minimal condition extending $p_\eta$ such that $p_\eta \in D_\beta$ and it belongs to the intersection of all open dense subsets of $\mathbb{P}_\beta$ which are elements of $N_\zeta.$ Since $N_\zeta \subseteq N_{\zeta+1},$ we have $p_\xi \in N_{\zeta+1}$ as well, as $\beta$ belongs to $N_0.$ If $\zeta$ is limit, then using the fact that $p_\eta \in D_\beta$ for all $\eta < \zeta$ we can define $p_\xi$ to be the “coordinatwise” union of $p_\eta$ over $\eta < \zeta.$ More precisely, for $\xi \in \cup_{\eta < \zeta} \text{supp}(p_\eta)$ we set

$$p_\xi(\eta) = \langle \bigcup_{\eta < \zeta} s_{\xi, \eta}, \bigcup_{\eta < \zeta} s^*_{\xi, \eta} \rangle$$

and

$$p_\xi(\xi) = \langle \bigcup_{\eta < \zeta} r_{\xi, \eta}, \bigcup_{\eta < \zeta} r^*_{\xi, \eta} \rangle,$$

where $p_\eta(\eta) = \langle s_{\xi, \eta}, s^*_{\xi, \eta} \rangle$ for all $\eta < \zeta$ provided that $\xi \in \{\rho \cdot \iota + \kappa, \rho \cdot \iota + \kappa + 1\}$ for some $\iota,$ and $p_0(\eta) = r_{\xi, \eta}$ for all $\eta < \zeta$ if $\xi$ is of the form $\rho \cdot \iota + \kappa + 2.$ For $\xi \in \cup_{\eta < \zeta} \text{supp}(p_\eta),$ we denote by $p_\xi(\eta)$ the $\mathbb{P}_\xi$-name $\tau$ which is forced by $1_{\mathbb{P}_\xi}$ to be the union of $p_\eta(\xi)$ over $\eta < \zeta.$

Since $p_\eta$ for $\eta < \zeta$ have been constructed by choosing $<_G$-minimal conditions fulfilling certain requirements, the sequence $(p_\eta : \eta < \zeta)$ is an element of $N_{\zeta+1},$ and hence $p_\xi \in N_{\zeta+1}$ as well. Thus, once we know that $p_\xi$ is a condition in $\mathbb{P}_\beta,$ it is a consequence from its definition that $p_\xi \in D_\beta \cap N_{\zeta+1}.$ In order to show that $p_\xi \in \mathbb{P}_\beta$ it is enough to establish by induction on $\xi \in \text{supp}(p_\xi)$ that $p_\xi \vert \xi \in \mathbb{P}_\xi.$ The only non-trivial case here is when $\xi$ has the form $\rho \cdot \alpha + \kappa + 2.$ Assuming that $p_\zeta \vert \rho \cdot \alpha + \kappa + 2 \in \mathbb{P}_\rho \alpha + \kappa + 2$ for some $\alpha,$ the equation $p_\zeta \vert \rho \cdot \alpha + \kappa + 2 \models p_\zeta(\rho \cdot \alpha + \kappa + 2) \in \mathbb{Q}_{\rho \alpha + \kappa + 2}$ can be established in the same way as above, using the fact that $p_\xi \vert \rho \cdot \alpha + \kappa + 2$ is strongly $(N_\eta, \mathbb{P}_{\rho \alpha + \kappa + 2})$-generic for all $\eta < \zeta$ and considering three cases depending on the height of a suitable model under consideration. It suffices to note that $q' = p_\gamma$ is as required. \hfill \square

2. $\xi$ is a limit ordinal of cofinality $\leq \gamma.$ Here we shall work in $L.$ We need the following auxiliary statement.

---

\footnote{We assume here that if $\xi \notin \text{supp}(p)$ then $p(\xi) = \langle \emptyset, \emptyset \rangle$ provided that $\xi = \rho \cdot \alpha + \zeta$ for some $\zeta \in \{\kappa, \kappa + 1\}$ and $p(\xi) = \emptyset$ otherwise.}
Claim 2.4. Suppose that \(( \dot{\gamma}_\beta \) holds and \( D_\beta \) is dense in \( \mathbb{P}_\beta \) for each \( \beta < \xi \), where \( \xi \) is a limit ordinal of cofinality \( \leq \gamma \). Then for every \( p \in \mathbb{P}_\xi \) and \( X_0 \in L_\gamma \) there exists an extension \( q \in D_\xi \cap N_{\gamma, \text{cof}(\xi)+1} \) of \( p \) such that \( q \upharpoonright \beta \) is strongly \( (N_{\gamma, \text{cof}(\xi)}, \mathbb{P}_\beta) \)-generic for all \( \beta < \xi \), where \( (N_\xi : \xi < \kappa) \) is the minimal sequence generated by \( \{p, X_0\} \).

Proof. Since \( p \in N_0 \), we have \( \xi \in N_0 \), and hence \( N_0 \) contains a continuous sequence \( \xi_0 < \xi_1 < \ldots \) cofinal in \( \xi \) of order type \( \text{cof}(\xi) \). Set \( p_0 = p \upharpoonright \xi_0 \) and assume that conditions \( (p_\eta : \eta < \zeta) \) have already been defined for some \( \zeta \leq \text{cof}(\xi) \) so that

(i) \( p_\eta \in N_{\gamma, \eta+1} \cap D_\xi \eta \) for all \( \eta < \zeta \);
(ii) \( p_\eta \upharpoonright \xi_\eta \leq p_\eta_0 \) for all \( \eta_0 < \eta < \zeta \);
(iii) \( p_\eta \upharpoonright \beta \) is strongly \( (N_{\gamma, \eta}, \mathbb{P}_\beta) \)-generic for all \( \eta < \zeta \) and \( \beta \leq \xi_\eta \).

Notice that (iii) is vacuous unless \( \beta \) is an element of \( N_{\gamma, \eta} \) because otherwise \( \mathbb{P}_\beta \not\subseteq N_{\gamma, \eta} \). If \( \zeta = \eta + 1 \), then let \( p_\zeta \) be the \(< \xi \)-minimal condition extending \( p_\eta, p_0 \upharpoonright [\xi_\eta, \xi_\zeta) \) so that (i)-(iii) hold. Its existence is guaranteed by (iv)_\zeta applied to \( X = N_{\gamma, \eta} \) and the inductive assumption that \( D_{\xi_\zeta} \) is dense in \( \mathbb{P}_{\xi_\zeta} \).

If \( \zeta \) is limit, then we define \( p_\zeta \) in exactly the same way as in Claim 2.3. In addition, almost literal repetition of the proof given in Claim 2.3 gives that (i)-(iii) are satisfied for all \( \eta, \eta_0, \eta_1 < \zeta \), the essential part here being to show that \( p_\zeta \in \mathbb{P} \). It suffices to set \( q = p_{\text{cof}(\xi)} \).

We are in a position now to prove the \( (< \kappa) \) distributivity of \( \mathbb{P}_\xi \). Moreover, the construction below gives a condition in \( D_\xi \) which lies in the intersection of \( \gamma \) many open dense subsets of \( \mathbb{P}_\xi \), and consequently it establishes that \( D_\xi \) is dense in \( \mathbb{P}_\xi \). Combined with Claim 2.3 this will complete the proof that the inductive assumption holds for \( \xi \).

Given \( p \in \mathbb{P}_\xi \) and fewer than \( \kappa \) open dense sets \( \{O_{\xi+1} : \zeta < \gamma\} \), let \( (N_\zeta : \zeta < \kappa) \) be the minimal sequence generated by \( \{p, (O_{\zeta+1} : \zeta < \gamma)\} \). Set \( \gamma' = \gamma \cdot \text{cof}(\xi), p = p_0 \), and assume that conditions \( (p_\eta : \eta < \zeta) \) have already been defined for some \( \zeta \leq \gamma \) so that

(iv) \( p_\eta \in N_{\gamma', \eta+1} \cap D_\xi \) for all \( \eta < \zeta \);
(v) \( p_\eta \leq p_\eta_0 \) for all \( \eta_0 < \eta < \zeta \);
(vi) \( p_\eta \upharpoonright \beta \) is strongly \( (N_{\gamma', \eta}, \mathbb{P}_\beta) \)-generic for all \( \eta < \zeta \) and \( \beta < \xi \); and
(vii) \( p_{\eta+1} \in O_{\eta+1} \) for all \( \eta + 1 < \zeta \).

If \( \zeta = \eta + 1 \), let \( p_\zeta \) be the \(< \xi \)-minimal condition extending \( p_\eta \) so that (iv)-(vii) hold for all \( \eta, \eta_0, \eta_1 < \zeta \). Its existence is guaranteed by Claim 2.4 applied to \( X = N_{\gamma', \eta} \) and \( p_\eta \). If \( \zeta \) is limit, then we define \( p_\zeta \) in exactly the same way as in Claim 2.3. Once we know that \( p_\zeta \in \mathbb{P}_\xi \), the verification of (iv)-(vi) is straightforward, whereas (vii) is vacuous. The verification that \( p_\zeta \in \mathbb{P}_\xi \) is exactly the same as in Claim 2.3, which in turn uses of course the ideas from the successor case. It suffices to note that \( p_\zeta \in \bigcap_{\zeta < \gamma} O_{\zeta+1} \).

3. \( \xi \) is a limit ordinal of cofinality \( \kappa \). Here we shall also work in \( L \). Given \( p \in \mathbb{P}_\xi \) and fewer than \( \kappa \) open dense sets \( \{O_{\xi+1} : \zeta < \gamma\} \), let \( (N_\zeta : \zeta < \kappa) \) be
the minimal sequence generated by \( \{ p_1 : O_{z+1} : z < \gamma \} \). Set \( \xi = \sup(N \cap \xi) \) for all \( z < \kappa, p = p_0 \), and assume that conditions \( \langle p_\eta : \eta < \zeta \rangle \) have already been defined for some \( \zeta \leq \gamma \) so that

(i) \( p_{\eta} \in N_{\gamma \cdot \eta + 1} \cap D_\xi \) for all \( \eta < \zeta \);

(ii) \( p_{\eta} \leq p_{\eta_0} \) for all \( \eta < \eta_0 < \zeta \);

(iii) \( p_{\eta} \upharpoonright \beta \) is strongly \( (N_{\gamma \cdot \eta}, \mathbb{P}_\beta) \)-generic for all \( \eta < \zeta \) and \( \beta < \xi_{\gamma \cdot \eta} \); and

(iv) \( p_{\eta + 1} \in \mathcal{O}_{\eta + 1} \) for all \( \eta + 1 < \zeta \).

Assume first that \( \zeta = \eta + 1 \). Let \( p_{\eta + 1}^\prime \) be the \( \langle L \rangle \)-minimal condition extending \( p_\eta \) so that \( p_{\eta + 1}^\prime \in \mathcal{O}_{\eta + 1} \). Then \( p_{\eta + 1} \in N_{\gamma \cdot \eta + 1} \). Let \( r_{\eta + 1} \in \mathcal{D}_{\xi_{\gamma \cdot (\eta + 1)}} \) be the \( \langle L \rangle \)-minimal element of \( \mathcal{D}_{\xi_{\gamma \cdot (\eta + 1)}} \) such that \( r_{\eta + 1} \upharpoonright \beta \) is strongly \( (N_{\gamma \cdot (\eta + 1)}, \mathbb{P}_\beta) \)-generic for all \( \beta < \xi_{\gamma \cdot (\eta + 1)} \). Its existence follows from the density of \( \mathcal{D}_{\xi_{\gamma \cdot (\eta + 1)}} \) and \( \langle \check{\xi}_{\gamma \cdot (\eta + 1)} \rangle \). Note that \( r_{\eta + 1}^\prime \in N_{\gamma \cdot (\eta + 1) + 1} \). Now set

\[
p_{\eta + 1} = r_{\eta + 1}^\prime \upharpoonright \mathcal{O}_{\eta + 1} \upharpoonright [\xi_{\gamma \cdot (\eta + 1)}, \xi].
\]

It is clear that \( p_{\eta + 1} \in N_{\gamma \cdot (\eta + 1) + 1} \) and conditions (ii)-(iv) hold. Since \( p_{\eta + 1} \in N_{\gamma \cdot \eta + 1} \), we have \( \text{sup} \langle p_{\eta + 1} \rangle \subset N_{\gamma \cdot \eta + 1} \cap \xi \subset \xi_{\gamma \cdot (\eta + 1)} \). Combining this with \( r_{\eta + 1}^\prime \in \mathcal{D}_{\xi_{\gamma \cdot (\eta + 1)}} \) we conclude that \( p_{\eta + 1} \in D_\xi \).

If \( \zeta \) is limit, then we define \( p_\lambda \) in exactly the same way as in Claim 2.3. Once we know that \( p_\lambda \in P_\xi \), the verification of (i)-(iii) is straightforward, whereas (iv) is vacuous. The verification that \( p_\lambda \in P_\xi \) is exactly the same as in Claim 2.3. It suffices to note that \( p_\gamma \in D_\xi \cap \bigcap_{\zeta < \gamma} O_{\xi \cdot \zeta + 1} \).

As in the case of \( \text{cof} \langle \xi \rangle \leq \gamma \) we have established the existence of a condition in \( D_\xi \) which lies in the intersection of given \( \gamma \) many open dense subsets of \( P_\xi \). Combined with Claim 2.3 this completes the proof that the inductive assumption holds for \( \xi \). \( \square \)

**Lemma 2.5.** Let \( p \in P_\xi \) for some \( \xi < \kappa^+ \) and \( \tau \) be a \( P_\xi \)-name. If \( p \Vdash_{P_\xi} \tau \) is a stationary subset of \( \kappa^+ \), then \( p \Vdash \tau \) is a stationary subset of \( \kappa^+ \).

In other words, every tail of the iteration \( \langle P_\xi, \check{Q}_\xi : \xi < \kappa^+ \rangle \) preserves stationary subsets of \( \kappa \).

**Proof.** In light of Lemma 2.2 we may restrict our attention to conditions \( p \in D_\xi \). Given \( p \in D_\xi \) and \( \zeta \in \text{sup} \langle p \rangle \), from now on we shall write simply \( p(\zeta) = a \) instead of \( p(\zeta) = \check{a} \).

Let \( \xi < \kappa^+ \) and \( G \) be a \( P_\xi \)-generic filter over \( L \). Note that \( L[G] \) has the same sequences of ordinals of length \( \kappa < \kappa \) as \( L \). From now on we shall work in \( L[G] \). Set \( P' = P_\xi^G \), \( D' = \{ p \upharpoonright [\xi, \kappa^+] : p \in D_\kappa, p \upharpoonright [\xi, \kappa^+] \in G \}, P'_\beta = P_{\xi(\beta)}^G \), and \( D'_\beta = \{ p \upharpoonright [\xi, \beta] : p \in D_\beta, p \upharpoonright [\xi, \beta] \in G \} \).

Fix a stationary subset \( S \) of \( \kappa^+ \) in \( L[G] \). Given any \( p \in P' \) and \( P' \)-name \( \check{C} \) such that \( p \Vdash \check{C} \) is a club in \( \kappa \), we shall construct \( q \in P' \) stronger than \( p \) such that \( q \Vdash \check{C} \cap S \neq \emptyset \).

Let us fix some large enough regular cardinal \( \theta \) and some large enough \( n \). Given a set \( X \in L_\theta[G] \), let \( N_0 \) be the least \( \Sigma_n \)-elementary submodel of \( L_\theta[G] \) such that \( \{ X \} \cup (\gamma + 1) \subset N_0 \). Least means here that \( N_0 \) is the closure of \( \{ X \} \cup (\gamma + 1) \) with respect to all \( \Sigma_n \) Skolem functions given by the
well-ordering $<_G$ of $L_\theta[G]$. Suppose that for some $\zeta < \kappa$ we have already constructed an increasing chain $(N_\zeta : \varepsilon < \zeta)$ of $\Sigma_n$-elementary submodels of $L_\theta[G]$. If $\zeta$ is limit then we set $N_\zeta = \bigcup_{\varepsilon < \zeta} N_\varepsilon$. If $\zeta = \zeta_0 + 1$ we let $N_\zeta$ be the minimal $\Sigma_\kappa$-elementary submodel of $L_\theta[G]$ such that $(\gamma + 1) \cup \{N_\theta \} \subset N_\zeta$. This completes the construction of the sequence $(N_\zeta : \zeta < \kappa)$ which will be called the $G$-minimal sequence generated by $X$ throughout the proof.

Let $\check{C} = \langle C_\varepsilon : \varepsilon \in \text{Lim}(\kappa) \rangle$ be a $\square_\gamma$ sequence and $(N_\zeta : \zeta < \kappa)$ be the $G$-minimal sequence generated by $\{P', G, S, \check{C}, \check{C}, \ldots \}$. Set $\kappa_\zeta = N_\zeta \cap \kappa$.

Since $S$ is stationary, we can find a limit ordinal $\zeta < \kappa$ such that $\kappa_\zeta \in S$. We shall find $q \leq p$ such that $q \models \kappa_\zeta \in \check{C}$. Set $\eta = \text{cof}(\zeta)$. Two cases are possible: $\eta > \omega$ and $\eta = \omega$. The latter one will be addressed on page 14.

1. $\eta > \omega$. Letting $(\kappa_\zeta : \beta < \eta)$ be the increasing enumeration of $\{\kappa_\zeta \} \cup \{\{\kappa_\nu : \nu < \zeta \} \cap C_\kappa \}$, we shall construct a decreasing sequence of conditions $(p_\beta : \beta < \eta)$ such that

(a) $p_{\beta + 1} \in \{O : O \in N_{\beta + 1} \}$ is open dense in $\mathbb{P}'$ for all $\beta < \eta$;

(b) $p_\beta \in N_{\zeta_\beta + 1} \cap D'$ for all $\beta < \eta$;

(c) For every $\beta < \eta$, $\lambda \in \text{supp}(p_\beta)$ of the form $\rho \cdot \alpha + \kappa + 2$, and $\nu < \zeta$ if $\kappa_\nu \in [p_\beta(\lambda)]$, then $p_\beta(\lambda)(\kappa_\nu) = 0$ if and only if $\nu \in \{\zeta_\mu : \mu < \eta\}$.

Then as a consequence of (a) and (b) we shall have

(d) $p_{\beta + 1} \models [\kappa_{\zeta_\beta}, \kappa_{\beta + 1}] \cap \check{C} \neq \emptyset$ for all $\beta < \eta$.

for all $\beta < \eta$. Let $p_0 = p$ and suppose that for some $\epsilon < \eta$ we have already constructed a decreasing sequence $(p_\beta : \beta < \epsilon)$ satisfying (a)-(c).

If $\epsilon = \beta + 1$ for some $\beta$, let $p_{\beta + 1}'$ be the $<G$-least condition $u \leq p_\beta$ in $D'$ such that for every $\lambda \in \text{supp}(u)$ of the form $\rho \cdot \alpha + \kappa + 2$ the following conditions hold:

(e) $\kappa_\beta \in [u(\lambda)]$;

(f) If $\kappa_\nu \in [u(\lambda)]$ for some $\nu < \zeta$, then $u(\lambda)(\kappa_\nu) = 0$ if and only if $\nu \in \{\zeta_\mu : \mu < \eta\}$;

(g) $[u(\lambda)] = [p_\beta(\lambda)] + \gamma$ and $u(\lambda) \upharpoonright ([p_\beta(\lambda)] + \gamma) \cap \{3\varepsilon : \varepsilon < \kappa \}$ is the $<G$-least code for a bijection between $\gamma$ and $\kappa_{\beta + 1}$.

It is clear that $p_{\beta + 1}' \in N_{\beta + 1}$. Since (g) makes the third condition of the definition of $\check{Q}_\lambda$ for $\lambda$ of the form $\rho \cdot \alpha + \kappa + 2$ vacuous for ordinals between $[p_\beta(\lambda)]$ and $\kappa_{\beta + 1} + \gamma$, we can find a condition $u \leq p_{\beta + 1}'$ in $D' \cap N_{\beta + 1}$ such that for every $\lambda \in \text{supp}(u)$ as above the following conditions hold:

(h) $\kappa_{\beta + 1} \in [u(\lambda)]$, and

(i) $u(\lambda)(\kappa_\nu) = 0$ if and only if $\nu \in \{\zeta_\mu : \mu < \eta\}$.

Let $p_{\beta + 1}''$ be the $<G$-least $u$ as above. Then $p_{\beta + 1}'' \in N_{\beta + 1}$. Now let $p_{\beta + 1}$ be the $<G$-least condition $w \in D'$ below $p_{\beta + 1}''$ so that $w \in \{O : O \in N_{\beta + 1}\}$ is open dense in $\mathbb{P}'$. It follows that $p_{\beta + 1}$ satisfies conditions (a)-(c) (and hence also (d)) with $\beta + 1$ instead of $\beta$.

If $\epsilon$ is limit then we define $p_\epsilon$ to be the “coordinatewise” union of $\{p_\beta : \beta < \epsilon\}$, see Claim 2.3. It follows from the construction of the sequence
\[ \langle \beta : \beta < \epsilon \rangle \] that \( p_\epsilon \in N_{\zeta +1} \). Indeed, \( p_\epsilon \) is determined by the sequence \( \langle \beta : \beta < \epsilon \rangle \) which has been constructed using \( C_{\kappa_\zeta} \cap \{ \kappa_v : v < \zeta_\epsilon \} \) by always choosing \( \kappa_0 \)-minimal conditions with certain properties. Since \( C_{\kappa_\zeta} \cap \{ \kappa_v : v < \zeta_\epsilon \} \) is \( \kappa_0 \)-minimal with \( \kappa_0 \) by the choice of \( \bar{C} \), we conclude that \( p_\epsilon \in N_{\zeta +1} \).

In order to show that \( p_\epsilon \in P' \) it is enough to establish by induction on \( \lambda \in \text{supp}(p_\epsilon) \) that \( p_\epsilon \upharpoonright \lambda \in P'_{\lambda} \). The only non-trivial case here is when \( \lambda \) has the form \( \rho \cdot (\alpha + 1) = \rho \cdot \alpha + \kappa + 3 \) for some \( \alpha \). In this case, assuming that \( p_\epsilon \upharpoonright (\lambda - 1) \in P'_{\lambda - 1} \), the equation

\[ p_\epsilon \upharpoonright (\lambda - 1) \models_{P'_{\lambda - 1}} p_\epsilon(\lambda - 1) \in \bar{Q}_{\lambda - 1} \]

can be established as follows: Given a \( P'_{\lambda - 3} \)-generic filter \( R \ni p_\epsilon \upharpoonright (\lambda - 3) \) over \( L[G] \), the strong \( (N_{\zeta_\epsilon}, P') \)-genericity of \( p_\epsilon \upharpoonright (\lambda - 1) \) (in \( L[G] \)) by the same argument as in item 1 of Lemma 2.2 implies that in \( L[G \ast R] \) we have

\[ \langle p_\epsilon(\lambda - 3), p_\epsilon(\lambda - 2) \rangle^{G \ast R} \models_{(\bar{Q}_{\lambda - 3} \ast \bar{Q}_{\lambda - 2})^{G \ast R}} p_\epsilon(\lambda - 1)^{G \ast R} \in \bar{Q}_{\lambda - 1}^{G \ast R}, \]

which yields \( p_\epsilon \upharpoonright \lambda \in P'_{\lambda} \). The only difference with the proof given in Lemma 2.2 is the case a) where suitable models \( M \) of height \( \text{Ord} \cap M > \text{Ord} \cap N_{\zeta_\epsilon} \) have to be treated (here \( N_{\zeta_\epsilon} \) is the Mostowski collapse of \( N_{\zeta_\epsilon} \)).

Now the sequence \( \{ \kappa_v : v < \zeta_\epsilon \} \) might have length larger than \( \gamma \). However, any such suitable model \( M \) still has a bijection between \( \gamma \) and \( (\gamma^+)_{N_{\zeta_\epsilon}} \) by the fact that \( M \) contains the sequence \( \{ \kappa_v : v < \zeta_\epsilon \} \cap C_{\kappa_\zeta} \) which has length \( \leq \gamma \) and is cofinal in \( \kappa_\zeta_\epsilon \). Since \( (\gamma^+)_{N_{\zeta_\epsilon}} = (\gamma^+)_{M} \) for suitable models as above, the latter is impossible, and hence such suitable models \( M \) are again ruled out.

The following statement completes the informal argument given above.

**Claim 2.6.** Let \( M \) be a suitable model of size \( \gamma \) containing \( p_\epsilon(\lambda - 3), p_\epsilon(\lambda - 2) \) and such that \( \text{Ord} \cap M > \text{Ord} \cap N_{\zeta_\epsilon} \). Then \( M \) contains the sequence \( \{ \kappa_v : v < \zeta_\epsilon \} \).

**Proof.** Let \( H = H(\lambda - 3) \ast H(\lambda - 2) \) be a \( (\bar{Q}_{\lambda - 3} \ast \bar{Q}_{\lambda - 2})^{G \ast R} \)-generic filter over \( L[G \ast R] \) containing \( \langle p_\epsilon(\lambda - 3), p_\epsilon(\lambda - 2) \rangle^{G \ast R} \) and \( \pi : N_{\zeta_\epsilon}[R \ast H] \rightarrow \bar{N} \) be the Mostowski collapsing function. Observe that by elementarity we have

\[ \bar{N} = \pi[N_{\zeta_\epsilon}][\pi(R) \ast \pi(H)] = \pi[N_{\zeta_\epsilon}][x^0_\alpha, x^1_\alpha] = L_{\text{Ord} \cap \bar{N}}[x^0_\alpha, x^1_\alpha], \]

where \( x^0_\alpha \) and \( x^1_\alpha \) are the unions of the first coordinates of all elements of \( \pi(H(\lambda - 3)) \) and \( \pi(H(\lambda - 2)) \) (equivalently, are the first coordinates of \( p_\epsilon(\lambda - 3) \) and \( p_\epsilon(\lambda - 2) \)), respectively. Indeed, letting \( X^0_\alpha \) and \( X^1_\alpha \) be the unions of the first coordinates of all elements of \( H(\lambda - 3) \) and \( H(\lambda - 2) \), we can easily conclude from the definition of \( \bar{P} \) that \( L[G \ast R \ast H] = L[X^0_\alpha, X^1_\alpha] \), and hence also \( N_{\zeta_\epsilon}[R \ast H] = N_{\zeta_\epsilon}[X^0_\alpha, X^1_\alpha] = (N_{\zeta_\epsilon} \cap L)[X^0_\alpha, X^1_\alpha]. \)

Since \( M \ni p_\epsilon(\lambda - 1) = p_\epsilon(\rho \cdot \alpha + \kappa + 2) \) and the latter is of the form \( r : \kappa_\zeta_\epsilon \rightarrow 2 \) such that \( r(3\tilde{\epsilon} + 1) = 1 \) iff \( \iota \in x^0_\alpha \) and \( r(3\tilde{\epsilon} + 2) = 1 \) iff \( \iota \in x^1_\alpha \), we conclude that \( x^0_\alpha, x^1_\alpha \in M \), and consequently

\[ \pi[N_{\zeta_\epsilon}][\pi(R) \ast \pi(H)] = L_{\text{Ord} \cap \bar{N}}[x^0_\alpha, x^1_\alpha] \in M \]
because Ord $\cap \bar{N} < \text{Ord} \cap M$. In $\pi[N_\zeta]$ we have that $\pi[N_{\zeta+1}]$ is the closure of $\{\pi[N_v]\}$ under $\Sigma_n$ Skolem functions of $\pi[N_\zeta]$ with respect to $<_{\pi(G)}$. Thus the sequence $\langle \pi[N_v] : v < \zeta \rangle$ is definable (as a class) over $\pi[N_\zeta]$, and hence the sequence

$$\langle \min(\text{Ord} \setminus \pi[N_v]) : v < \zeta \rangle = \langle \kappa_v : v < \zeta \rangle$$

is definable over $\pi[N_\zeta]$. As a result, this sequence is an element of $M$. \hfill $\Box$

2. $\eta = \omega$. In this case let $C'_\zeta \subset \{\kappa_\mu : \mu < \zeta\}$ be a cofinal subset of $\kappa_\zeta$ of order type $\omega$ which is an element of $N_{\zeta+1}$. Using $C'_\zeta$ instead of $C_\zeta$, we can repeat the argument from case 1 and construct a decreasing sequence $\langle p_\beta : \beta \leq \eta \rangle$ satisfying conditions (a)-(d).

In both of the cases considered above we have $p_\eta \leq p_0 = p$ and $p_\eta$ forces that $\hat{C}$ has nonempty intersection with $[\kappa_{\zeta_0} \wedge \kappa_{\zeta+1}]$ for all $\beta < \eta$, and hence it forces that $\kappa_\zeta = \sup \{\kappa_\beta : \beta < \eta\}$ is an element of $\hat{C}$. Since $\kappa_\zeta \in S$ this completes our proof. \hfill $\Box$ Lemma 2.5

Let us denote by $\text{Supp}_\kappa^+$ the set of all $\xi \in \kappa^+$ of the form $\alpha \cdot \rho + \zeta$ for some $\zeta < \kappa$ and set $\text{Supp}_\kappa = \kappa^+ \setminus \text{Supp}_\kappa^+$.

Let $p, q \in D$. We say that $q \leq^* p$ if $q \leq p$, $\text{supp}_\kappa(p) = \text{supp}_\kappa(q)$ and $q \upharpoonright \text{supp}_\kappa(q) = p \upharpoonright \text{supp}_\kappa(p)$. Suppose that $q \leq p$. We shall define a new condition $\eta p$ called the reduction of $q$ to $p$ by induction as follows. Suppose that $\eta p \upharpoonright \xi$ has been already defined. If $\xi \in \text{Supp}_\kappa$ then $(\eta p)(\xi) = p(\xi)$. If $\xi \in \text{Supp}_\kappa^+$ then $(\eta p)(\xi)$ is a $\mathbb{P}_\xi$-name $\tau$ such that $q \upharpoonright \xi \forces \tau = q(\xi)$ and $u \forces \tau = p(\xi)$ for all $\mathbb{P}_\xi \ni u \leq \eta p \upharpoonright \xi$ which are incompatible with $q \upharpoonright \xi$. A direct verification shows that $\eta p \in \mathbb{P}$ and $q \leq \eta p \leq^* p$.

For a pair $c = (a, b)$ we shall use the following notation: $a = c_0, b = c_1$. From now on we shall consider only conditions $p \in D$ such that $\forces_{\mathbb{P}} p(\xi) \in \hat{Q}_\xi$ for all $\xi \in \text{supp}_\kappa^+(p)$. It is easy to see that for every $q \in D$ there exists $p \in D$ with this property such that $p \leq q \leq p$.

The next lemma shows, in particular, that $\mathbb{P}$ does not collapse $\kappa^+$. Its proof is patterned after that of [5, Proposition 2.3]. Here our choice of the support comes into play.

**Lemma 2.7.** Let $p \in \mathbb{P}$ and $\mu < \kappa^+$ be an ordinal of the form $\rho \cdot \alpha + \zeta$ with $\zeta < \kappa$ such that $p \forces \zeta \notin T(F(\alpha))$. Suppose also that $\hat{C}$ is a $\mathbb{P}$-name for a club in $\kappa^+$. Then there exists $q \leq p$ such that $q \forces_{\mathbb{P}} S_\mu \cap \hat{C} \neq \emptyset$. In particular, if $G$ is a $\mathbb{P}$-generic filter such that $\zeta \notin T(F(\alpha))^G$, then $S_\mu$ remains stationary in $L[G]$.

**Proof.** Without loss of generality we may assume that $p \in D$. Let $\langle M_i : i < \kappa^+ \rangle$ be an increasing chain of elementary submodels of $L_\theta$ of size $\kappa$, where $\theta$ is big enough, such that

(i) $M_i \supset [M_i]^\kappa$ for all $i \in \kappa^+$;
(ii) $M_i = \bigcup_{j \leq i} M_j$ for all $i \in \kappa^+$ of cofinality $\kappa$; and
(iii) $\kappa \cup \{p, \mathbb{P}, \hat{C}, \alpha, \ldots\} \subset M_0$.
Now a standard Fodor argument yields \( i \in \kappa^+ \) such that \( i = M_i \cap \kappa^+ \in S_\mu \) and \( i \not\in S_\xi \) for any \( \xi \in M_i \setminus \{ \mu \} \). Let \( \langle (O_v, \phi_v) : v < \kappa \rangle \in M_\kappa^* \) be a sequence in which all pairs \((O, \phi) \in M_i\) appear cofinally often, where \( O \) is an open dense subset of \( P \) and \( \phi \) is a function of size \( \leq \gamma \) such that dom(\( \phi \)) \( \subseteq i \), \( \phi(\xi) \in [\kappa]^{\leq \gamma} \times [\kappa^+]^{\leq \gamma} \) if \( \xi \) is of the form \( \rho \cdot \beta + \kappa \) or \( \rho \cdot \beta + \kappa + 1 \), and \( \phi(\xi) \in 2^{<\kappa} \) if \( \xi \) is of the form \( \rho \cdot \alpha + \kappa + 2 \). Let also \( \langle i_v : v < \kappa \rangle \) be an increasing sequence of ordinals cofinal in \( i \).

Construct by induction on \( v \) a \( \leq^* \)-decreasing sequence \( \langle q^v : v \leq \kappa \rangle \in D^{\kappa+1} \) such that \( \langle q^v : v < \kappa \rangle \in (D \cap M_\kappa)^\kappa \) as follows. Set \( q^0 = \emptyset \) and suppose that \( \langle q^\eta : \eta < v \rangle \) has been already constructed. If \( v \) is limit then we set \( q^v(\xi) = p(\xi) \) if \( \xi \in \text{Supp}_\kappa \) and let \( q^v(\xi) \) be a \( P_\xi \)-name which is forced by \( q^\eta \) to be the union of all \( q^\eta(\xi), \eta < v \), together with its supremum. Since the \( S_\xi \)'s consist of ordinals of cofinality \( \kappa \) for all \( \xi < \kappa^+ \), we conclude that \( q^v \in P \) provided that \( v < \kappa \). Now suppose that \( v = \eta + 1 \). Let us first consider the case that there exists a condition \( r \in O_\eta \cap D \) stronger than \( q^0 \), 

\[
\text{Claim 2.8. If } \xi \in \text{Supp}_{\kappa^+} \cap M_i \text{ and } q^\kappa \upharpoonright \xi \text{ is } (M_i, P^\xi)\text{-generic, then } q^\kappa \upharpoonright (\xi + 1) \in P_{\xi+1}.
\]

\[\begin{align*}
&\quad \text{(iv) } \text{dom}(\phi_\eta) \subseteq \text{dom}(\psi); \\
&\quad \text{(v) } \vDash \psi(\xi) \leq \phi_\eta(\xi) \text{ for all } \xi \in \text{dom}(\phi_\eta); \\
&\quad \text{(vi) } \psi(\xi)_0 = \phi_\eta(\xi)_0 \text{ for all } \xi \in \text{dom}(\phi_\eta) \text{ of the form } \rho \cdot \beta + \kappa \text{ or } \rho \cdot \beta + \kappa + 1.
\end{align*}\]

In this case we fix such a condition \( r^\eta \in M_i \) and set \( q^\eta_0 = r^\eta q^\eta \). If there is no such condition \( r \) then we set \( q^\eta_0 = q^\eta \). Now let \( q^v \leq^* q^\eta_0 \) be such that for all \( \xi \in \text{Supp}_{\kappa^+} \cap (q^\eta_0) \), \( \vDash \xi \upharpoonright \xi q^\eta(\xi) = q^\eta(\xi) \cup \{ \max(q^\eta(\xi)) + i_v \} \) if \( \xi \in T(F(\beta)) \) and \( q^v(\xi) = \emptyset \) otherwise, where \( \xi = \rho \cdot \beta + \zeta \).

We claim that \( q^\kappa \in P \) and it is \((M_i, P)\)-generic. We shall prove this in several steps.

\[\text{Claim 2.8. If } \xi \in \text{Supp}_{\kappa^+} \cap M_i \text{ and } q^\kappa \upharpoonright \xi \text{ is } (M_i, P^\xi)\text{-generic, then } q^\kappa \upharpoonright (\xi + 1) \in P_{\xi+1}.
\]

\begin{proof}
It suffices to show that \( r \vDash \xi q^\kappa(\xi) \cap S_\xi = \emptyset \) for every \( r \leq q^\kappa \upharpoonright \xi \) which forces \( \zeta \in T(F(\beta)) \), where \( \xi = \rho \cdot \beta + \zeta \). Suppose, contrary to our claim, that there exists \( r \leq q^\kappa \upharpoonright \xi \) such that \( r \vDash \xi \zeta \in T(F(\beta)) \) but

\[r \vDash \xi \bigcup_{\nu < \kappa} q^\nu(\xi) \cup \{ \sup(\bigcup_{\nu < \kappa} q^\nu(\xi)) \} \cap S_\xi \neq \emptyset.\]

Then \( \xi \neq \mu \). Indeed, otherwise \( r \leq q^\kappa \upharpoonright \mu \leq \rho \upharpoonright \mu \), and the latter forces \( \zeta \notin T(F(\alpha)) \) by our assumptions. Thus \( r \vDash \zeta \notin T(F(\alpha)) \), and hence \( r \vDash \zeta \notin T(F(\beta)) \) because \( \langle \xi, \beta, \zeta \rangle = \langle \mu, \alpha, \zeta \rangle \), which contradicts the choice of \( r \).

Without loss of generality we may assume that \( r \vDash \xi \sup(\bigcup_{\nu < \kappa} q^\nu(\xi)) = j \) for some \( j \).

Note that \( j \leq i \) because \( r \) is \((M_i, P^\xi)\)-generic and therefore forces \( \max q^\nu(\xi) < i \) for each \( \nu \). By the definition of the \( q^\nu \)'s we have that \( r \vDash \zeta \max q^\nu(\xi) \geq i_v \) for all \( \nu < \kappa \) and therefore \( i \leq j \), so \( i = j \). But (1) is possible only if \( j \) belongs to \( S_\xi \) and since \( \xi \) belongs to \( M_i \setminus \{ \mu \} \), we have \( j \neq i \) by our choice of \( i \), contradiction.
\end{proof}
Claim 2.9. Suppose that \( j \leq i \) and \( q^\kappa \upharpoonright \xi \) is \((M_i, \mathbb{P}_\xi)\)-generic for all \( \xi < j \).

- If \( j < i \), then \( q^\kappa \upharpoonright j \) is \((M_i, \mathbb{P}_j)\)-generic;
- If \( j = i \), then \( q^\kappa \upharpoonright j = q^\kappa \) is \((M_i, \mathbb{P})\)-generic.

Proof. Let us first consider the case \( j < i \). It follows that \( q^\kappa \upharpoonright j \in \mathbb{P}_j \), the case of a successor \( j \) is handled by Claim 2.8.

Fix an open dense subset \( E \subset M_i \) of \( \mathbb{P}_j \) and \( w \leq q^\kappa \upharpoonright j \). We need to show that there exists \( w_1 \in E \cap M_i \) such that \( w \) and \( w_1 \) are compatible. Without loss of generality, \( w \in \mathcal{D} \cap E \).

Consider the set \( K = \text{supp}_{\kappa^+}(w) \cap M_i \) and note that \( K \subset M_i \) and \( K \subset j \). For every \( \xi \in K \) let \( \phi(\xi) = w(\xi) \) if \( \xi \) is of the form \( \rho \cdot \beta + \kappa + 2 \) and \( \phi(\xi) = \langle w(\xi)_0, w(\xi)_1 \cap M_i \rangle \) otherwise. Observe that \( \phi \in M_i \). Let \( O \) be the set of those \( r \in \mathbb{P} \) such that \( r \upharpoonright j \in E \). Then \( O \) is an open dense subset of \( \mathbb{P} \). Let \( \eta < \kappa \) be such that \( \langle O, \phi \rangle = \langle O_{\eta}, \phi_{\eta} \rangle \) and \( v = \eta + 1 \). It follows from the above that we have made the non-trivial choice in the construction of \( q^\kappa \). More precisely, there exists \( r \in O_{\eta} \cap \mathcal{D} \) (namely \( w \) extended by \( q^\eta \upharpoonright [j, \kappa^+] \)) such that conditions (iv)-(vi) are satisfied. Thus there exists \( r^\eta \in O \cap \mathcal{D} \cap M_i \) satisfying (iv)-(vi) such that \( q^\eta \upharpoonright j \). In particular, \( w \leq q^\kappa \upharpoonright j \) and \( r^\eta \upharpoonright j \in E \cap M_i \). We claim that \( w_1 = r^\eta \upharpoonright j \) is compatible with \( w \). Let us define a sequence \( w_2 \) of length \( j \) as follows:

- (vi) \( w_2(\xi) = w(\xi) \) if \( \xi \in \text{Supp}_{\kappa^+} \);
- (vii) \( w_2(\xi) = \langle w(\xi)_0, w(\xi)_1 \cup r^\eta(\xi)_1 \rangle \) if \( \xi \in \text{supp}_{\kappa}(w) \) is of the form \( \rho \cdot \alpha + \kappa \) or \( \rho \cdot \alpha + \kappa + 1 \);
- (ix) \( w_2(\xi) = w(\xi) \) if \( \xi \) is of the form \( \rho \cdot \alpha + \kappa + 2 \).

We are left with the task to show that \( w_2 \in \mathbb{P}_j \), since then it becomes straightforward that \( w_2 \) is a lower bound for \( w \) and \( w_1 \). We shall show by induction on \( \xi < j \) that if \( w_2 \upharpoonright \xi \in \mathbb{P}_\xi \) then \( w_2 \upharpoonright \xi \vdash w_2(\xi) \in \mathbb{P}_\xi \). In light of our convention regarding conditions in \( \mathcal{D} \) made before Lemma 2.7 we have to consider only the case \( \xi \in \text{supp}_{\kappa}(w) \). By (ix) and \( w_2 \upharpoonright \xi \leq w \upharpoonright \xi, w_1 \upharpoonright \xi \) we may further restrict ourselves to \( \xi \)'s in \( \text{supp}_{\kappa}(w) \) of the form \( \rho \cdot \alpha + \kappa \) or \( \rho \cdot \alpha + \kappa + 1 \). In the latter case \( w_2 \upharpoonright \xi \), being a lower bound of \( w_1 \upharpoonright \xi = r^\eta \upharpoonright \xi \), forces both \( w(\xi) \) and \( r^\eta(\xi) \) to be elements of \( \mathbb{Q}_\xi \). Moreover, \( w_2 \upharpoonright \xi \) forces \( r^\eta(\xi) \) and \( w(\xi) \) to be compatible in \( \mathbb{Q}_\xi \) (because so are any two conditions in the almost disjoint coding forcing with the same first coordinate), and \( w_2(\xi) \) defined as in (vii) to be their largest lower bound. In particular, \( w_2 \upharpoonright \xi \vdash w_2(\xi) \in \mathbb{Q}_\xi \), which completes our proof in case of \( j < i \).

The case \( j = i \) can be proved by almost literal repetition of the above proof: We just have to take \( O = E \) and replace most of the instances of \( j \) for \( \kappa^+ \) in it (or, alternatively, remove them). However, we shall present this proof for the sake of completeness.

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\(^9_{\text{w(\xi)_0 = r^\eta(\xi)_0 = \phi_\eta(\xi)_0 \text{ in this case.}}}\)

\(^{10}_{\text{w(\xi) = r^\eta(\xi) = \phi_\eta(\xi) \text{ in this case.}}}\)
Fix an open dense subset $E \in M_i$ of $\mathbb{P}$ and $w \leq q^\kappa$. We need to show that there exists $w_1 \in E \cap M_i$ such that $w$ and $w_1$ are compatible. Without loss of generality, $w \in D \cap E$. Let $K, \phi, \eta, v$ be such as in the previous case. It follows from the above that we have made the non-trivial choice in the construction of $q^\nu$. More precisely, there exists $r \in O_\eta \cap D$ (namely $w$) such that conditions (iv)-(vi) are satisfied. Thus there exists $r^\eta \in E \cap D \cap M_i$ satisfying (iv)-(vi) such that $q^\nu \leq r^\eta p^\eta$. In particular, $w \leq q^\nu \leq r^\eta p^\eta$ and $r^\eta \in E \cap M_i$. We claim that $w_1 = r^\eta$ is compatible with $w$. Let us define a sequence $w_2$ of length $\kappa^+$ as follows:

$\ast$ \hspace{1cm} $w_2(\xi) = w(\xi)$ if $\xi \in \text{Supp}_\mu$;

$\ast$ \hspace{1cm} $w_2(\xi) = \langle w(\xi)_0, w(\xi)_1 \cup r^\eta(\xi)_1 \rangle$ if $\xi \in \text{supp}_\kappa(w)$ is of the form $\rho \cdot \beta + \kappa$ or $\rho \cdot \beta + \kappa + 1$;

$\ast$ \hspace{1cm} $w_2(\xi) = w(\xi)$ if $\xi$ is of the form $\rho \cdot \beta + \kappa + 2$.

The fact that $w_2 \in \mathbb{P}$ can be verified in exactly the same way as in the case $j < i$, and then it becomes straightforward that $w_2$ is a lower bound for $w$ and $w_1$. This completes our proof.

By induction on $j \leq i$ using Claim 2.9 we can prove that $q^\kappa$ is $(M_i, \mathbb{P})$-generic. Since $\check{C} \in M_i$ this implies $q^\kappa \Vdash i \in \check{C}$. It remains to note that $i \in S_{\mu}$ and $q^\kappa \leq p$.\hfill\square

Let $\mathbb{H}$ be a poset. An $\mathbb{H}$-name $\check{f}$ is called a nice name for an element of $\kappa^\kappa$ if $\check{f} = \bigcup_{v \in \kappa} \{ \langle \langle v, \eta^*_v \rangle, p \rangle : p \in A_v(\check{f}) \}$, where $A_v(\check{f})$ is a maximal antichain in $\mathbb{H}$ for all $v \in \kappa$ and $\eta^*_v \in \kappa$ for all $p \in A_v(\check{f})$. Then $p \Vdash \check{f}(v) = \eta^*_v$ for all $v \in \kappa$ and $p \in A_v$. From now on we will assume that all names for an element of $\kappa^\kappa$ are nice.

**Lemma 2.10.** Let $f = \bigcup_{v \in \kappa} \{ \langle \langle v, \eta^*_v \rangle, p \rangle : p \in A_v \}$ be a nice name for an element of $\kappa^\kappa$. Then for every $p \in \mathbb{P}$ there exists $q \leq p$ and a $\mathbb{P}$-name $\sigma \subset \check{f}$ of size $|\sigma| \leq \kappa$ such that $q \Vdash \check{f} = \sigma$.

**Proof.** Let $(M_i : i < \kappa^+)$ be such as in Lemma 2.7, where instead of (iii) we require $\kappa \cup \{ p, \mathbb{P}, \check{f}, \ldots \} \subset M_0$. As it has been established in the proof of Lemma 2.7, there exists $i < \kappa^+$ and a $(M_i, \mathbb{P})$-generic condition $q \leq^* p$. In particular, $A_v \cap M_i$ is predense below $q$, and hence no elements of $A_v \setminus M_i$ are compatible with $q$. It follows from the above that $q \Vdash \check{f} = \sigma$, where $\sigma = \bigcup_{v \in \kappa} \{ \langle \langle v, \eta^*_v \rangle, p \rangle : p \in A_v \setminus M_i \}$.

The same proof as above also works for $\mathbb{P}_\xi$ when $\xi < \kappa^+$.

**Corollary 2.11.** The poset $\mathbb{P}_\xi$ has a dense subset of size $\kappa^+$ for every $\xi \leq \kappa^+$.

**Proof.** We shall prove by induction on $\xi \leq \kappa^+$ that there exists a $\leq^*$-dense subset $D^*_\xi$ of $D_\xi$ of size $\kappa^+$.

The successor case is easily handled by Lemma 2.10. Notice that it is essential here that the generic condition $q$ considered in its proof can be chosen $\leq^*$-below the given one.
Suppose that $\text{cof}(\xi) = \eta \leq \kappa$ and fix an increasing cofinal in $\xi$ sequence $\langle \xi_v : v < \eta \rangle$ of ordinals such that $\xi_0 = 0$. Let $p \in D_\xi$ and $M$ be an elementary submodel of $L_\theta$ of size $\kappa$, where $\theta$ is big enough, such that $M \supseteq [M]^\gamma$ and $\kappa \cup \{p, \mathbb{P}_\xi, \langle \xi_v : v < \eta \rangle, \ldots \} \subseteq M$. By a standard Fodor argument we may additionally assume that $i \notin S_\mu$ for all $\mu \in M$, where $i = M \cap \kappa^+$: this can be ensured by picking $M$ out of an increasing continuous chain of elementary submodels of $L_\theta$ like in Lemma 2.7. Let also $\langle i_v : v < \kappa \rangle$ be an increasing sequence of successor ordinals cofinal in $i$. By the inductive assumption we can construct by induction on $v$ a sequence $\langle q^v : v < \eta \rangle \in M^\#$ such that the following conditions hold:

(i) $q^v \in D_{i_v}^\#$;

(ii) $q^{v+1} \leq^* q^v \rest p \restriction [\xi_v, \xi_{v+1})$; and

(iii) If $v$ is limit, then $q^v$ is $\leq^*$-below the condition $r^v \in \mathbb{P}_{i_v}$ defined as follows: for all $\mu \in \text{Supp}_\kappa \cap \xi_v$, $p \Vdash q^v(\mu) = \bigcup_{\nu < v} q^\nu(\mu) \cup \{\sup(\bigcup_{\nu < v} q^\nu(\mu)) + i_v \}$ if $\xi \in T(F(\beta))$ and $r^v(\mu) = \emptyset$ otherwise, where $\mu = p \cdot \beta + \xi$; $r^v(\mu) = p(\mu)$ for all $\mu \in \text{Supp}_\kappa \cap \xi_v$.

Now let $r^0$ be defined in the same way as in item (iii) above. Observe that $r^0 \in D_{\xi}$: This is obvious if $\eta < \kappa$ and follows from $i \notin \bigcup_{\mu \in M \cap \xi} S_\mu$ if $\eta = \kappa$.

In addition, $r^0 \leq^* p$ by the construction and it is uniquely determined by the sequence $\langle q^v : v < \eta \rangle \in \bigcup_{\mu \leq \xi} D_\mu^\#$. Now, it suffices to note that there are at most $(\kappa^+)^\kappa = \kappa^+$ such sequences.

And finally, the case $\xi = \kappa^+$ is straightforward because the supports of conditions have size $\leq \kappa$.

Combining Lemma 2.10 with Corollary 2.11 we conclude that $2^\kappa = \kappa^+$ holds in $V^\mathbb{P}_\xi$ for all $\xi \leq \kappa^+$. Recall that our main poset $\mathbb{P}$ depends on a particular bookkeeping function $F : \kappa^+ \rightarrow L$, so we may write $\mathbb{P}_F$ instead of $\mathbb{P}$. The following statement is a direct corollary of Lemma 2.10 and Corollary 2.11.

**Corollary 2.12.** There exists a bookkeeping function $F : \kappa^+ \rightarrow L$ such that for every $\mathbb{P}_F$-name $\sigma$ for a subset of $\kappa$ and $p \in \mathbb{P}_F$ there exists $\alpha < \kappa^+$ such that $F(\alpha)$ is a $\mathbb{P}_F$-name, and a condition $q \in \mathbb{P}_F$ below $p$ which forces $\sigma = F(\alpha)$.

From now on we shall fix a bookkeeping function $F_0$ with the properties described in Corollary 2.12 and assume that $\mathbb{P} = \mathbb{P}_{F_0}$. Combining Lemmas 2.5 and 2.7 we obtain the following

**Corollary 2.13.** Let $G$ be a $\mathbb{P}$-generic filter over $L$ and $\xi \leq \kappa^+$ be an ordinal of the form $\rho \cdot \alpha + \zeta$ for some $\zeta < \kappa$. Then $S_\xi$ is non-stationary in $L[G]$ iff $F(\alpha)^G$ is a stationary subset of $\kappa$ and $\zeta \in T(F(\alpha)^G)$.

The following statement is reminiscent of [4, Lemma 4].

**Lemma 2.14.** Let $G$ be $\mathbb{P}$-generic over $L$ and $S$ a subset of $\kappa$ in $L[G]$. If $S$ is stationary, then there exists $Y \in [\kappa]^\kappa$ such that for every suitable model
M of size \( \gamma \) containing \( Y \cap (\gamma^+)^M \), the set \( S \cap (\gamma^+)^M \) belongs to \( M \) and there is \( \mu < (\gamma^+)^M \) such that for all \( \zeta \in T(S) \cap (\gamma^+)^M \) we have \( M \vDash S_{\mu + \zeta} \) is not stationary.

**Proof.** Using Corollary 2.12 we may find \( \alpha < \kappa^+ \) such that \( S = F(\alpha)^G \). We claim that \( Y_\alpha \) (this is the subset of \( \kappa \) added in Case 4 of the definition of \( P \)) is as required. Indeed, let \( M \) be a suitable model of size \( \gamma \) containing \( Y_\alpha \cap (\gamma^+)^M \). Then by the definition of \( Q_{\alpha+\kappa+2} \) we know that \( S \cap (\gamma^+)^M \in M \) and \( M \vDash \psi(\gamma^+, \gamma^{++}, \mu, S \cap (\gamma^+)^M, X_\alpha \cap (\gamma^+)^M) \) for some \( \mu < (\gamma^{++})^M \), where \( X_\alpha = \{ v < \kappa : 3v + 1 \in Y_\alpha \} \). It suffices to analyze the statement of \( \psi \). \( \square \)

The next fact resembles [4, Lemma 5].

**Lemma 2.15.** Let \( G \) be a \( P \)-generic over \( L \) and \( S \) be a subset of \( \kappa \) in \( L[G] \). If there exists \( Y \in [\kappa]^{\kappa^+} \) such that for every suitable model \( M \) of size \( \gamma \) containing \( Y \cap (\gamma^+)^M \), there is \( \mu < (\gamma^{++})^M \) such that for all \( \zeta \in T(S) \cap (\gamma^+)^M \) we have \( M \vDash S_{\mu + \zeta} \) is not stationary, then \( S \) is stationary in \( \kappa \).

**Proof.** Let \( N \) be an elementary submodel of \( L[G] \) of size \( \gamma \) containing \( (\gamma + 1) \cup \{ S, Y \} \), where \( \theta \) is a large enough cardinal. Let \( M \) be the Mostowski collapse of \( N \) and \( \pi : N \to M \) be the collapsing function. Then

\[
M \vDash \exists \mu < \pi(\kappa^+) \forall \zeta \in \pi(T(S)) (S_{\mu + \zeta} \text{ is not stationary in } \pi(\kappa^+)),
\]

which implies

\[
N \vDash \exists \alpha < \kappa^+ \forall \zeta \in T(S) (S_{\alpha + \zeta} \text{ is not stationary in } \kappa^+),
\]

and hence in \( L[G] \) there exists \( \alpha < \kappa^+ \) such that for all \( \zeta \in T(S) \) the set \( S_{\alpha + \zeta} \) is not stationary in \( \kappa^+ \). This means that \( P \) destroys the stationarity of \( S_{\alpha + \zeta} \) for some \( \zeta \), and hence Corollary 2.13 implies that \( F(\alpha)^G \) is a stationary subset of \( \kappa \) and \( S_{\alpha + \zeta} \) is non-stationary in \( L[G] \) if \( \zeta \in T(F(\alpha)^G) \). It follows from the above that \( T(S) \subseteq T(F(\alpha)^G) \) which gives \( S = F(\alpha)^G \) and thus completes our proof. \( \square \)

Theorem 1.1(1) is a direct consequence of Lemmas 2.14 and 2.15, as they easily imply that in \( V^P \) we have the \( \Sigma_1 \) definition of the complement of \( NS_{\kappa} \) presented on page 3.

The proof of Theorem 1.1(2) is completely analogous to that of the first part. In this case we consider the same iteration but proceed until \( \kappa^{++} \). In order to be able to do this we need a suitable sequence \( \langle S_\alpha : \alpha < \kappa^{++} \rangle \) of mutually almost disjoint stationary subsets of \( \kappa^+ \). It may be obtained in the same way as in the first part, the only difference being that now we have to use the diamond to "convert" all subsets of \( \kappa^+ \) (previously we restricted ourselves to singletons) into stationary subsets of \( \kappa^+ \). Then we can repeat the same proof with \( \kappa^+ \) replaced with \( \kappa^{++} \) whenever the length of the iteration is concerned. The only new thing here will occur in Corollary 2.11. The same proof shows that it remains true for all \( \xi < \kappa^{++} \). The poset \( P_{\kappa^{++}} \)
will obviously have size (i.e., a dense subset of size) \( \kappa^{++} \). By a standard argument it has \( \kappa^{++} \)-c.c. Indeed, in order to prove this it is enough to basically replace \( \omega \) with \( \kappa \) in the proof of [1, Theorem 2.10], and be a little bit more careful with the choice of elementary submodels. More precisely, given \( \{ r_\xi : \xi < \kappa^{++} \} \subset \mathbb{P}_{\kappa^{++}} \), for every \( \xi \) choose an elementary submodel \( M_\xi \ni r_\xi \) of \( L_\lambda \) of size \( \kappa \) for some big enough \( \lambda \) such that \( [M_\xi]^{< \lambda} \cup \{ \mathbb{P}_{\kappa^{++}} \} \subset M_\xi \) and there exists a \( (M_\xi, \mathbb{P}_{\kappa^{++}}) \)-generic condition\(^{11}\) below \( r_\xi \), and apply the fact that \( \kappa^{++} \)-many of these submodels have the same isomorphism type to find \( \xi_1 \neq \xi_2 \) in \( \kappa^{++} \) such that \( r_{\xi_1} \) is compatible with \( r_{\xi_2} \). The existence of the \( M_\xi \)'s is established in the proof of Lemma 2.7.

3. Final remarks and open problems

In this section we shall consider the set \( \kappa^\omega \) with the \( (\kappa^\omega) \)-box topology, i.e., the topology with a base \( \{ [s] : s \in \kappa^{\kappa^\omega} \} \), where \( [s] = \{ x \in \kappa^\omega : x \text{ extends } s \} \). Following [9] we say that a subset \( A \) of \( \kappa^\omega \) is meager if it is a union of \( \kappa \) many nowhere dense subsets. \( A \) is said to have the Baire property if \( A \Delta O \) is meager for some open subset \( O \) of \( \kappa^\omega \). It is well-known [9, Theorem 4.2] (see also [7, Theorem 49]) that \( \text{NS}_\kappa \) does not have the Baire property, even though it is \( \Sigma_1^1 \) definable. This is one of the main differences with the classical case \( \kappa = \omega \).

One may however hope that there is an analogy between the Baire property of \( \Delta_1^1 \) definable subsets of \( \kappa^\omega \) and that of \( \Delta_1^3 \) definable subsets of \( \omega^\omega \): informally, in the uncountable case there is no need for an extra quantifier to express that a relation under consideration is well-founded. It turns out that there is no such analogy, as we can see using the model constructed in the proof of Theorem 1.1\(^{12}\). Recall that in the classical setting \( \kappa = \omega \), the Baire property of all \( \Delta_1^3 \) definable sets of reals is equivalent to the statement that for every real \( x \) there exists a Cohen real \( y \) over \( L[x] \), see [11].

**Proposition 3.1.** In the model constructed in the proof of Theorem 1.1(1) for every \( X \subset \kappa \) there is \( Y \subset \subset \kappa \) which is Add(\( \kappa, 1 \))-generic over \( L[X] \), where Add(\( \kappa, 1 \)) = \( 2^{< \kappa} \) ordered by extension. Thus the \( \kappa \)-analogue of the condition guaranteeing the Baire property of all \( \Delta_1^3 \) definable subsets does not imply the Baire property of all \( \Delta_1^3 \) definable subsets of \( \kappa^\omega \), as witnessed by \( \text{NS}_\kappa \).

**Proof.** By Corollary 2.12 it is enough to show that posets \( \dot{Q}_\xi \) add Add(\( \kappa, 1 \)) generics over \( L^{\mathbb{P}_\xi} \) for cofinally many \( \xi \in \kappa^+ \). For every \( (\kappa, \kappa) \)-complete filter \( F \) on \( \kappa \) there is a natural poset \( \mathbb{M}(F) \) (“\( \mathbb{M} \)” comes from “Mathias”) producing a pseudointersection of \( F \). This poset consists of all pairs \( \langle s, F \rangle \in [\kappa]^{< \kappa} \times \mathbb{F} \) where \( \langle s', F' \rangle \) extends \( \langle s, F \rangle \) if and only if \( s' \) end-extends \( s \), \( F' \subset F \), and \( s' \setminus s \subset F \). Observe that for every \( \xi \) of the form \( \rho \cdot \alpha + \kappa \), in \( V^{\mathbb{P}_\xi} \) the poset \( \dot{Q}_\xi \) is order isomorphic to \( \mathbb{M}(F) \) for the \( (\kappa, \kappa) \)-complete filter on \( \kappa \).

\(^{11}\)In [1] one can take any \( M_\xi \ni r_\xi \) because the poset under consideration is proper.

\(^{12}\)We would like to thank Yurii Khomskii for asking us whether such an analogy holds.
generated by \( \{ \kappa \setminus A_\nu : \nu \in D_\alpha \} \). The following statement may be thought of as folklore. We have learned it from an unpublished manuscript of Brendle.

**Claim 3.2.** Let \( \mathcal{F} \) be a \((\prec \kappa)\)-complete filter on \( \kappa \) such that there exists a function \( f : [\kappa]^2 \to 2 \) for which \( f([F]^2) = 2 \) for all \( F \in \mathcal{F} \). Then there exists a Add(\( \kappa, 1) \)-generic filter in \( V[\mathbb{M}(\mathcal{F})] \).

**Proof.** Let \( G \) be a \( \mathbb{M}(\mathcal{F}) \)-generic and \( g = \bigcup \{ s : \exists F \in \mathcal{F}(s, F) \in G \} \). Set \( c(\alpha) = f(\gamma_{2\alpha}, \gamma_{2\alpha+1}) \), where \( \{ \gamma_\alpha : \alpha < \kappa \} \) is the increasing enumeration of \( g \). We claim that \( \{ c \upharpoonright \alpha : \alpha \in \kappa \} \) is Add(\( \kappa, 1) \)-generic. Indeed, let \( D \subseteq \text{Add}(\kappa, 1) \) be dense and \( \langle s, F \rangle \in \mathbb{M}(\mathcal{F}) \) be such that the order type of \( s \) equals \( 2\alpha \) for some \( \alpha \in \kappa \). Thus \( \langle s, F \rangle \) determines \( c \upharpoonright \alpha \), say \( \langle s, F \rangle \models c \upharpoonright \alpha = \sigma \). By the density of \( D \) there exists an extension \( \tau \in D \) of \( \sigma \). Since \( f([F \setminus \xi]^2) = 2 \) for all \( \xi \in \kappa \), we can easily find an end-extension \( t \) of \( s \) such that \( t \setminus s \subseteq F \), order type of \( t \) equals \( 2\beta \), where \( \beta = \text{dom}(\tau) \), and \( t, F \setminus \sup t + 1 \models c \upharpoonright \beta = \tau \). This completes our proof. \( \Box \)

In our case \( \kappa \) is a successor cardinal. In particular it is not measurable. It suffices to note that for every \((\prec \kappa)\)-complete filter \( \mathcal{F} \) which is not an ultrafilter there exists a function \( f \) as in the claim above. Indeed, take \( A \subseteq \kappa \) such that each element of \( \mathcal{F} \) intersects both \( A \) and \( \kappa \setminus A \) and set \( f(\{ \alpha, \beta \}) = 1 \) iff \( \{ \alpha, \beta \} \subseteq A \) or \( \{ \alpha, \beta \} \subseteq \kappa \setminus A \). \( \Box \)

Instead of arguing as in Proposition 3.1 we could just change the construction of \( \mathbb{P} \) by letting \( \mathbb{Q}_\xi \) be the \( \mathbb{P}_\xi \)-name for \text{Add}(\( \kappa, 1) \) for cofinally many \( \xi \in \kappa^+ \). It is easy to check that this would not interfere with the proof of the \( \Delta_1 \)-definability of \( \text{NS}_{\kappa} \).

Finally we mention two open questions related to Theorem 1.1 whose solutions seem to require essentially different approaches.

**Problem 3.3.** Is it consistent that \( \text{NS}_{\kappa} \) is \( \Delta_1 \)-definable over \( H(\kappa^+) \) and \( 2^\kappa \geq \kappa^{+++} \)?

**Problem 3.4.** Is \( 2^\gamma \geq \gamma^{+++} \) together with \( \text{NS}_{\gamma^+} \) being \( \Delta_1 \)-definable over \( H(\gamma^+) \) consistent? What if \( \gamma = \omega \)? In the latter case, can we additionally have MA instead of \( \neg \text{CH} \)? In case the answer to some of these questions is affirmative, can the corresponding consistency be forced over \( L \)?

Let us note that the existence of a collection \( S \) of stationary subsets of \( \omega_1 \) such that \( |S| = \omega_1 \) and each stationary subset of \( \omega_1 \) contains some \( S \subseteq S \), which may be thought of as a strong form of the \( \Delta_1 \)-definability of the \( \text{NS}_{\omega_1} \), implies the existence of a Suslin tree, see, e.g., [6, Theorem 5.28]. Thus it contradicts MA.

**References**


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