PRODUCTS OF HUREWICZ SPACES IN THE LAVER MODEL

DUŠAN REPOVŠ AND LYUBOMYR ZDOMSKYY

Abstract. We prove that in the Laver model for the consistency of the Borel's conjecture, the product of any two metrizable spaces with the Hurewicz property has the Menger property.

1. Introduction

A topological space X has the *Menger* property (or, alternatively, is a Menger space) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the collection $\{\cup \mathcal{V}_n : n \in \omega\}$ is a cover of X. This property was introduced by Hurewicz, and the current name (the Menger property) is used because Hurewicz proved in [5] that for metrizable spaces his property is equivalent to one property of a base considered by Menger in [9]. If in the definition above we additionally require that $\{\cup \mathcal{V}_n : n \in \omega\}$ is a γ-*cover* of X (this means that the set $\{n \in \omega : x \notin \cup \mathcal{V}_n\}$ is finite for each $x \in X$), then we obtain the definition of the *Hurewicz property* introduced in [6]. Each σ-compact space is obviously a Hurewicz space, and Hurewicz spaces have the Menger property. Contrary to a conjecture of Hurewicz the class of metrizable spaces having the Hurewicz property appeared to be much wider than the class of σ-compact spaces [7, Theorem 5.1]. The properties of Menger and Hurewicz are classical examples of combinatorial covering properties of topological spaces which are nowadays called *selection principles*. This is a rapidly growing area of general topology, see, e.g., [14]. For instance, Menger and Hurewicz spaces found applications in such areas as forcing [4], Ramsey theory in algebra [15], and the combinatorics of discrete subspaces [1].

This paper is devoted to the preservation of the Hurewicz property by finite products. This topic is perhaps as old as the properties themselves because the preservation by products is one of the central questions in topology. There are two reasons why a product of Hurewicz spaces X, Y may fail to be Hurewicz. In the first place, $X \times Y$ may simply fail to be a Lindelöf space, i.e., it might have an open cover $\mathcal{U}$ without countable subcover. Then $X \times Y$ is not even a Menger space. This may indeed happen: in ZFC there

2010 *Mathematics Subject Classification.* Primary: 03E35, 54D20. Secondary: 54C50, 03E05.

Key words and phrases. Menger space, Hurewicz space, semifilter, Laver forcing.

The first author was partially supported by the Slovenian Research Agency grant P1-0292-0101. The second author would like to thank the Austrian Science Fund FWF (Grant I 1209-N25) for generous support for this research.

are two normal spaces X, Y with a covering property much stronger than the Hurewicz one such that $X \times Y$ does not have the Lindelöf property, see [13, §3]. However, the above situation becomes impossible if we restrict our attention to metrizable spaces. This second case, on which our paper concentrates, turned out to be sensitive to the ambient set-theoretic universe: under CH there exists a Hurewicz space whose square is not Menger, see [7, Theorem 2.12]. The above result has been achieved by a transfinite construction of length ω_1, using the combinatorics of the ideal of measure zero subsets of reals. This combinatorics turned out [12, Theorem 43] to require much weaker set-theoretic assumptions than CH. In particular, under the Martin Axiom there are Hurewicz subspaces of the irrationals whose product is not Menger.

The following theorem, which is the main result of this paper, shows that an additional assumption in the results from [7, 12] mentioned above is really needed. In addition, it implies that the affirmative answer to [7, Problem 2] is consistent, see [14, § 2] for the discussion of this problem.

Theorem 1.1. *In the Laver model for the consistency of the Borel's conjecture, the product of any two Hurewicz spaces has the Menger property provided that it is a Lindelöf space. In particular, the product of any two Hurewicz metrizable spaces has the Menger property.*

This theorem seems to be the first "positive" consistency result related to the preservation by products of selection principles weaker than the σ-compactness, in which no further restrictions[1] on the spaces are assumed. The proof is based on the analysis of continuous maps and names for reals in the model of set theory constructed in [8]. The question whether the product of Hurewicz metrizable spaces is a Hurewicz space in this model remains open. It is worth mentioning here that in the Cohen model there are Hurewicz subsets of $\mathbb{R}$ whose product has the Menger property but fails to have the Hurewicz one, see [11, Theorem 6.6].

2. Proofs

We shall first prove Proposition 2.4 below, a formally more general statement than the second part of Theorem 1.1. It involves the following

Definition 2.1. A topological space X is called *weakly concentrated* if for every collection $\mathsf{Q} \subset [X]^\omega$ which is cofinal with respect to inclusion, and for every function $R : \mathsf{Q} \to \mathcal{P}(X)$ assigning to each $Q \in \mathsf{Q}$ a G_δ-set $R(Q)$ containing Q, there exists $\mathsf{Q}_1 \in [\mathsf{Q}]^{\omega_1}$ such that $X \subset \bigcup_{Q \in \mathsf{Q}_1} R(Q)$.

The following lemma is the key part of the proof of Theorem 1.1. Its proof is reminiscent of that of [10, Theorem 3.2]. We will use the notation from [8] with only differences being that smaller conditions in a forcing poset are supposed to carry more information about the generic filter, and the ground model is denoted by V.

[1] The requirement that the product must be Lindelöf is vacuous for metrizable spaces. Let us note that nowadays the study of selection principles concentrates mainly on sets of reals.

A subset C of ω_2 is called an ω_1-*club* if it is unbounded and for every $\alpha \in \omega_2$ of cofinality ω_1, if $C \cap \alpha$ is cofinal in α then $\alpha \in C$.

Lemma 2.2. *In the Laver model every Hurewicz subspace of $\mathcal{P}(\omega)$ is weakly concentrated.*

Proof. We work in $V[G_{\omega_2}]$, where G_{ω_2} is $\mathbb{P}_{\omega_2}$-generic and $\mathbb{P}_{\omega_2}$ is the iteration of length ω_2 with countable supports of the Laver forcing, see [8] for details.

It is well-known that a space $X \subset \mathcal{P}(\omega)$ is Hurewicz if and only if $f[X]$ is bounded with respect to $\leq^*$ for every continuous $f : X \to \omega^\omega$, see [7, Theorem 4.4] or [6]. Let us fix a Hurewicz space $X \subset \mathcal{P}(\omega)$. The Hurewicz property is preserved by F_σ-subspaces because it is obviously preserved by closed subspaces and countable unions. Therefore by a standard argument (see, e.g., the proof of [3, Lemma 5.10]) there exists an ω_1-club $C \subset \omega_2$ such that for every $\alpha \in C$ and continuous $f : F \to \omega^\omega$ (coded) in $V[G_\alpha]$, where F is an F_σ-subspace of X (coded) in $V[G_\alpha]$, there exists $b \in \omega^\omega \cap V[G_\alpha]$ such that $f(x) \leq^* b$ for all $x \in F$.

Let $\mathsf{Q} \subset [X]^\omega$ be cofinal with respect to inclusion. Fix a function $R : \mathsf{Q} \to \mathcal{P}(X)$ assigning to each $Q \in \mathsf{Q}$ a G_δ-set $R(Q)$ containing Q. By the same argument there exists an ω_1-club $D \subset \omega_2$ such that $\mathsf{Q} \cap V[G_\alpha] \in V[G_\alpha]$, $R \restriction (\mathsf{Q} \cap V[G_\alpha]) \in V[G_\alpha]$ for[2] all $\alpha \in D$, and for every $Q_0 \in [X \cap V[G_\alpha]]^\omega \cap V[G_\alpha]$ there exists $Q \in \mathsf{Q} \cap V[G_\alpha]$ such that $Q_0 \subset Q$.

Let us fix $\alpha \in C \cap D$. We claim that $X \subset W$, where $W = \bigcup\{R(Q) : Q \in \mathsf{Q} \cap V[G_\alpha]\}$. Suppose that, contrary to our claim, there exists $p \in G_{\omega_2}$ and a $\mathbb{P}_{\omega_2}$-name $\dot{x}$ such that $p \Vdash \dot{x} \in \dot{X} \setminus \dot{W}$. By [8, Lemma 11] there is no loss of generality in assuming that $\alpha = 0$. Applying [8, Lemma 14] to a sequence $\langle \dot{a}_i : i \in \omega \rangle$ such that $\dot{a}_i = \dot{x}$ for all $i \in \omega$, we get a condition $p' \leq p$ such that $p'(0) \leq^0 p(0)$, and a finite set U_s of reals for every $s \in p'(0)$ with $p'(0)\langle 0 \rangle \leq s$, such that for each $\varepsilon > 0$, $s \in p'(0)$ with $p'(0)\langle 0 \rangle \leq s$, and for all but finitely many immediate successors t of s in $p'(0)$ we have

$$p'(0)_t \hat{\ } p' \restriction [1, \omega_2) \Vdash \exists u \in U_s \, (|\dot{x} - u| < \varepsilon).$$

Fix $Q \in \mathsf{Q} \cap V$ containing $X \cap \bigcup\{U_s : s \in p'(0), s \geq p'(0)\langle 0 \rangle\}$ and set $F = X \setminus R(Q)$. Note that F is an F_σ-subset of X coded in V. It follows that $p' \Vdash \dot{x} \in \dot{F}$ because p' is stronger than p that forces $\dot{x} \notin \dot{W} \supset \dot{X} \setminus \dot{F}$. Consider the map $f : F \to \omega^S$, where $S = \{s \in p'(0) : s \geq p'(0)\langle 0 \rangle\}$, defined as follows:

$$f(y)(s) = [1/\min\{|y - u| : u \in U_s\}] + 1$$

for[3] all $s \in S$ and $y \in F$. Since F is disjoint from Q which contains all the U_s's, f is well-defined. Since both F and f are coded in V, there exists $b \in \omega^S \cap V$ such that $f(y) \leq^* b$ for all $y \in F$.

It follows from $p' \Vdash \dot{x} \in \dot{F}$ that $p' \Vdash \dot{f}(\dot{x}) \leq^* b$, and hence there exists $p'' \leq p$ and a finite subset S_0 of S such that $p'' \Vdash \dot{f}(\dot{x})(s) \leq b(s)$ for all $S \setminus S_0$. By replacing p'' with $p''(0)_s \hat{\ } p'' \restriction [1, \omega_2)$ for some $s \in p''(0)$, if necessary, we

[2] Here by $R \restriction (\mathsf{Q} \cap V[G_\alpha])$ we mean the map which assigns to a $Q \in \mathsf{Q} \cap V[G_\alpha]$ the code of $R(Q)$.

[3] Here $[a]$ is the largest integer not exceeding a.

may additionally assume that $p''(0)\langle 0\rangle \in S\setminus S_0$. Letting $s''=p''(0)\langle 0\rangle$, we conclude from the above that $p''\Vdash \dot f(\dot x)(s'')\le b(s'')$, which means that

$$p''\Vdash \min\{|\dot x-u| : u\in U_{s''}\}\ge 1/b(s'').$$

On the other hand, by our choice of p' and $p''\le p'$ we get that for all but finitely many immediate successors t of s'' in $p''(0)$ we have

$$p''(0)_t{}^\frown p''\restriction [1,\omega_2)\Vdash \exists u\in U_{s''}\ |\dot x-u|<1/b(s'')$$

which means $p''(0)_t{}^\frown p''\restriction [1,\omega_2)\Vdash \min\{|\dot x-u| : u\in U_{s''}\}<1/b(s'')$ and thus leads to a contradiction. □

The next lemma may probably be considered as folklore. We present its proof for the sake of completeness.

Lemma 2.3. *Let $Y\subset\mathcal{P}(\omega)$ be Hurewicz and $Q\subset\mathcal{P}(\omega)$ countable. Then for every G_δ-subset O of $\mathcal{P}(\omega)^2$ containing $Q\times Y$ there exists a G_δ-subset $R\supset Q$ such that $R\times Y\subset O$.*

Proof. Without loss of generality we shall assume that O is open. Let us write Q in the form $\{q_n : n\in\omega\}$ and set $O_n=\{z\in\mathcal{P}(\omega) : \langle q_n,z\rangle\in O\}\supset Y$. For every n find a cover $\mathcal{U}_n$ of Y consisting of clopen subsets of $\mathcal{P}(\omega)$ contained in O_n. Let $\langle \mathcal{U}'_k : k\in\omega\rangle$ be a sequence of open covers of Y such that each $\mathcal{U}_n$ appears in it infinitely often. Applying the Hurewicz property of Y we can find a sequence $\langle\mathcal{V}_k : k\in\omega\rangle$ such that $\mathcal{V}_k\in[\mathcal{U}_k]^{<\omega}$ and $Y\subset\bigcup_{k\in\omega}Z_k$, where $Z_k=\bigcap_{m\ge k}\cup\mathcal{V}_m$. Note that each Z_k is compact and $Z_k\subset O_n$ for all $n\in\omega$ (because there exists $m\ge k$ such that $\mathcal{U}'_m=\mathcal{U}_n$, and then $Z_k\subset\cup\mathcal{V}_m\subset O_n$). Thus $Q\times Y\subset Q\times(\bigcup_{k\in\omega}Z_k)\subset O$. Since Z_k is compact, there exists for every k an open $R_k\supset Q$ such that $R_k\times Z_k\subset O$. Set $R=\bigcap_{k\in\omega}R_k$ and note that $R\supset Q$ and $R\times Y\subset R\times\bigcup_{k\in\omega}Z_k\subset O$. □

Let A be a countable set and $x,y\in\omega^A$. As usually, $x\le^* y$ means that $\{a\in A : x(a)>y(a)\}$ is finite. The smallest cardinality of an unbounded with respect to $\le^*$ subset of ω^ω is denoted by $\mathfrak{b}$. It is well-known that $\omega_1<\mathfrak{b}$ in the Laver model, see [2] for this fact as well as systematic treatment of cardinal characteristics of reals.

The second part of Theorem 1.1 is a direct consequence of Lemma 2.2 and the following

Proposition 2.4. *Suppose that $\mathfrak{b}>\omega_1$. Let $Y\subset\mathcal{P}(\omega)$ be a Hurewicz space and $X\subset\mathcal{P}(\omega)$ weakly concentrated. Then $X\times Y$ is Menger.*

Proof. Fix a sequence $\langle\mathcal{U}_n : n\in\omega\rangle$ of covers of $X\times Y$ by clopen subsets of $\mathcal{P}(\omega)^2$. For every $Q\in[X]^\omega$ fix a sequence $\langle\mathcal{W}^Q_n : n\in\omega\rangle$ such that $\mathcal{W}^Q_n\in[\mathcal{U}_n]^{<\omega}$ and $Q\times Y\subset\bigcap_{n\in\omega}\bigcup_{m\ge n}\cup\mathcal{W}^Q_m$. Letting $O_Q=\bigcap_{n\in\omega}\bigcup_{m\ge n}\cup\mathcal{W}^Q_m$ and using Lemma 2.3, we can find a G_δ-subset $R_Q\supset Q$ such that $R_Q\times Y\subset O_Q$. Since X is weakly concentrated, there exists $\mathsf{Q}\subset[X]^\omega$ of size $|\mathsf{Q}|=\omega_1$ such that $R=\bigcup\{R_Q : Q\in\mathsf{Q}\}$ contains X as a subset. Let us fix $x\in X$ and find $Q\in\mathsf{Q}$ such that $x\in R_Q$. Then $\{x\}\times Y\subset R_Q\times Y\subset O_Q$. Therefore for every $\langle x,y\rangle\in X\times Y$ there exists $Q\in\mathsf{Q}$ such that $\langle x,y\rangle\in O_Q=\bigcap_{n\in\omega}\bigcup_{m\ge n}\cup\mathcal{W}^Q_m$. Let us write $\mathcal{U}_n$ in the form $\{U^n_k : k\in\omega\}$ and for every

$Q \in \mathsf{Q}$ fix a real $b_Q \in \omega^\omega$ with the property $\mathcal{W}^Q_n \subset \{U^n_k : k \leq b_Q(n)\}$. Since $|\mathsf{Q}| = \omega_1 < \mathfrak{b}$, there exists $b \in \omega^\omega$ such that $b_Q \leq^* b$ for all $Q \in \mathsf{Q}$. It follows from the above that $X \times Y \subset \bigcup_{n\in\omega} \bigcup_{k\leq b(n)} U^n_k$, which completes our proof. □

A family $\mathcal{F} \subset [\omega]^\omega$ is called a *semifilter* if for every $F \in \mathcal{F}$ and $X^* \supset F$ we have $X \in \mathcal{F}$, where $F \subset^* X$ means $|F \setminus X| < \omega$.

The proof of the first part of Theorem 1.1 uses characterizations of the properties of Hurewicz and Menger obtained in [16]. Let $u = \langle U_n : n \in \omega\rangle$ be a sequence of subsets of a set X. For every $x \in X$ let $I_s(x, u, X) = \{n \in \omega : x \in U_n\}$. If every $I_s(x, u, X)$ is infinite (the collection of all such sequences u will be denoted by $\Lambda_s(X)$), then we shall denote by $\mathcal{U}_s(u, X)$ the smallest semifilter on ω containing all $I_s(x, u, X)$. By [16, Theorem 3], a Lindelöf topological space X is Menger (Hurewicz) if and only if for every $u \in \Lambda_s(X)$ consisting of open sets, the semifilter $\mathcal{U}_s(u, X)$ is Menger (Hurewicz). The proof given there also works if we consider only those $\langle U_n : n \in \omega\rangle \in \Lambda_s(X)$ such that all U_n's belong to a given base of X.

Proof of Theorem 1.1. Suppose that X, Y are Hurewicz spaces such that $X \times Y$ is Lindelöf and fix $w = \langle U_n \times V_n : n \in \omega\rangle \in \Lambda_s(X \times Y)$ consisting of open sets. Set $u = \langle U_n : n \in \omega\rangle$, $v = \langle V_n : n \in \omega\rangle$, and note that $u \in \Lambda_s(X)$ and $v \in \Lambda_s(Y)$. It is easy to see that

$$\mathcal{U}_s(w, X \times Y) = \{A \cap B : A \in \mathcal{U}_s(u, X), B \in \mathcal{U}_s(v, Y)\},$$

and hence $\mathcal{U}_s(w, X \times Y)$ is a continuous image of $\mathcal{U}_s(u, X) \times \mathcal{U}_s(v, Y)$. By [16, Theorem 3] both of latter ones are Hurewicz, considered as subspaces of $\mathcal{P}(\omega)$, and hence their product is a Menger space by Proposition 2.4 and Lemma 2.2. Thus $\mathcal{U}_s(w, X \times Y)$ is Menger, being a continuous image of a Menger space. It suffices to use [16, Theorem 3] again. □

References

[1] Aurichi, L.F., *D-spaces, topological games, and selection principles,* Topology Proc. **36** (2010), 107–122.

[2] Blass, A., *Combinatorial cardinal characteristics of the continuum*, in: *Handbook of Set Theory* (M. Foreman, A. Kanamori, and M. Magidor, eds.), Springer, 2010, pp. 395–491.

[3] Blass, A.; Shelah, S., *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed,* Ann. Pure Appl. Logic **33** (1987), 213–243.

[4] Chodounský, D.; Repovš, D.; Zdomskyy, L., *Mathias forcing and combinatorial covering properties of filters*, J. Symb. Log. **80** (2015), 1398–1410.

[5] Hurewicz, W., *Über die Verallgemeinerung des Borellschen Theorems*, Math. Z. **24** (1925), 401–421.

[6] Hurewicz, W., *Über Folgen stetiger Funktionen,* Fund. Math. **9** (1927), 193–204.

[7] Just, W.; Miller, A.W.; Scheepers, M.; Szeptycki, P.J., *The combinatorics of open covers. II,* Topology Appl. **73** (1996), 241–266.

[8] Laver, R., *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), 151–169.

[9] Menger, K., *Einige Überdeckungssätze der Punktmengenlehre, Sitzungsberichte*. Abt. 2a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik (Wiener Akademie) **133** (1924), 421–444.

[10] Miller, A.W.; Tsaban, B., *Point-cofinite covers in the Laver model,* Proc. Amer. Math. Soc. **138** (2010), 3313–3321.

[11] Miller, A.W.; Tsaban, B.; Zdomskyy, L., *Selective covering properties of product spaces, II: gamma spaces,* Trans. Amer. Math. Soc. **368** (2016), 2865–2889.

[12] Scheepers, M.; Tsaban, B., *The combinatorics of Borel covers,* Topology Appl. **121** (2002), 357–382.

[13] Todorčević, S., *Aronszajn orderings. Djuro Kurepa memorial volume,* Publ. Inst. Math. (Beograd) (N.S.) **57**(71) (1995), 29–46.

[14] Tsaban, B., *Selection principles and special sets of reals,* in: *Open problems in topology II* (edited By Elliott Pearl), Elsevier Sci. Publ., 2007, pp. 91–108.

[15] Tsaban, B., *Algebra, selections, and additive Ramsey theory*, preprint. `http://arxiv.org/pdf/1407.7437.pdf`.

[16] Zdomskyy, L, *A semifilter approach to selection principles,* Comment. Math. Univ. Carolin. **46** (2005), 525–539.

Faculty of Education, and Faculty of Mathematics and Physics, University of Ljubljana, Kardeljeva Pl. 16, Ljubljana, Slovenia 1000.
E-mail address: `dusan.repovs@guest.arnes.si`
URL: `http://www.fmf.uni-lj.si/~repovs/index.htm`

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Strasse 25, A-1090 Wien, Austria.
E-mail address: `lzdomsky@gmail.com`
URL: `http://www.logic.univie.ac.at/~lzdomsky/`