The Nikodym property in the Sacks model

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Abstract. We prove that if $\mathcal{A}$ is a $\sigma$-complete Boolean algebra in a ground model $V$ of set theory, then $\mathcal{A}$ has the Nikodym property in every side-by-side Sacks forcing extension $V[G]$, i.e. every pointwise bounded sequence of measures on $\mathcal{A}$ in $V[G]$ is uniformly bounded. This gives a consistent example of a class of infinite Boolean algebras with the Nikodym property and of cardinality strictly less than the continuum.

1. Introduction

Let $\mathcal{A}$ be a Boolean algebra. A sequence of measures $\langle \mu_n : n \in \omega \rangle$ on $\mathcal{A}$ is pointwise bounded if $\sup_{n \in \omega} |\mu_n(A)| < \infty$ for every $A \in \mathcal{A}$ and it is uniformly bounded if $\sup_{n \in \omega} \|\mu_n\| < \infty$. The Nikodym Boundedness Theorem states that if $\mathcal{A}$ is $\sigma$-complete, then every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded. This principle, due to its numerous applications, is one of the most important results in the theory of vector measures, see Diestel and Uhl [7, Section I.3].

Since $\sigma$-completeness is rather a strong property of Boolean algebras, Schachermayer [11] made a detailed study of the Nikodym theorem and introduced the Nikodym property for general Boolean algebras.

Definition 1.1. A Boolean algebra $\mathcal{A}$ has the Nikodym property if every pointwise bounded sequence of measures on $\mathcal{A}$ is uniformly bounded.

The property has been studied by many authors, e.g. Darst [5], Seever [12], Haydon [9], Molto [10], Freniche [8], Aizpuru [1, 2] or Valdivia [14].

Let us pose the following question. Let $V$ be a model of ZFC+CH and $\mathcal{A} \in V$ be a $\sigma$-complete Boolean algebra of cardinality equal to the continuum $c$. Let $\mathbb{P}$ be a notion of forcing preserving $\omega_1$ and $G$ its generic filter over $V$. Assume that in the extension $V[G]$ the CH does not hold. Then, $\mathcal{A}$ will have cardinality $\omega_1$ in $V[G]$, and hence it will no longer be $\sigma$-complete. However, will $\mathcal{A}$ still have the Nikodym property?

Brech [4, Theorem 3.1] proved that if $\mathbb{P}$ is the side-by-side Sacks forcing $S^\kappa$ for some regular cardinal number $\kappa$, then $\mathcal{A}$ will have the Grothendieck property in $V[G]$, i.e. every sequence of measures in $V[G]$ which is weak* convergent on $\mathcal{A}$ is also weakly convergent. The Nikodym and Grothendieck properties are closely related to each other, see e.g. Schachermayer [11]. Thus, motivated by Brech’s result, we studied the preservation of the Nikodym property by the Sacks forcing $S^\kappa$ and proved that if $\mathcal{A}$ is a $\sigma$-complete Boolean algebra in $V$, then $\mathcal{A}$ has the Nikodym property in the $S^\kappa$-generic extension $V[G]$ (Theorem 3.3).

Our result has one important consequence. In Sobota [13], the first author studied the relation between the Nikodym property and cardinal characteristics of the continuum. In particular, a construction of a Boolean algebra with the

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The Stone duality theorem states that the dual space of a measure space is isometrically isomorphic with the space of all measures on a Boolean algebra. Every measure has a unique Borel extension (denoted also by \( \mu \)) onto the space \( K_A \), preserving the variation of \( \mu \). This theorem is a cornerstone in the study of Boolean algebras and measures, providing a bridge between algebraic and measure-theoretic properties. For more information concerning measure theory and Banach spaces, see the book of Diestel [6].

V always denotes the set-theoretic universum. By \( S^\kappa \) we denote the side-by-side product of \( \kappa \) many Sacks forcings \( S \) for some uncountable regular cardinal number \( \kappa \). Regarding all other notions related to the Sacks forcing, we follow the paper of Baumgartner [3]. If \( s \in S \) and \( p \in s \), then \( s[p] = \{ q \in s : q \subseteq p \text{ or } p \subseteq q \} \subseteq S \). If \( n \in \omega \), then \( l(n,s) \) denotes the \( n \)-th forking level of \( s \).

Let \( s, s' \in S^\kappa \), \( F \in [\text{dom}(s)]^{<\omega} \) and \( n \in \omega \). We put \( l(F,n,s) = \{ \sigma : \text{dom}(\omega) = F \text{ and } \forall \alpha \in F : \sigma(\alpha) \in l(n,s(\alpha)) \} \). Note that \( |l(F,n,s)| = 2^{\omega |F|} \). We write \( s' \leq_F n \) \( s \) if \( s' \leq s \) and \( l(F,n,s') = l(F,n,s) \). If \( \sigma : F \to 2^{<\omega} \) is such that \( \sigma(\alpha) \in s(\alpha) \) for every \( \alpha \in F \), then we write \( s|F \) for a condition defined as \( (s|F)(\alpha) = \sigma(\alpha) \).

2. Anti-Nikodym sequences in the Sacks model

In this section, assuming in a forcing extension the existence of sequences of measures on a ground model Boolean algebra \( A \) which are pointwise bounded but not uniformly bounded, we build (Proposition 2.9) in the ground model a special antichain in \( A \) which will be crucial in proving the main theorem of the paper — Theorem 3.3.

**Definition 2.1.** A sequence \( \langle \mu_n : n \in \omega \rangle \) of measures on a Boolean algebra \( A \) is called anti-Nikodym if it is pointwise bounded but not uniformly bounded.

**Lemma 2.2.** If a sequence \( \langle \mu_n : n \in \omega \rangle \) of measures on a Boolean algebra \( A \) is anti-Nikodym, then there exists a point \( t \in K_A \) such that for every clopen neighborhood \( U \subseteq A \) of \( t \) we have \( \sup_{n \in \omega} \| \mu_n \downharpoonright U \| = \infty \).

The point \( t \) will be called a Nikodym concentration point of the sequence \( \langle \mu_n : n \in \omega \rangle \).

**Proof.** Assume that for every point \( t \in K_A \) there exists \( A_t \subseteq A \) such that \( t \in [A_t] \) and \( \langle \mu_n \downharpoonright A_t : n \in \omega \rangle \) is uniformly bounded. Then, by compactness of \( K_A \) there exist \( t_1, \ldots, t_n \in K_A \) such that \( A_{t_1} \vee \ldots \vee A_{t_n} = 1_A \). This in turn implies that

\[
\sup_{n \in \omega} \| \mu_n \| = \sup_{n \in \omega} |\mu_n| (1_A) \leq \sup_{n \in \omega} |\mu_n| (A_{t_1}) + \ldots + \sup_{n \in \omega} |\mu_n| (A_{t_n}) = \sup_{n \in \omega} \| \mu_n \downharpoonright A_{t_1} \| + \ldots + \sup_{n \in \omega} \| \mu_n \downharpoonright A_{t_n} \| < \infty,
\]
which is a contradiction, since $\langle \mu_n: \ n \in \omega \rangle$ is not uniformly bounded. 

(Nota that in the above proof we did not use the pointwise boundedness of $\langle \mu_n: \ n \in \omega \rangle$.)

\textbf{Lemma 2.3.} Let $\langle \mu_n: \ n \in \omega \rangle$ be an anti-Nikodym sequence on $\mathcal{A}$ and let $t \in K_\mathcal{A}$ be its Nikodym concentration point. Assume that $t \in [A]$ for some $A \in \mathcal{A}$. Then, for every positive real number $\rho$ and natural number $M$ there exist an element $B \in \mathcal{A}$ and a natural number $n > M$ such that:

- $B \leq A$ and $t \in [A \setminus B]$,
- $|\mu_n(B)| > \rho$.

\textbf{Proof.} Since $\langle \mu_n: \ n \in \omega \rangle$ is anti-Nikodym and $t \in [A]$, there exist $C \leq A$ and $n > M$ such that

$$|\mu_n(C)| > \sup_{m \in \omega} |\mu_m(A)| + \rho$$

and hence

$$|\mu_n(A \setminus C)| = |\mu_n(C) - \mu_n(A)| \geq |\mu_n(C)| - |\mu_n(A)| > \rho.$$

If $t \in [C]$, then put $B = A \setminus C$, otherwise put $B = C$. \hfill \Box

To the end of this section let $\mathcal{A}$ be a ground model infinite Boolean algebra.

\textbf{Lemma 2.4.} Let $A_0, \ldots, A_k \in \mathcal{A}$, $K$, $M$, $N \in \omega$. Let $\langle \mu_n: \ n \in \omega \rangle$ be a sequence of names for measures on $\mathcal{A}$ and $i$ a name for a point in $K_\mathcal{A}$. Let $s \in \mathcal{S}$ force that $\langle \mu_n: \ n \in \omega \rangle$ is anti-Nikodym, $i$ is its Nikodym concentration point and $i \notin \bigcup_{j=0}^k [A_j]$.

Then, there exist a sequence $B_1, \ldots, B_K$ of pairwise disjoint elements of $\mathcal{A}$ disjoint with $1_\mathcal{A} \setminus \bigcup_{j=0}^k A_j$, a sequence $n_K > \ldots > n_1 > M$ of natural numbers and a condition $s^* \leq s$ forcing for every $1 \leq i \leq K$ that $i \notin [B_i]$ and

$$|\hat{\mu}_{n_i}(B_i)| > \sum_{j=0}^k |\hat{\mu}_{n_i}(A_j)| + N + 2.$$

\textbf{Proof.} Use Lemma 2.3 inductively $K$ times to obtain sequences $B_1, \ldots, B_K \in \mathcal{A}$, $n_K > \ldots > n_1 > M$ and $s_K \leq \ldots \leq s_1 \leq s$ such that for every $1 \leq i \leq K$ the element $B_i$ is disjoint with $\bigcup_{j=0}^i A_j$ and the condition $s_i$ forces that $i \notin [B_i]$ and

$$|\hat{\mu}_{n_i}(B_i)| > \sum_{j=0}^i |\hat{\mu}_{n_i}(A_j)| + N + 2.$$

Let $s^* = s_K$. \hfill \Box

\textbf{Lemma 2.5.} Let $K, P \in \omega$. Let $\mu_1, \ldots, \mu_K$ be a sequence of $K$ measures on $\mathcal{A}$. Assume that $K \cdot \|\mu_j\| < P$ for every $1 \leq j \leq K$. Then, for every $Q > K \cdot P$ and every pairwise disjoint elements $C_1, \ldots, C_Q$ of $\mathcal{A}$ there exist natural numbers $k_1 < \ldots < k_{Q - K \cdot P}$ such that

$$|\mu_j|(C_{k_l}) < 1/K$$

for every $1 \leq j \leq K$ and $1 \leq l \leq Q - K \cdot P$.

\textbf{Proof.} Let $Q > K \cdot P$ and $C_1, \ldots, C_Q$ be an antichain in $\mathcal{A}$. Assume that there exist $k_1 < \ldots < k_P$ such that

$$|\mu_j|(C_{k_l}) \geq 1/K$$
for some $1 \leq j \leq K$ and every $1 \leq l \leq P$. Then, we have:

$$\|\mu_j\| \geq \sum_{l=1}^{P} |\mu_j| (C_k) \geq P \cdot 1/K > K \cdot \|\mu_j\| \cdot 1/K = \|\mu_j\|,$$

a contradiction, so for every $1 \leq j \leq K$ there must exist at most $P - 1$ elements $B_i$’s such that

$$|\mu_j| (C_k) \geq 1/K.$$

Hence, the thesis of the lemma holds for some $Q - K \cdot (P - 1) \geq Q - K \cdot P$ elements $B_i$’s. □

The following lemma is standard, cf. Baumgartner [3, Lemmas 1.5–1.8].

**Lemma 2.6.** Let $s \in S^\kappa$, $N \in \omega$ and $F_N \in [\text{dom}(s)]^{<\omega}$.

a) $\{s(\sigma): \sigma \in l(F_N, N, s)\}$ is an antichain in $S^\kappa$ and $s = \bigcup_{\sigma \in l(F_N, N, s)} s|\sigma$.

b) If $\sigma \in l(F_N, N, s)$ and $p \leq s|\sigma$, then there exists $q \leq_F N s$ such that $q|\sigma = p$.

c) If $D \subseteq S^\kappa$ is open dense below $s$, then there exists $q \leq_F N s$ such that $q|\sigma \in D$ for every $\sigma \in l(F_N, N, s)$.

□

**Lemma 2.7.** Let $A_0, \ldots, A_k, M, N, (\tilde{\mu}_n: n \in \omega), l$ and $s$ be as in the assumptions of Lemma 2.4. Let $F_N \in [\text{dom}(s)]^{<\omega}$. Put $K = |l(F_N, N, s)|$ and enumerate $l(F_N, N, s) = \langle \sigma_i: 1 \leq i \leq K \rangle$.

Then, there exist a condition $s^* \leq_{F_N, N} s$, a sequence $B_1, \ldots, B_K$ of pairwise disjoint elements of $A$ disjoint with $1_A \setminus \bigcup_{j=0}^{k} A_j$ and a sequence $n_K > \ldots > n_1 > M$ such that for every $1 \leq i \leq K$ the condition $s^*|\sigma_i$ forces that:

- $|\tilde{\mu}_{n_i}(B_i)| > \sum_{j=0}^{k} |\tilde{\mu}_{n_i}(\tilde{A}_j)| + \sum_{j=1}^{i-1} |\tilde{\mu}_{n_i}(\tilde{B}_j)| + \check{N} + 2$,

- $|\tilde{\mu}_{n_i}| \bigg(\bigvee_{i=1}^{n_K} \tilde{B}_i\bigg) < 1$,

- $i \notin \bigcup_{i=1}^{K} [\tilde{B}_i]$.

**Proof.** The proof basically goes by induction in $K$ steps — each step for one $\sigma_i$ ($1 \leq i \leq K$). We start as follows — by Lemmas 2.4 and 2.6.b) there exist a sequence of conditions $s_1 \leq_{F_N, N} s$, a family $\mathcal{R}_1 = \{B_1, \ldots, B_K\}$ of pairwise disjoint elements of $A$ disjoint with $1_A \setminus \bigcup_{j=0}^{k} A_j$, a sequence $n^1_K > \ldots > n^1_1 > M$ of natural numbers and a natural number $\check{P}_1 > 0$ such that for every $1 \leq j \leq K$ we have:

- $s_1|\sigma_1 \models |\tilde{\mu}_{n^1_1}(\tilde{B}^1_1)| > \sum_{l=0}^{k} |\tilde{\mu}_{n^1_1}(\tilde{A}_l)| + \check{N} + 2$,

- $s_1|\sigma_1 \models \check{K} \cdot \|\tilde{\mu}_{n^1_1}\| < \check{P}_1$, and

- $s_1|\sigma_1 \models i \notin \bigcup_{i \in \mathcal{R}^1} [B_i]$.

Assume now that for some $1 \leq L < K$ we have found:

- a sequence of conditions $s_L \leq_{F_N, N} \ldots \leq_{F_N, N} s_1 \leq_{F_N, N} s$, for every $1 \leq i \leq L$ a sequence of families $\mathcal{R}^i_L \subseteq \ldots \subseteq \mathcal{R}^i_1 \subseteq \mathcal{P}^i \subseteq A$ of pairwise disjoint non-zero elements of $A$ with $\mathcal{R}^i_L \neq \emptyset$ and $\mathcal{R}^i = \{B^i_1, \ldots, B^i_K\}$,

- a sequence of natural numbers $n^L_K > \ldots > n^L_1 > n^{L-1}_K > \ldots > n^{L-1}_1 > \ldots > n^1_K > \ldots > n^1_1 > M$, and

- a sequence of natural numbers $P_L > \ldots > P_1 > 0$,

such that:
(i) for every $1 \leq i \leq L$ and $1 \leq j \leq K$ we have:

$$s_i | \sigma_i | \not\vdash | \mu_{n_j} (\hat{B}_j) | > \sum_{l=0}^{k} | \mu_{n_j} (\hat{A}_l) | + \sum_{l=1}^{i-1} \sum_{B \in \mathcal{B}_i} | \mu_{n_j} (B) | + \hat{N} + 2, \text{ and}$$

$$s_i | \sigma_i | \not\vdash \hat{K} \cdot | \mu_{n_j} | < \hat{P}_i;$$

(ii) for every $1 \leq j \leq i \leq L$ we have:

$$s_{i} | \sigma_{j} | \not\vdash i \notin \bigcup_{l=1}^{i} \bigcup_{B \in \mathcal{B}_i} [B];$$

(iii) for every $1 \leq l < i \leq L, 1 \leq j \leq K$ and $B \in \mathcal{B}_i$ we have:

$$s_{i} | \sigma_{l} | \not\vdash | \mu_{n_j} (B) | < 1/\hat{K}.$$

Let us now construct $s_{L+1} \leq_{F_{N,N}} s_{L}, \mathcal{B}_{L+1}^1 \subseteq \mathcal{B}_L^1, \ldots, \mathcal{B}_{L+1}^L \subseteq \mathcal{B}_L^L, \mathcal{B}_{L+1} \subseteq \mathcal{B}_{L+1} \subseteq A, n_{K}^{L+1} > \ldots > n_{1}^{L+1} > n_{K}^{L+1}$ and $P_{L+1} > P_L$ satisfying also the properties (i)–(iii).

First, we modify a bit the condition $s_{L}$. By density, there exists $p \leq s_{L} | \sigma_{L+1}$ such that for every $1 \leq i \leq L$ either there exists unique $1 \leq j \leq K$ such that $p \vdash i \in [\hat{B}_j^i]$, or for every $B \in \mathcal{B}_L^L$ we have $p \vdash i \notin [\hat{B}]$. In the former case put $\mathcal{B}_{L+1}^i = \{ \hat{B}_j^i \}$, in the latter $\mathcal{B}_{L+1} = \mathcal{B}_L$. By Lemma 2.6.b), there exists $q \leq_{F_{N,N}} s_{L} \text{ such that } q | \sigma_{L+1} = p$. Note that

$$q | \sigma_{L+1} \vdash i \notin \bigcup_{j=0}^{k} [\hat{A}_j] \cup \bigcup_{l=1}^{L} \bigcup_{B \in \mathcal{B}_L^L} [B].$$

By Lemmas 2.4 and 2.6.b), there exist a condition $r \leq_{F_{N,N}} q$, a family $\mathcal{C} = \{ C_1, \ldots, C_Q \}$ of pairwise disjoint elements of $A$ disjoint with $1_A \setminus \left( \bigvee_{j=1}^{k} A_j \right. \vee \bigvee_{l=1}^{L} \bigvee_{B \in \mathcal{B}_L^L} [B], \right)$ where $Q = K \cdot L \cdot P_L + K$, a sequence $m_0 > \ldots > m_1 > n_{K}^{L+1}$ of natural numbers and a natural number $P_{L+1}$ such that for every $1 \leq j \leq Q$ we have:

$$r | \sigma_{L+1} \vdash | \mu_{m_j} (C_j) | > \sum_{l=0}^{k} | \mu_{m_j} (\hat{A}_l) | + \sum_{l=1}^{i} \sum_{B \in \mathcal{B}_{L+1}} | \mu_{m_j} (B) | + \hat{N} + 2, \text{ and}$$

$$r | \sigma_{L+1} \vdash \hat{K} \cdot | \mu_{m_j} | < \hat{P}_{L+1}, \text{ and}$$

$$r | \sigma_{L+1} \vdash i \notin \bigcup_{j=1}^{Q} [C_j].$$

We now define $s_{L+1}$ out of $r$ in two steps. In the first step, by induction, the inequality (2) and Lemmas 2.5 and 2.6.b), we get a sequence $\mathcal{C}_L \subseteq \ldots \subseteq \mathcal{C}_1 \subseteq \mathcal{C}$ with $\mathcal{C}_L = K$, a sequence $k_K > \ldots > k_1$ of natural numbers and a sequence of conditions $p_L \leq_{F_{N,N}} \ldots \leq_{F_{N,N}} p_1 \leq_{F_{N,N}} r$ such that $\mathcal{C}_L = \{ C_{k_1}, \ldots, C_{k_K} \}$ and for every $1 \leq i \leq L, 1 \leq j \leq K$ and $C \in \mathcal{C}_i$ we have:

$$p_i | \sigma_i | \not\vdash | \mu_{n_j} (C) | < 1/\hat{K}.$$ 

For every $1 \leq j \leq K$ write $B_{j}^{L+1} = C_{k_j}$ and $n_{j}^{L+1} = m_{k_j}$, and put $\mathcal{B}_{L+1}^{K} = \{ B_{j}^{L+1}, \ldots, B_{K}^{L+1} \}$.

In the second step, by induction and again Lemma 2.6.b), we get a sequence $t_L \leq_{F_{N,N}} \ldots \leq_{F_{N,N}} t_1 \leq_{F_{N,N}} p_L$ such that for every $1 \leq i \leq L$ either there
exists $1 \leq j_i \leq K$ such that $t_i|\sigma_i \models i \in [\bar{B}_{j_i}^{L+1}]$, or for every $1 \leq j \leq K$ we have $t_i|\sigma_i \models i \notin [\bar{B}_{j_i}^{L+1}]$. Put:

$B_{L+1}^T = \mathcal{B} \setminus \{B_{j_i}^{L+1}: t_i|\sigma_i \models i \in [\bar{B}_{j_i}^{L+1}], 1 \leq i \leq L\}$ and

$s_{L+1} = t_L$.

Note that by (7) and (9), for every $1 \leq i \leq L + 1$ we have:

$s_{L+1} \models i \notin \bigcup_{B \in B_{L+1}^T} \bar{B}_i^T$.

After the $K$-th step of the induction has been finished, we are left with the non-empty collections $\mathcal{B}_1^K, \ldots, \mathcal{B}_K^K$ (some of them may be singletons), the sequence $n_i^K > n_i^{K-1} > \ldots > n_i^1 > n_i^0 > M$ and the conditions $s_K \leq F_N, \ldots \leq F_N, N < s_1 \leq F_N, N$. From each $\mathcal{B}_i^K$ pick one element $B_i^i$. Then, for every $1 \leq i \leq K$ by (1) and (6) we have:

$s_K|\sigma_i \models |\bar{\mu}_{n_i}^i (\bar{B}_i^i)| > \sum_{j=0}^{k} |\bar{\mu}_{n_i}^j (\bar{A}_j)| + \sum_{j=1}^{i-1} |\bar{\mu}_{n_i}^j (\bar{B}_j^T)| + \bar{N} + 2$,

and by (4) and (8):

$s_K|\sigma_i \models |\bar{\mu}_{n_i}^i (\bigvee_{j=i+1}^K \bar{B}_j^i) = \sum_{j=i+1}^K |\bar{\mu}_{n_i}^j (\bar{B}_j^i)| < K : 1/K = 1$,

and finally by (3), (5) and (10):

$s_K|\sigma_i \models i \notin \bigcup_{j=1}^K \bar{B}_j^i$.

Put:

$s^* = s_K$

and for every $1 \leq i \leq K$:

$B_i = B_i^i$ and $n_i = n_i^i$.

By Lemma 2.6.a) we immediately obtain the following corollary.

**Corollary 2.8.** Let $A_0, \ldots, A_K$, $K$, $M$, $N$, $\langle \bar{\mu}_n: n \in \omega \rangle$, $i$, $s$ and $F_N$ be as in the assumptions of Lemma 2.7.

Then, there exist a condition $s^* \leq F_N, N$ such, a sequence $B_1, \ldots, B_K$ of pairwise disjoint elements of $A$ disjoint with $1_A \setminus \bigcup_{j=0}^{K} A_j$ and a sequence $n_i^K > \ldots > n_i^1 > M$ such that $s^*$ forces that $i \notin \bigcup_{i=1}^K \bar{B}_i$ and that there exists $1 \leq i \leq K$ for which it holds:

$|\bar{\mu}_n (\bar{B}_i)| > \sum_{j=0}^{k} |\bar{\mu}_n (\bar{A}_j)| + \sum_{j=1}^{i-1} |\bar{\mu}_n (\bar{B}_j)| + \bar{N} + 2$

and

$|\bar{\mu}_n (\bigvee_{j=i+1}^K \bar{B}_j) < 1$. 

□
Proposition 2.9. Let \( \langle \mu_n : n \in \omega \rangle \) be a sequence of names for measures on \( A \).
Let \( s \in S^* \) force that \( \langle \hat{\mu}_n : n \in \omega \rangle \) is anti-Nikodym.
Then, there exists:

- an increasing sequence \( \langle K_N : N \in \omega \rangle \) of natural numbers,
- a sequence \( \langle B_i^N : 1 \leq i \leq K_N, N \in \omega \rangle \) of pairwise disjoint elements of \( A \),
- a sequence \( \langle n_i^N : 1 \leq i \leq K_N, N \in \omega \rangle \) in \( \omega \) such that \( n_i^N > M_{K_N} > \ldots > n_i^N \)
for every \( N > M \), and
- a condition \( s^* \leq s \) forcing for every \( N \in \omega \) that there exist \( 1 \leq i \leq K_N \) such that:

\[
|\hat{\mu}_{n_i^N}(B_i^N)| > \sum_{M=0}^{N-1} \sum_{j=1}^{K_M} |\hat{\mu}_{n_i^N}(B_j^M)| + \sum_{j=1}^{i-1} |\hat{\mu}_{n_i^N}(B_j^N)| + \hat{N} + 2
\]
and

\[
|\hat{\mu}_{n_i^N}(\bigvee_{j=i+1}^{K_N} B_j^N)| < 1.
\]

Proof. The conclusion follows by the inductive use of Corollary 2.8 (to obtain an appropriate fusion sequence \( \langle s_N : N \in \omega \rangle \) of conditions in \( S^* \)) and the ultimate use of the fusion lemma (to obtain a fusion condition \( s^* \in S^* \) such that \( s^* \leq s_{F_N,N} \)) for every \( N \in \omega \); see Baumgartner [3, Lemma 1.8]).

3. Main result

Throughout this section \( A \) is a ground model \( \sigma \)-complete Boolean algebra, i.e. \( A \in V \) and \( A \) is \( \sigma \)-complete in \( V \).

Lemma 3.1. Let \( X \in [\omega]^\omega \) and \( X = \bigcup_{k \in \omega} X_k \) be an infinite partition of \( X \) into infinite subsets. For every measure \( \mu \) on \( A \) and an antichain \( \langle B_N : N \in \omega \rangle \) in \( A \) there exists \( L \in \omega \) such that

\[
|\mu|\left( \bigvee_{N \in X_k} B_N \right) < 1
\]
for every \( k > L \).

Proof. Since \( \mu \) is finitely additive and bounded, we have:

\[
\sum_{k \in \omega} |\mu|\left( \bigvee_{N \in X_k} B_N \right) \leq |\mu|\left( \bigvee_{N \in \omega} B_N \right) \leq |\mu|(1_A) < \infty.
\]

Lemma 3.2. Let \( \langle B_N : N \in \omega \rangle \in V \) be an antichain in \( A \) and \( X \in [\omega]^\omega \cap V \).
Let \( s \in S^* \) be a condition, \( N \in \omega \), \( F_N \subseteq [\text{dom}(s)]^{<\omega} \) and \( \hat{\mu}_1, \ldots, \hat{\mu}_K \) names for measures on \( A \). Assume that \( s \) forces that \( \hat{\mu}_1, \ldots, \hat{\mu}_K \) are measures. Then, there exists a condition \( s^* \leq s_{F_N,N} \) and a set \( X' \in [X]^\omega \cap V \) such that for every \( 1 \leq i \leq K \) we have:

\[
s^* \models |\hat{\mu}_i|\left( \bigvee_{M \in X'} B_M \right) < 1.
\]

Proof. Let \( X = \bigcup_{k \in \omega} X_k \) be an infinite partition of \( X \) into infinite sets. By Lemma 3.1 the following set is open dense below \( s \):

\[
D = \left\{ p \leq s : \forall 1 \leq i \leq K \exists L \in \omega \forall k > L : p \models |\hat{\mu}_i|\left( \bigvee_{M \in X_k} B_M \right) < 1 \right\}.
\]
By Lemma 2.6.c) there exists \( s^* \leq_{F_N,N} s \) such that \( s^*|\sigma \in D \) for every \( \sigma \in l(F_N,N,s) \). Hence, for every \( \sigma \in l(F_N,N,s) \) there exists \( L_\sigma \) \( \omega \) such that for every \( k > L_\sigma \) the condition \( s^*|\sigma \) forces that:
\[
|\bar{\mu}_k|\left( \bigvee_{M \in X_k} \bar{B}_M \right) < 1.
\]

Let \( L = \max \{ L_\sigma : \sigma \in l(F_N,N,s) \} + 1 \). Put \( X' = X_L \) and appeal to Lemma 2.6.a).

We are now in the position to prove the main theorem of this paper.

**Theorem 3.3.** Let \( G \) be an \( S^* \)-generic filter over \( V \). Then, in \( V[G] \) the Boolean algebra \( A \) has the Nikodym property.

**Proof.** Working in \( V[G] \) assume that \( A \) does not have the Nikodym property. Then, there exists an anti-Nikodym sequence \( \langle \mu_n : n \in \omega \rangle \) of measures on \( A \). Let \( t \in K_A \) be its Nikodym concentration point.

Now and to the end of the proof, let us work in the ground model \( V \). Let \( \langle \mu_n : n \in \omega \rangle \) be a sequence of names for measures in the sequence \( \langle \mu_n : n \in \omega \rangle \) and \( t \) a name for \( t \). There exists a condition \( s \in G \) forcing that \( \langle \mu_n : n \in \omega \rangle \) is anti-Nikodym on \( A \) and \( t \) is its Nikodym concentration point.

Let \( \langle K_N : N \in \omega \rangle, \langle B_i^N : 1 \leq i \leq K_N, N \in \omega \rangle, \langle n_i^N : 1 \leq i \leq K_N, N \in \omega \rangle \) and \( s^* \leq s \) be given by Proposition 2.9. We will find a condition \( s^{**} \leq s^* \) and a set \( Y \in [\omega]^{\omega} \cap V \) such that \( s^{**} \) forces that
\[
\bar{B} = \bigvee_{N \in Y} B_i^N \in \bar{A}
\]
and
\[
\sup_{n \in \omega} |\bar{\mu}_n(\bar{B})| = \infty,
\]
which will contradict the fact that \( s \) forces that \( \langle \mu_n : n \in \omega \rangle \) is pointwise bounded.

To obtain \( s^{**} \) and \( Y \) we follow by induction and use Lemma 3.2 to construct a fusion sequence \( \langle s_N : N \in \omega \rangle \) of conditions such that \( s_0 = s^* \) and for every \( N \in \omega \) we have \( s_{N+1} \leq_{F_N,N} s_N \), where \( F_N = \{ \alpha_i^k : i, k < N \} \) and \( \text{dom}(s_N) = \{ \alpha_i^k : k \in \omega \} \), and a decreasing sequence \( \langle X_N : N \in \omega \rangle \) of infinite subsets of \( \omega \) such that:

- \( X_0 = \omega \) and for every \( N \in \omega \) we have \( \min X_N < \min X_{N+1} \), and
- for every \( N \in \omega \) and \( L = \min X_N \) the condition \( s_N \) forces that:
\[
|\bar{\mu}_n|\left( \bigvee_{M \in X_{N+1}} B_j^M \right) < 1
\]
for every \( 1 \leq i \leq K_L \).

Let \( s^{**} \in S^* \) be such a condition that \( s^{**} \leq_{F_N,N} s_N \) for every \( N \in \omega \) (see Baumgartner [3, Lemma 1.8]). Put:
\[
Y = \{ \min X_N : N \in \omega \}
\]
and
\[
B = \bigvee_{N \in Y} B_i^N.
\]
Then, $B \in \mathcal{A}$ and, since $(X_N : N \in \omega)$ is decreasing, $s^{**}$ forces that for every $N \in Y$ and $1 \leq i \leq K_N$ the following inequality holds:

$$|\hat{\mu}_{n,N}(\bigvee_{M \in Y}^{K_M} \hat{B}^M_j)| < 1.$$  

Finally, since $s^{**} \leq s^*$, $s^{**}$ forces for every $N \in Y$ that there exists $1 \leq i \leq K_N$ such that

$$|\hat{\mu}_{n,N}(B^N_i)| > \sum_{M \in N}^{K_M} \sum_{j=1}^{K_N} |\hat{\mu}_{n,N}(\hat{B}^M_j)| + \sum_{j=1}^{i-1} |\hat{\mu}_{n,N}(\hat{B}^N_j)| + N + 2$$

and hence:

$$|\hat{\mu}_{n,N}(\hat{B})| = |\hat{\mu}_{n,N}(\bigvee_{M \in Y}^{K_M} \hat{B}^M_j) + \hat{\mu}_{n,N}(\bigvee_{j=1}^{i-1} \hat{B}^N_j) + \hat{\mu}_{n,N}(\bigvee_{j=1}^{K_N} \hat{B}^N_j)| \geq$$

$$\geq |\hat{\mu}_{n,N}(B^N_i)| - \sum_{M \in N}^{K_M} \sum_{j=1}^{K_N} |\hat{\mu}_{n,N}(\hat{B}^M_j)| - \sum_{j=1}^{i-1} |\hat{\mu}_{n,N}(\hat{B}^N_j)| -$$

$$- |\hat{\mu}_{n,N}(\bigvee_{j=1}^{K_N} \hat{B}^N_j)| - |\hat{\mu}_{n,N}(\bigvee_{M \in Y}^{K_M} \hat{B}^M_j)| \geq$$

$$\geq N + 2 - 1 - 1 = N.$$  

Thus, $s^{**}$ forces that for every $N \in \omega$ there exists $n$ such that $|\hat{\mu}_{n}(\hat{B})| > N$ and hence $s^{**}$ forces that $\sup_{n \in \omega} |\hat{\mu}_{n}(\hat{B})| = \infty$. □

Since the forcing $S^\ast$ preserves $\omega_1$ and $\kappa = \varepsilon$ in any $S^\ast$-generic extension (see Baumgartner [3, Theorems 1.11 and 1.14]), we immediately obtain the following corollary.

**Corollary 3.4.** Assume that $V$ is a model of ZFC+$\text{CH}$. If $G$ is an $S^\ast$-generic filter, then in $V[G]$ the relations $\omega_1 < \kappa = \varepsilon$ hold and $\mathcal{A}$ is an example of a Boolean algebra with the Nikodym property and of cardinality $\omega_1$.

Schachermayer [11, Theorem 2.5] proved that if a Boolean algebra $\mathcal{A}$ has simultaneously the Nikodym property and the Grothendieck property, then $\mathcal{A}$ has the Vitali–Hahn–Saks property, i.e. every pointwise convergent sequence of measures on $\mathcal{A}$ is uniformly exhaustive. Thus, Theorem 3.3 and Breč’s result [4, Theorem 3.1] imply together that if $\mathcal{A}$ is a $\sigma$-complete Boolean algebra in the ground model $V$, then it has the Vitali–Hahn–Saks property in the $S^\ast$-generic extension $V[G]$. In particular, as in Corollary 3.4, this yields a simple consistent example of a Boolean algebra with the Vitali–Hahn–Saks property and of cardinality strictly less than $\varepsilon$. 
References


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